4 Unit Distance Representations

Fix a graph $G = (V, E)$ with $|V| = n$.

Representation: A family of vectors $\{x_v\}_{v \in V}$ in $\mathbb{R}^m$ is a *representation* of G. We view G as drawn in $\mathbb{R}^m$ with each vertex v at the point x_v and each edge as a straight line segment connecting its endpoints. It is *unit distance* if $||x_i - x_j|| = 1$ whenever $ij \in E$.

Problem: Find a unit distance representation which minimizes $\max_{i \in V} ||x_i||$.

Model: As usual in semidefinite programming, we want to find a family of vectors $\{x_v\}_{v \in V}$, but we must encode the program in terms of the associated Gram matrix X, given by $X_{ij} = x_i \cdot x_j$. In this case, our objective is equivalent to

$$\text{minimize } \max_{i \in V} X_{ii}$$

while the constraint for unit distance is given by

$$1 = ||x_i - x_j||^2 = (x_i - x_j) \cdot (x_i - x_j) = X_{ii} - 2X_{ij} + X_{jj}$$

Though this does not have the form of our usual SDP, we can rewrite it to be so. Set

$$Y = \begin{bmatrix} s & & & & \\ & r_1 & & 0 & \\ & & \ddots & & \\ & 0 & & r_n & \\ & & & & X \end{bmatrix}$$

Now, $Y \succeq 0$ if and only if $s, r_1, \ldots, r_n \geq 0$ and $X \succeq 0$, so our problem is equivalent to the following SDP (which is in the usual form).

$$\begin{gathered} \text{minimize } s \\ \text{s.t. } Y \succeq 0 \\ s = X_{ii} + r_i \text{ for every } i \in V \\ X_{ii} - 2X_{ij} + X_{jj} = 1 \text{ whenever } ij \in E \end{gathered}$$

Kneser Graph: The *Kneser Graph* $Kn(n, k)$ has vertex set the collection of all k element subsets of $\{1, \ldots, n\}$ with two vertices adjacent if they are disjoint.

Theorem 4.1 *An optimal representation for Petersen* $\cong Kn(5, 2)$ *is given by assigning each vertex* $S \subseteq \{1, \ldots, 5\}$ *of* $Kn(5, 2)$ *to the vector* $x^S \in \mathbb{R}^5$ *given by*

$$x_i^S \begin{cases} \frac{3}{10} & \text{if } i \in S \\ -\frac{1}{5} & \text{otherwise} \end{cases}$$

Proof: Let Γ be the group consisting of all permutation matrices in $\mathbb{R}^{V \times V}$ which correspond to an automorphism of our graph. If X is an optimal solution to our SDP, then we can form a new optimal solution by averaging over the automorphism group as follows:

$$X' = \frac{1}{|\Gamma|} \sum_{Q \in \Gamma} Q X Q^\top$$

Since the automorphism group of Petersen is transitive on vertices, edges, and nonedges, it follows that X' has the form

$$X' = aA + bJ + cI$$

where A is the adjacency matrix, and $a, b, c \in \mathbb{R}$. So, to solve our SDP, we may restrict our attention to matrices which have this form. This gives us the following equivalent problem with three variables $a, b, c \in \mathbb{R}$

$$\min\{b + c : \ aA + bJ + cI \succeq 0 \text{ and } 2c - 2a = 1\}$$

The vector 1 is an eigenvector of $aA + bJ + cI$ of eigenvalue $3a + 10b + c$, while every other eigenvector of Petersen with eigenvalue λ is an eigenvector of $aA + bJ + cI$ of eigenvalue $\lambda a + c$. Now, the condition that our matrix is semidefinite is equivalent to all of these eigenvalues being nonnegative, and this is just a collection of linear constraints on a, b, c. So in other words, we have now reduced our problem to a three variable LP. Solving this yields the claimed answer. $\square$