Unexpected behaviour of crossing sequences

Matt DeVos  Bojan Mohar∗†
Robert Šámal‡§

Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
email: \{mdevos,mohar,rsamal\}@sfu.ca

December 14, 2007

Abstract
The \(n^{th}\) crossing number of a graph \(G\), denoted \(cr_n(G)\), is the minimum number of crossings in a drawing of \(G\) on an orientable surface of genus \(n\). We prove that for every \(a > b > 0\), there exists a graph \(G\) for which \(cr_0(G) = a\), \(cr_1(G) = b\), and \(cr_2(G) = 0\). This provides support for a conjecture of Archdeacon et al. and resolves a problem of Salazar.

1 Introduction
Planarity is ubiquitous in the world of structural graph theory, and perhaps the two most obvious generalizations of this concept—crossing number, and embeddings in more complicated surfaces—are topics which have been thoroughly researched. Despite this, relatively little work has been done on the common generalization of these two: crossing numbers of graphs drawn on surfaces. This subject seems to have been introduced in [4], and studied further in [1]. Following these authors, we define for every nonnegative integer \(i\) and every graph \(G\), the \(i^{th}\) crossing number, \(cr_i(G)\), (and also the \(i^{th}\) nonorientable crossing number, \(\tilde{c}r_i(G)\)) to be the minimum number of crossings in a drawing of \(G\) in the orientable (nonorientable, respectively) surface of genus \(i\). Observe that \(cr_i(G) = 0\) (respectively, \(\tilde{c}r_i(G) = 0\)) if and only

∗Supported in part by the Research Grant P1–0297 of ARRS (Slovenia), by an NSERC Discovery Grant (Canada) and by the Canada Research Chair program.
†On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.
‡Supported by PIMS postdoctoral fellowship.
§On leave from Institute for Theoretical Computer Science (ITI), Charles University, Prague, Czech Republic.
if \( i \) is greater or equal to the genus (resp., nonorientable genus) of \( G \). This gives, for every graph \( G \), two finite sequences of integers, \((cr_0(G), cr_1(G), \ldots, 0)\) and \((\tilde{c}r_0(G), \tilde{c}r_1(G), \ldots, 0)\), both of which end up with a zero. The first of these is the orientable crossing sequence of \( G \), the second the nonorientable crossing sequence of \( G \).

The natural question is to characterize crossing sequences of graphs. This is the focus of both [4] and [1]. If we are given a drawing of a graph in a surface \( S \) with at least one crossing, then modifying our surface in the neighborhood of this crossing by either adding a crosscap or a handle gives rise to a drawing of \( G \) in a higher genus surface with one crossing less. It follows from this that every orientable and nonorientable crossing sequence is strictly decreasing until it hits 0. This necessary condition was conjectured to be sufficient in [1].

**Conjecture 1.1 (Archdeacon, Bonnington, and Širáň)**

If \((a_1, a_2, \ldots, 0)\) is a sequence of nonnegative integers which strictly decreases until 0, then there is a graph whose crossing sequence (nonorientable crossing sequence) is \((a_1, a_2, \ldots, 0)\).

To date, there has been very little progress on this appealing conjecture. For the special case of sequences of the form \((a, b, 0)\), Archdeacon, Bonnington, and Širáň [1] constructed some interesting examples for both the orientable and nonorientable cases. We shall postpone discussion of their examples for the oriented case until later, but let us highlight their result for the nonorientable case here.

**Theorem 1.2 (Archdeacon, Bonnington, and Širáň)** If \(a\) and \(b\) are integers with \(a > b > 0\), then there exists a graph \(G\) with nonorientable crossing sequence \((a, b, 0)\).

It has been believed by many that such a result cannot hold for the orientable case. For the most extreme special case \((N, N - 1, 0)\), where \(N\) is a large integer, Salazar asked [3] if this sequence could really be the crossing sequence of a graph. The following almost contradictory quote of Dan Archdeacon illustrates why such crossing sequences are counterintuitive:

If \(G\) has crossing sequence \((N, N - 1, 0)\), then adding one handle enables us to get rid of no more than a single crossing, but by adding the second handle, we get rid of many. So, why would we not rather add the second handle first?

Our main theorem is an analogue of Theorem 1.2 for the orientable case, and its special case \(a = N\), \(b = N - 1\) resolves a question of Salazar [3].

**Theorem 1.3** If \(a\) and \(b\) are integers with \(a > b > 0\), then there exists a graph \(G\) whose orientable crossing sequence is \((a, b, 0)\).

Quite little is known about constructions of graphs for more general crossing sequences. Next we shall discuss the only such construction we know of. Consider a sequence \(a = (a_0, a_1, \ldots, a_g)\) and define the sequence \((d_1, \ldots, d_g)\) by the rule \(d_i = a_{i-1} - a_i\). Then, roughly speaking, \(d_i\) is the number of crossings which can be saved by adding the \(i^{th}\) handle. It seems intuitively clear that sequences for which \(d_1 \geq d_2 \geq \cdots \geq d_g\) should be crossing sequences, since here we receive diminishing returns for each extra handle we use. Indeed, Širáň [4] constructed a graph with crossing sequence \(a\) whenever \(d_1 \geq d_2 \geq \cdots \geq d_g\).
Constructing graphs for sequences which violate the above condition is rather more difficult. For instance, it was previously open whether there exist graphs with crossing sequence \((a, b, 0)\) where \(a/b\) is arbitrarily close to 1. The most extreme examples are due to Archdeacon, Bonnington and Širáň [1] and have \(a/b\) approximately equal to \(6/5\). Although our main theorem gives us a graph with every possible crossing sequence of the form \((a, b, 0)\), we don’t know what happens for longer sequences. The following problem asks for graphs where the first \(s\) handles save only an epsilon fraction of what is saved by the \(s + 1\)st handle.

**Problem 1.4** For every positive integer \(s\) and every \(\varepsilon > 0\), construct a graph \(G\) for which \(\text{cr}_0(G) - \text{cr}_s(G) \leq \varepsilon (\text{cr}_s(G) - \text{cr}_{s+1}(G))\).

A possible approach to this seems to be to combine the example of [4] and ours. In particular, it may be useful to consider how to embed a disjoint union of two graphs, \(G_1\) and \(G_2\). We can always “use part of the surface for \(G_1\) and the other part for \(G_2\)”, leading to

\[
\text{cr}_i(G_1 \cup G_2) \leq \min_j (\text{cr}_j(G_1) + \text{cr}_{i-j}(G_2)).
\]

We don’t know when/whether this inequality is in fact an equality. Perhaps surprisingly, we were not even able to decide the case \(i = 1\).

**Problem 1.5** Let \(G\) be a graph with connected components \(G_1\) and \(G_2\). Is there an optimal drawing of \(G\) on torus, such that no edge of \(G_1\) crosses an edge of \(G_2\)?

Our primary family of graphs can be constructed with relatively little machinery, so we shall introduce them here in the introduction. We will however use a couple of gadgets which are common in the study of crossing numbers. Let us pause here to define them precisely. A **special graph** is a graph \(G\) together with a distinguished subset \(T \subseteq E(G)\) of thick edges, a subset \(U \subseteq V(G)\) of rigid vertices and a family \(\{\pi_u\}_{u \in U}\) of prescribed local rotations for the rigid vertices. Here, \(\pi_u\) describes the cyclic ordering of the ends of edges incident with \(u\). A **drawing** of a special graph \(G\) in a surface \(\Sigma\) is a drawing of the underlying graph \(G\) with the added property that for every \(u \in U\), the local rotation of the edges incident with \(u\) given by this drawing either in the local clockwise or counterclockwise order matches \(\pi_u\).

The **crossing number** of a drawing of the special graph \(G\) is \(\infty\) if there is an edge in \(T\) which contains a crossing, and otherwise it is the same as the crossing number of the drawing of the underlying graph. We define the **crossing number** of a special graph \(G\) in a surface \(\Sigma\) to be the minimum crossing number of a drawing of \(G\) in \(\Sigma\), and \(\text{cr}_i(G)\) to be the crossing number of \(G\) in a surface of genus \(i\). In the next section, we shall prove the following result.

**Lemma 1.6** If \(G\) is a special graph with crossing sequence \(a\), then there exists an (ordinary) simple graph with crossing sequence \(a\).

This result permits us to use special graphs in our constructions. Indeed, starting in the third section, we shall consider special graphs on par with ordinary ones, and we shall drop the term special. When defining a (special) graph with a diagram, we shall use the convention that thick edges are drawn thicker, and vertices which are marked with a box
instead of a circle have the distinguished rotation scheme as given by the figure. With this terminology, we can now introduce our principal family of graphs.

The \( n \)th hamburger graph \( H_n \) is a special graph with \( 3n + 8 \) vertices. Its thick edges form a cycle \( C = qv_1 \ldots v_nrr's'u_n \ldots u_1tt'q'q \) of length \( 2n + 8 \) together with two additional thick edges \( \tau_0 = qr \) and \( \tau_1 = st \). See Figure 1. In addition to these, \( H_n \) has \( n \) special vertices \( u'_i \) (for odd values of \( i \)) and \( v'_i \) (for even values of \( i \)) with rotation as shown in the figure. These vertices are of degree 4 and they lie on paths \( r_1 = q'v'_2v'_4\ldots v'_mr' \) (where \( m = n \) if \( n \) is even and \( m = n - 1 \) otherwise) and \( r_2 = t'u'_1u'_3\ldots u'_ls' \) (where \( l = n \) if \( n \) is odd and \( m = n - 1 \) otherwise). These two paths will be referred to as the rows of \( H_n \). Each \( u'_i \) and each \( v'_i \) is adjacent to \( u_i \) and \( v_i \), and the 2-path \( c_i = u_iu'_iv_i \) (or \( c_i = u_i'v'_iv_i \), depending on the parity of \( i \)) is called a column of \( H_n \), \( i = 1, \ldots, n \).

We claim that the hamburger graph \( H_n \) has crossing sequence \((n, n - 1, 0)\) whenever \( n \geq 5 \) (or \( n = 3 \)). Although this does not handle all possible sequences of the form \((a, b, 0)\), as discussed above, these are in some sense the most difficult and counterintuitive cases. Indeed, a rather trivial modification of these will be used to get all possible sequences.

Since it is quite easy to sketch proofs of \( cr_0(H_n) = n \) and \( cr_2(H_n) = 0 \), let us pause to do so here (rigorous arguments will be given later). The first of these equalities follows from the observation that every row must meet every column in any planar drawing in which thick edges are crossing-free. The second equality follows from the observation that \( H_n \) minus the thick edges \( \tau_0, \tau_1 \) is a graph which can be embedded in the sphere. Using an extra handle for each of \( \tau_0, \tau_1 \) gives an embedding of the whole graph in a surface of genus 2. Of course, it is possible to draw \( H_n \) in the torus with only \( n - 1 \) crossings by starting with the drawing in the figure and then adding a handle to remove one crossing. In the third section we shall show that these are indeed optimal drawings (for \( n = 3 \) and \( n \geq 5 \)).

2 Gadgets

The goal of this section is to establish Lemma 1.6 which permits us to use special graphs in our constructions. The first part (dealing with thick edges) appears in [1], we include it for readers convenience.
Thick edges

For every $e \in E(G)$ choose positive integer $w(e)$ and replace $e$ by $L_e$—a copy of $L_{w(e)}$—whenever $w(e) > 1$. Let $G'$ be the resulting graph. We claim, that the crossing number of $G'$ is the same as the “weighted crossing number” of $G$: each crossing of edges $e_1, e_2$ is counted $w(e_1)w(e_2)$-times. Obviously, $cr(G')$ is at most that, as we can draw each $L_e$ sufficiently close to where $e$ was drawn. Moreover, there is an optimal drawing of this form (which proves the converse inequality): Given an optimal embedding of $G'$, consider the subgraph $L_e$ and from the $w(e)$ paths of length 2 between its “end-points” pick the one, that is crossed the least number of times. We can draw the whole subgraph $L_e$ close to this path without increasing the number of crossings.

This shows that we can “simulate weighted crossing number” by crossing number of a modified graph. In particular, we can let $w(e) = 1$ for each ordinary edge and $w(e) > cr(G)$ for each thick edge $e$ of $G$. This proves Lemma 1.6 for graphs with thick edges.

Rigid vertices

Suppose that we are considering drawings in surfaces of Euler genus $\leq g$; put $n = 3g + 2$. Let $G$ be a graph with rigid vertices. We replace each rigid vertex $v$ by a graph $V_v$—copy of $V_{n,\deg(v)}$. That is, we add $n$ nested thick cycles of length $d = \deg(v)$ around $v$ as shown in Figure 3 for $d = 6$ and $n = 5$. When doing this, the cycles meet the edges incident with $v$ in the same order as requested by the local rotation $\pi_v$ around $v$. If an edge incident with $v$ is thick, then all edges in $G'$ arising from it are thick too (as indicated in the figure for one of the edges). Call the resulting graph $G'$.

We claim that the crossing number of $G'$ (graph with thick edges but no rigid vertices) is the same as that of $G$. Any drawing of $G$ that respects the rotations at each rigid vertex can be extended to a drawing of $G'$ without any new crossing; in this drawing all $n$ thick cycles in each $V_v$ are contractible and $v$ is contained in the disc that any of them is bounding. We
will show, that there is an optimal drawing of $G'$ of this “canonical” type.

Let us consider an optimal drawing (respecting thick edges) of $G'$ in $S$ (of genus $\leq g$). Let $v$ be a rigid vertex of $G$, and consider the inner $n - 1$ out of the $n$ thick cycles in $V_v$. No edge of these cycles is crossed; so by [2, Proposition 4.2.6], either one of these cycles is contractible in $S$, or two of them are homotopic.

Suppose first, that one of the cycles, $Q$, is contractible. Since $Q$ separates the graph into two connected components, either the disk $D$ bounded by $Q$ or its exterior contains no vertex or edge of $G'$ apart from some cycles and edges of $V_v$. Let us assume that this is the interior of $D$. Now delete the drawing of all thick cycles in $V_v$ except $Q$, and delete the drawing of all $\deg(v)$ paths from $Q$ to $v$. Now think of $Q$ as the outermost cycle of $V_v$ and draw the rest on $V_v$ inside $D$ without crossings.

Suppose next, that two of the cycles, $Q_1$ and $Q_2$ are homotopic (and that $Q_1$ is closer to $v$ in $G'$). We cut $S$ along $Q_1$, and patch the two holes with a disc. This simplifies the surface, so if we can draw $G'$ on it without new crossings, we get a contradiction. Such drawing of $G'$ indeed exists, as we may delete the drawing of all of $V_v$ that is “inside” $Q_1$ and draw it in one of the new discs.

By performing such a change to each rigid vertex, we obtain an optimum drawing of $G'$ which is canonical. Consequently, it gives rise to a legitimate drawing of the special graph $G$, and which is also optimal for $G$. This shows that Lemma 1.6 holds also when there are special vertices.

### 3 Hamburgers

The hamburger graphs $H_n$ have been defined in the introduction. We have redrawn $H_n$ (for $n = 5$) again in Figure 4 where we have given names to numerous subgraphs of it. We have previously defined the rows $r_1, r_2$ and columns $c_1, \ldots, c_n$. For convenience we add rows $r_0$ and $r_3$ and columns $c_0$ and $c_{n+1}$ (see Figure 4). The cycle $C$ has two trivial bridges (the thick edges $\tau_0$ and $\tau_1$) and two other bridges. The first, denoted by $B_1$, consists of the row $r_1$ together with all columns $c_i$ with $i$ even. The second one is denoted by $B_2$ and consists of the row $r_2$ and columns $c_i$ with $i$ odd.

The goal of this section is to prove Theorem 1.3, showing the existence of a graph with crossing sequence $(a, b, 0)$ for every $a > b > 0$. The hamburger graphs $H_n$ have all of the key features of interest. To get every possible crossing sequence $(a, b, 0)$, we will also require a slightly more general class of graphs. For every $n, k \in \mathbb{N}$ with $n \geq 3$, we define the graph $H_{n,k}$, which is obtained from $H_n$ by adding $k$ duplicates of the second column $c_2$ as shown in Figure 5 for the case of $n = 4$ and $k = 3$. Note that $H_n \cong H_{n,0}$.

We shall denote by $S_g$ ($g \geq 0$) the orientable surface of genus $g$.

**Lemma 3.1** $cr_2(H_{n,k}) = 0$ for every $n, k \in \mathbb{N}$ with $n \geq 3$.

**Proof:** To draw $H_n$ in the double torus $S_2$, start by embedding $H_n - \tau_0 - \tau_1$ in the sphere $S_0$. Now, use one handle to route the edge $\tau_0$, and another handle for $\tau_1$. □
Figure 4: Main constituents of the graph $H_n$ (for $n = 5$)

Figure 5: The graph $H_{n,k}$ (for $n = 4$ and $k = 3$)
Lemma 3.2 \( cr_0(H_{n,k}) = n + k \) for every \( n, k \in \mathbb{N} \) with \( n \geq 3 \).

Proof: Consider a drawing of \( H_{n,k} \) in the sphere. If this drawing has finite crossing number, the cycle \( C \) must be embedded as a simple closed curve which separates the surface into two discs \( D_1, D_2 \) and is not crossed by any edge. Moreover, both thick edges \( \tau_0 \) and \( \tau_1 \) are drawn in the same disc, say \( D_2 \). Now every column of \( B_1 \) crosses the row \( r_2 \) and every column of \( B_2 \) crosses the row \( r_1 \), so we have at least \( n + k \) crossings. Since \( H_{n,k} \) is drawn in \( S_0 \) with \( n + k \) crossings in Figure 5, we conclude that \( cr_0(H_{n,k}) = n + k \) as required. \( \square \)

Not surprisingly, the situation when drawing our graphs \( H_n \) on the torus is considerably more complicated to analyze. By drawing \( H_n \) in the plane with \( n \) crossings and then using a handle to remove one crossing, we see that \( cr_1(H_n) \leq n - 1 \) for all \( n \geq 3 \). For \( n \geq 5 \), we shall prove that this is the best which can be achieved. For \( n \leq 4 \), however, there is some exceptional behavior (cf. Lemma 3.4).

When dealing with drawings on the torus, we will frequently use the following simple fact about curves drawn on torus (for reference see e.g. [2, Proposition 4.2.6]).

Fact. Any two simple closed curves on the torus of different (and nontrivial) homotopy types cross.

Lemma 3.3 For every optimal drawing of \( H_n \) (in a given surface), each column \( c_i \) (1 \( \leq i \leq n \)) is a simple curve.

Proof: It is easy to see that in every optimal drawing, every edge is represented by a simple curve. Let us now consider a column \( c_i = v_i v'_i u_i \) (or similarly for \( v_i u'_i u_i \)) and suppose that the edges \( e = v_i v'_i \) and \( f = u_i v'_i \) cross. Suppose that \( e \) is represented by the simple curve \( \alpha(t), 0 \leq t \leq 1 \), where \( \alpha(0) = v_i \) and \( \alpha(1) = v'_i \). Similarly, let \( f \) be represented by the simple curve \( \beta(t), 0 \leq t \leq 1 \), where \( \beta(0) = u_i \) and \( \beta(1) = v'_i \). Let \( \alpha(t') = \beta(t') \) (0 \( < t' < 1 \)) be where they cross. Now let \( \tilde{\alpha}(t) = \alpha(t) \) for \( t \leq t' \) and \( \tilde{\alpha}(t) = \beta(t) \) for \( t \geq t' \). Change similarly \( \beta \) to \( \tilde{\beta} \). Then the crossing becomes a touching of the two curves, which can be eliminated yielding a drawing with fewer crossings. Observe that the local rotation at the special vertex \( v'_i \) changes from clockwise to anticlockwise but this is still consistent with the requirement for this special vertex. Therefore the new drawing contradicts the optimality of the original one. \( \square \)

It will be convenient for us to classify different types of drawings of \( H_n \) in the torus depending on the drawing of the thick subgraph \( C + \tau_0 + \tau_1 \). In Figure 6 we have listed five possible embeddings of \( C + \tau_0 + \tau_1 \) in \( S_1 \), where \( \tau_0 \) and \( \tau_1 \) are drawn with broken lines. We shall say that a drawing of \( H_n \) is of type A, B, C, D, or E if the induced drawing of \( C + \tau_0 + \tau_1 \) is as in the corresponding part of Figure 6. To keep the number of cases limited, we only consider drawings of \( C + \tau_0 + \tau_1 \) up to its symmetry: that is, we don’t specify, which edge is \( \tau_0 \) and which is \( \tau_1 \), and also which column is \( c_0 \). As \( H_n \) does not have that much symmetry, we will have to be careful about this when dealing with the cases. Note also that \( C + \tau_0 + \tau_1 \) has embeddings in \( S_1 \) which are not of types A–E; they do not extend to a drawing of \( H_n \).
of finite crossing number, though. In the proof we will consider a particular drawing of $H_n$. If $A, B \subseteq G$ are subgraphs of $H_n$, then we shall denote by $Cr(A \mid B)$ the total number of crossings of an edge from $A$ with an edge from $B$, where crossings of an edge $e \in E(A \cap B)$ with another edge $f \in E(A \cap B)$ are counted only once. In particular, the total number of crossings of a graph $G$ is equal to $Cr(G \mid G)$.

Lemma 3.4 $cr_1(H_n) = n - 1$ if $n = 3$ or $n \geq 5$, while $cr_1(H_4) = 2$. Furthermore, Figure 7(a)–(e) shows the only drawings of $H_3$ in the torus with two crossings and the added property that at least one row has no crossings. Figure 8 displays the unique drawing of $H_4$ in the torus with two crossings. (The words “the only” and “unique” are meant up to homeomorphism of the torus and automorphism of the (unlabeled) graph $H_3$, resp. $H_4$.)

Proof: We proceed by induction on $n$. Consider a drawing $D$ of $H_n$ in a surface $S$ homeomorphic to the torus, such that $D$ yields minimum crossing number. We shall frequently use the inductive assumption for $n - 1$ and $n - 2$, since by deleting the edges of the column $c_1$, the column $c_n$, or two consecutive columns $c_i$ and $c_{i+1}$ we obtain a new graph which is a subdivision of $H_{n-1}$ or $H_{n-2}$ (assuming $n \geq 3$). This technique will be used throughout the proof. It is also worth noting that after applying this operation to $D$, the drawing of the smaller hamburger graph is of the same type as the drawing $D$.

The cycle $C$ is not crossed in $D$, so we may cut our surface along this curve. This leaves us with a drawing of $H_n$ in a closed bordered surface—which we shall denote $S'$—where each edge of $C$ appears twice on the boundary. We shall use $C^1$ and $C^2$ to denote these copies.

Essential to our proof is an analysis of the homotopy behavior of the rows and columns. To make this precise, let us now choose a point $N$ in the interior of the row $r_0$, $S$ in the interior of $r_3$, $E$ in the interior of $c_0$ and $W$ in the interior of $c_{n+1}$. (Actually, for each of these points we have two copies: $N^1$ and $N^2$, etc. But we will avoid distinguishing these if there is no danger of confusion). For each column $c_i$ ($0 \leq i \leq n + 1$) let $c_i^+$ be a simple
Figure 7: Exceptional drawings of $H_3$

Figure 8: Exceptional type B drawing of $H_4$
curve in $S'$ obtained by extending $c_i$ along the appropriate copies of the rows $r_0$ and $r_3$ so that it has ends $N$ and $S$. We shall focus our attention on the homotopy types in $S'$ of the curves $c^+_i$ where $N$ and $S$ are the fixed end points: we say that $c_i^+$ and $c_j^+$ are homotopic if $c_i^+$ may be continuously deformed to $c_j^+$ in the surface $S' \setminus \{\tau_0, \tau_1\}$, while keeping their endpoints fixed. Note that $c_i^+$ and $c_j^+$ can only be homotopic if $c_i$ and $c_j$ are connecting the same copies of $N$ and $S$—that is they attach on the same side of $C$ in the original surface $S$. Also note, that for $i = 0$ or $i = n + 1$ we actually have two copies of $c_i$, so we should be speaking of, e.g., $c_0^+$ and $c_0^{+2}$. We will refrain from this distinction whenever possible to keep the notation clearer—so when saying $c_0^+$ and $c_1^+$ are homotopic we will actually mean that $c_0^{+s}$ is homotopic with $c_1^+$ for some $s \in \{1, 2\}$.

We proceed similarly for the rows of $H_n$. Unfortunately, we do not know whether $r_1$ and $r_2$ are drawn as simple curves. For $j = 0, 1, 2, 3$ choose a simple curve $r_j^+$ contained in $r_j$ with the same ends as $r_j$, and extend $r_j^+$ along the columns $c_0$ and $c_{n+1}$ to a simple curve with ends $E$ and $W$, and denote this curve by $r_j^+$. As with the columns, we shall say that $r_j^+$ and $r_k^+$ are homotopic if one can be continuously deformed to the other in $S'$ while keeping the ends $E$ and $W$ fixed.

**Claim 1:** If $c_i^+$ and $c_{i+1}^+$ are homotopic ($1 \leq i < n$), then $Cr(r_j \mid c_i \cup c_{i+1}) \geq 1$ for $j = 1, 2$. Similarly, if $r_1^+$ and $r_2^+$ are homotopic, then $Cr(r_1 \cup r_2 \mid c_i) \geq 1$ for every $1 \leq i \leq n$.

The closed curve $\varphi$ obtained by following $c_i^+$ from $S$ to $N$ and then $c_{i+1}^+$ from $N$ to $S$ is contractible, and apart from its intersection with the cycle $C$, it intersects itself only at finitely many points. The row $r_j$ must cross either $c_i^+$ or $c_{i+1}^+$ (it cannot only touch it as their common vertex has prescribed local rotation), and must then form another crossing with $\varphi$. A similar argument holds for the rows.

**Claim 2:** If $C$ is contractible in $S$, then the drawing $D$ is of type A.

In this case, the surface $S'$ is disconnected, with one component a disc $D$, and the other component $S''$ homeomorphic to $S_1$ minus a disc. If both $B_1$ and $B_2$ are drawn in $D$, then we have at least $n$ crossings (as in Lemma 3.2). If only one of $B_1$ or $B_2$, say $B_1$ is drawn in $D$, then $B_2$ and the edges $\tau_0$ and $\tau_1$ are drawn in $S''$ (else the crossing number is infinite). Consider the curves $\tau_0 \cup r_0$ and $\tau_1 \cup r_3$ in $S''$. If either of these is contractible, then $B_2$ must cross it (yielding infinite crossing number). Otherwise they must be freely homotopic noncontractible curves in $S''$, so $\tau_0 \cup c_0 \cup \tau_1 \cup c_{n+1}$ is a contractible curve. Therefore $B_2$ must cross one of them, yielding again infinitely many crossings. Thus, we may assume that both $\tau_0$ and $\tau_1$ are drawn in the disc $D$ and $B_1$ and $B_2$ are drawn in $S''$ so our drawing is of type A.

**Claim 3:** If $C$ is not contractible, then the drawing $D$ is of type B, C, D, or E (up to homeomorphism).

If $C$ is not contractible, the surface $S'$ is a cylinder bounded by two copies of the cycle $C$. If both $\tau_0$ and $\tau_1$ have all of their ends on the same copy of $C$, we must have a drawing of type B or C. If one has both ends on one copy of $C$, and the other has both ends on the other copy of $C$, then there are infinitely many crossings, unless the drawing is of type
D. Finally, if one of $\tau_0$ or $\tau_1$, say $\tau_0$ has its ends on distinct copies of $C$, then the crossing number will be infinite unless $\tau_1$ has both ends on the same copy of $C$ giving us a drawing of type E.

In light of Claims 2 and 3, it now suffices to consider drawings of type A, B, C, D, and E which we shall treat one by one.

**Case 1:** Type A.

Let us first suppose that $n \geq 4$. If there exists $1 \leq i \leq n$ so that $c_i^+$ is homotopic to $c_0^+$, then either $c_1$ crosses $c_i$, or $c_1^+$ is homotopic to $c_0^+$. In the latter case, $c_1$ crosses $r_1$. So, in short, $Cr(c_1 \mid H_n) \geq 1$ and by removing this column and applying induction, we deduce that there are at least $n - 1$ crossings in our drawing. Note here that the resulting drawing of $H_{n-1}$ is still of type A, so it must have at least $n - 1$ crossings, even if $n = 5$. Thus, we may assume that $c_i^+$ is not homotopic to $c_0^+$ for any $1 \leq i \leq n$. By a similar argument, $c_i^-$ is not homotopic to $c_{n+1}^+$. If there exist $i, j \in \{1, \ldots, n\}$ with $c_i^+$ not homotopic to $c_j^-$, then $c_i^+$ and $c_j^-$ cross, and further, $Cr(c_k \mid c_i \cup c_j) \geq 1$ for every $k \in \{1, \ldots, n\}$ with $k \neq i, j$. This implies that we have at least $n - 1$ crossings, as desired. The only other possibility is that $c_i^+$ and $c_j^-$ are homotopic for every $i, j \in \{1, \ldots, n\}$. In this case, it follows from Claim 1 (applied to $c_1^+$ and $c_2^+$, $c_2^-$ and $c_3^+$, . . .) that there are at least $n - 1$ crossings.

Suppose now that $n = 3$. If $c_2^+$ is homotopic with $c_1^+$ or $c_3^+$, then it follows from Claim 1 that each row has at least one crossing, and we are done. Thus, we may assume that $c_2^+$ has distinct homotopy type from that of $c_1^+$ and from that of $c_3^+$. If $c_2^+$ is homotopic with either $c_0^+$ or $c_4^+$, then $Cr(c_2 \mid r_2) \geq 1$, and $Cr(c_2 \mid c_i) \geq 2$ for either $i = 1$ or $i = 3$, since $c_i^+$ is not homotopic to $c_2^+$. Thus, we may assume that $c_2^+$ has homotopy type distinct from that of $c_0^+, c_1^+, c_2^+, c_4^+$. If $c_1^+$ is homotopic with $c_0^+$ and $c_3^+$ is homotopic with $c_1^+$, then $Cr(r_1 \mid c_1 \cup c_3) \geq 2$, and assuming there are at most two crossings, we must have a drawing equivalent to that in Figure 7(a). Thus, we may now assume (without loss of generality) that $c_1^+$ is not homotopic with $c_0^+$ . It follows from this that $Cr(c_1 \mid c_2) \geq 1$. If $r_1^+$ is homotopic with either $r_0^+$ or $r_3^+$, then $Cr(r_1 \mid c_i) \geq 1$ for $i = 1, 3$ so we have three or more crossings. Thus, we may assume that $r_1^+$ has homotopy type distinct from that of $r_0^+, r_3^+, r_2^+$ (the last assumption is from Claim 1). If $Cr(r_1 \mid r_2) \geq 1$, then we have nothing more to prove, so we may assume that these rows do not cross, and this implies that $r_2^+$ must be homotopic with $r_3^+$, so $Cr(r_2 \mid c_2) \geq 1$. If $c_3^+$ is not homotopic with $c_1^+$, then $Cr(c_2 \mid c_3) \geq 1$, which gives us 3 crossings. Otherwise, $c_3^+$ is homotopic with $c_1^+$, so $Cr(r_1 \mid c_3) \geq 1$ and again we have 3 crossings. This completes the proof of Case 1.

In all the remaining cases, we have that $S'$ is a cylinder, and in our figures we have drawn $S'$ with the boundary component $C^1$ on the top and $C^2$ on the bottom. It will be convenient to decide which parts of $C^1$ ($C^2$, resp.) correspond to $c_0$, $r_0$, $c_{n+1}$, and $r_3$, respectively. If a given drawing does not follow this decision, we will relabel the rows and/or the columns. (In cases B and D we don’t actually have to do this, instead we may use an appropriate symmetry of the torus.) The drawback is that this may slightly change the graph. The change, if any, is the same as switching the roles of $r_1$ and $r_2$. More precisely, we will always have one of the following:
(A) $r_1$ has a common vertex with even numbered columns and $r_2$ with odd columns (the original version).

(B) $r_1$ has a common vertex with odd columns and $r_2$ with even ones.

**Case 2:** Type B.

Here all of the column curves $c_i^+$ have ends $N^2$ and $S^2$. Recall that these are copies of $N$ and $S$ drawn at the “bottom copy” $C^2$ of $C$. Since all of these curves are simple, it follows that for every $1 \leq i \leq n$, the curve $c_i^+$ is either homotopic with the simple curve $N^2-W^2-S^2$ in $C^2$ (we shall call this homotopy type $\ell$), or with the simple curve $N^2-E^2-S^2$ in $C^2$ (homotopy type $r$). Let $a = a_1 a_2 \ldots a_n$ be the word given by the rule that $a_i$ is the homotopy type of $c_i^+$. We now have the following simple crossing property.

**P1.** If $a_i = r$ and $a_j = \ell$ where $1 \leq i < j \leq n$, then $Cr(c_i \mid c_j) \geq 2$.

If there exists $1 \leq i \leq n$ so that $Cr(c_i \mid H_n) \geq 4$, then $n \geq 5$ (otherwise the drawing is not optimal), and by removing $c_i$ and either $c_{i-1}$ or $c_{i+1}$ and applying the theorem inductively to the resulting graph, we deduce that there are at least $4 + cr_1(H_{n-2}) \geq n$ crossings in our drawing, a contradiction. It follows from this and P1, that either $a = \ell^r r^{n-i}$ or $a = \ell^r \ell r n-i-2$.

We now split into subcases depending on $n$.

Suppose first that $n = 3$. If $a_1 = a_2 = \ell$ or $a_2 = a_3 = r$, then it follows from Claim 1 that $Cr(r_j \mid c_1 \cup c_2 \cup c_3) \geq 1$ for $j = 1, 2$ and we are finished. Otherwise, $a$ must be $\ell r \ell$ or $r \ell r$ and $Cr(c_2 \mid c_1 \cup c_3) \geq 2$. These configurations are possible, but require that our drawing is equivalent with the one in Figure 7(b)—this comes from $a = \ell r \ell$, if $a = r \ell r$ we get a mirror image.

If $n = 4$, let us first assume that $a = \ell^i r^{4-i}$. Applying Claim 1 for the columns $c_1, c_2$ and $c_3, c_4$ resolves the cases when $a$ is one of $\ell^4$, $r^4$, or $\ell^2 r^2$ (each gives at least four crossings—a contradiction). Suppose that $a = \ell^3 r$ (or, with the same argument, $a = \ell r^3$). It follows from Claim 1 that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$ and $Cr(c_2 \cup c_3 \mid r_1 \cup r_2) \geq 2$, so the only possibility for fewer than three crossings is that our drawing has 2 crossings, both of which are between $c_2$ and the rows $r_1$ and $r_2$. But then $c_2$ does not cross $c_1$ or $c_3$, so $c_2$ is separated from $c_0$ by $c_1^+$ and $c_3^+$, a contradiction.

Next suppose that $a = \ell^r \ell r^{2-i}$. If $a = \ell^2 r \ell$, then it follows from P1 that $Cr(c_3 \mid c_4) \geq 2$ and from Claim 1 that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$, so we have at least four crossings—a contradiction. Similarly $a = r \ell r^2$ is impossible. The only remaining possibility is $a = r \ell \ell r$. In this case, we have $Cr(c_2 \mid c_3) \geq 2$, so the only possibility is that there are exactly two crossings, both between $c_2$ and $c_3$. This case can be realized, but requires that our drawing is equivalent to that of Figure 8.

Lastly, suppose that $n \geq 5$. If $a \in \{\ell^i r^{n-i}, \ell^r \ell r^{n-i-2}\}$, then either $a_1 = a_2 = \ell$ or $a_{n-1} = a_n = r$. Since these arguments are similar, we shall consider only the former case. Now, it follows from Claim 1 that $Cr(c_1 \cup c_2 \mid r_1 \cup r_2) \geq 2$, so removing the first two columns gives us a drawing of $H_{n-2}$ with at least two crossings less than in our present drawing of $H_n$. By applying our theorem inductively to this new drawing, we find that the only possibility for less than $n-1$ crossings is that $n = 6$ and $a = \ell^3 r \ell r$. In this case, we have
\( Cr(c_4 \mid c_5) \geq 2 \), so we may eliminate two crossings by removing columns 4 and 5. This leaves us with a drawing of a graph isomorphic to \( H_4 \) as above with the pattern \( \ell^3 r \). It follows from our earlier analysis, that this drawing has at least three crossings. This completes the proof of Case 2.

**Case 3:** Type C.

Now each column curve has one end on the segment of \( C^2 \) between \( q^2 \) and \( r^2 \). As above, every curve \( c_i^+ \) with both ends on \( C^2 \) must be homotopic with either the simple curve \( N^2-W^2-S^2 \) in \( C^2 \) (we shall call this homotopy type \( \ell \)), or with the simple curve \( N^2-E^2-S^2 \) in \( C^2 \) (homotopy type \( r \)).

The homotopy types of the other column curves will be represented by integers. Since \( S' \) is a cylinder, we may choose a continuous deformation \( \Psi \) of \( S' \) onto the 1-sphere \( S^1 \) with the property that \( C^1 \) and \( C^2 \) map bijectively to \( S^1 \), and \( N^2 \) and \( S^1 \) map to the same point \( x \in S^1 \). Now, each curve \( c_i^+ \) maps to a closed curve in \( S^1 \) from \( x \) to \( x \), and for an integer \( \alpha \in \mathbb{Z} \), we say that \( c_i^+ \) has homotopy type \( \alpha \) if the corresponding curve in \( S^1 \) has (counterclockwise) winding number \( \alpha \). It follows that \( c_i^+ \) and \( c_j^+ \) are homotopic if and only if they have the same homotopy type. As before, we let \( a = a_1 a_2 \ldots a_n \) be the word given by the rule that \( a_i \) is the homotopy type of \( c_i^+ \). We now have the following useful crossing properties:

If \( 1 \leq i < j \leq n \) then

1. \( Cr(c_i \mid c_j) \geq |a_i - a_j - 1| \) if \( a_i, a_j \in \mathbb{Z} \).
2. \( Cr(c_i \mid c_j) \geq 2 \) if \( a_i = r \) and \( a_j = \ell \).
3. \( Cr(c_i \mid c_j) \geq 1 \) if either \( a_i = r \) and \( a_j \in \mathbb{Z} \) or \( a_i \in \mathbb{Z} \) and \( a_j = \ell \).

By choosing \( \Psi \) appropriately, we may further assume that the smallest integer \( 1 \leq i \leq n \) for which \( a_i \in \mathbb{Z} \) satisfies \( a_i = 0 \). Again, we split into subcases depending on \( n \).

Suppose first that \( n = 3 \). Note that every column of type \( r \) or \( \ell \) separates the segment \( q^2 t^2 \) on \( C^2 \) from \( r^2 s^2 \). Consequently, \( Cr(r_1 \cup r_2 \mid c_i) \geq 1 \) whenever \( a_i \in \{\ell, r\} \). Next consider the rows: each of \( r_1^+ \) and \( r_2^+ \) is homotopic to either \( r_0^+ \) or \( r_3^+ \). If \( r_1^+ \) and \( r_2^+ \) are homotopic then Claim 1 implies that there are at least three crossings. Hence, we may assume that \( r_1^+ \) is homotopic to \( r_0^+ \) and \( r_2^+ \) to \( r_3^+ \) (the other possibility yields two crossings and each row crossed).

(A) First suppose that \( r_1 \) has a vertex in common with \( c_2 \). (Recall the discussion before Case 2 about automorphisms of embedding types A–E versus automorphisms of \( H_n \).) Then \( c_i \) \( (i = 1,3) \) crosses \( r_1 \). If \( a_i \in \mathbb{Z} \) for \( i \in \{1,3\} \), then \( c_i \) also crosses \( r_2 \) because of the requirement about the local rotations at special vertices \( u'_1 \) and \( u'_3 \). Hence we have at least three crossings, unless \( a = \ell_0 \ell, \ell_0 r, r_0 \ell, \) or \( r_0 r \). Each of these, except \( \ell_0 r \) gives at least three crossings by (P3). The remaining case (with exactly two crossings, each between \( r_1 \) and one of \( c_1, c_3 \)) is possible, but only as it appears in Figure 7(c).

(B) Next suppose that \( r_1 \) has a vertex in common with \( c_1 \) and \( c_3 \). In this case \( c_2 \) crosses \( r_1 \) and (unless \( a_2 \in \{\ell, r\} \)) also \( r_2 \) (because of the local rotations at special vertices). Thus we may assume that \( a_2 \in \{\ell, r\} \). If \( a_i \in \{\ell, r\} \) for \( i = 1 \) or 3, then \( c_i \) crosses \( r_2 \). So we may
assume $a_1, a_3 \in \mathbb{Z}$ and (by P3) $c_2$ crosses either $c_1$ or $c_3$. The only way how to realize this case with two crossings is as in Figure 7(d) (or its “mirror image”).

Suppose now that $n \geq 4$. If either $c_1$ or $c_n$ is crossed, then we delete it and use the induction hypothesis. If none of them is crossed, then both $a_1$ and $a_n$ are integers (otherwise $Cr(c_1 \cup c_n \mid r_1 \cup r_2) \geq 1$ as above). It follows that $a_1 = 0$, and $a_n = -1$ (otherwise $c_1$ and $c_n$ cross). Now there is no value for $a_2$ to avoid crossing with either $c_1$ or $c_n$. This completes the proof of Case 3.

Case 4: Type D.

In this case, every column has one end on $r_2$ and one end on $r_1$. We define the homotopy types of curves $c_i^+$ as in the previous case. Again, $c_i^+$ and $c_j^+$ are homotopic if and only if they have the same homotopy type. As before, we let $a = a_1a_2\ldots a_n$ be the word given by the rule that $a_i$ is the homotopy type of $c_i^+$. And as before, we have the following useful crossing property:

P1. $Cr(c_i \mid c_j) \geq |a_i - a_j - 1|$ if $1 \leq i < j \leq n$.

Suppose first that $n \geq 4$. If the first column $c_1$ does not cross any other columns, then $a = 01^{n-1}$. Similarly, if the last column does not cross any other columns, then $a = 0^{n-1}1$. Since these cases are mutually exclusive for $n \geq 4$, we may assume (without loss) that the first column contains a crossing. Then we may remove it and apply induction.

If $n = 3$, we proceed as follows. Using P1 (and the convention $a_1 = 0$) we get that the number of crossings between the columns is at least $|a_2 + 1| + |a_3 + 1| + |a_2 - a_3 - 1| \geq |a_2 + 1| + |a_2|$ (using the triangle inequality). Symmetrically, we get another lower bound for the number of crossings: $|a_3 + 1| + |a_3 + 2|$. If any of these bounds is at least 3, we are done. It follows that $a_2 \in \{0, -1\}$ and $a_3 \in \{-1, -2\}$. Now, if there are two consecutive columns with the same homotopy type, then each row will cross some of these columns, and we are done. Consequently $a = 0, -1, -2$. It follows that $Cr(c_1 \mid c_3) \geq 1$. If $c_2$ crossed either $c_1$ or $c_3$, then it would have to cross the column twice—which would yield too many crossings. Similarly, if $Cr(c_1 \mid c_3) > 1$, then $Cr(c_1 \mid c_3) \geq 3$ and we would have too many crossings. It follows that the three columns $c_1, c_2, c_3$ are drawn as in Figure 9. Now we have that $c_1$ and $c_3$ separate $c_2$ from $c_0^1, c_0^2, c_{n+1}^1,$ and $c_{n+1}^2$. It follows that $Cr(r_1 \mid c_1 \cup c_3) \geq 2$ giving us too many crossings.
Figure 10: Towards type E drawings of $H_3$

**Case 5:** Type E.

In this case, every curve $c_i^+$ must have one end in $r_2$ and the other end in either $r_0$ or $r_0'$. In the first case, we say that $c_i^+$ has homotopy type 0 and in the second we say it has type $\ell$. It is immediate that any two such curves of the same type are homotopic. As usual, we let $a = a_1a_2\ldots a_n$ be the word given by the rule that $a_i$ is the homotopy type of $c_i^+$. The following rule indicates some forced crossing behavior.

P1. $Cr(c_i | c_j) \geq 1$ if $a_i = 0$ and $1 \leq i < j \leq n$.

Let us first treat the case when $n \geq 4$. If the last column $c_n$ contains at least one crossing, then we may remove it and apply induction to obtain a contradiction. Otherwise, (P1) implies that $a = \ell^n$ or $a = \ell^n0$. If $n \geq 5$, then it follows from Claim 1 that $Cr(c_1 \cup c_2 | r_1 \cup r_2) \geq 2$ so we may remove the first two columns and apply induction to get a contradiction. If $n = 4$ and $a = \ell^4$, then Claim 1 gives us at least four crossings—a contradiction. The only remaining case is $a = \ell^30$. If there are fewer than 3 crossings, then (again by applying Claim 1 twice) there are exactly two, and both occur on $c_2$. However, in this case $Cr(r_i | c_1 \cup c_3) = 0$ for $i = 1$ and for $i = 2$, again a contradiction.

Finally, suppose that $n = 3$. If there are two consecutive columns with the same homotopy type, then we are finished, so we may assume $a = 0\ell0$ or $a = \ell0\ell$. In the former case, we have $Cr(c_1 | c_2 \cup c_3) \geq 2$, so we may assume that there are exactly two crossings, and the columns must be drawn as in Figure 10(a). However, it is impossible to complete this drawing to a drawing of $H_3$ with fewer than three crossings. In the latter case we have $Cr(c_2 | c_3) \geq 1$ and the total number of crossings is at most two. It is possible to complete this to a drawing of $H_3$ with two crossings in two distinct ways. One of them is shown in Figure 10(b) and has crossings on both rows, $r_1$ and $r_2$. The second one is exceptional—it is shown in Figure 7(e). Because of symmetries, we also need to consider the case where $r_1$ and $r_2$ “interchange their roles” (see version (B) mentioned in the discussion preceding Case 2). Then we get another drawing of $H_3$ with two crossings; it is shown in Figure 10(c). Let us observe that in this case $r_1$ and $r_2$ cross each other.

This completes the proof of Lemma 3.4. □

Let us define the graph $H_3^+$ in the same way as $H_3$ except that we have three rows instead of two. See Figure 11.
Lemma 3.5 The graph $H^+_3$ has crossing sequence $(4, 3, 0)$ and the graph $H_{n,k}$ has crossing sequence $(n + k, n - 1, 0)$ for every $n \geq 3$ and $k \geq 0$, except when $n = 4$ and $k = 0$.

Proof: First we shall prove that $H^+_3$ has crossing sequence $(4, 3, 0)$. It follows from an argument as in Lemma 3.2 that $cr_0(H^+_3) = 4$. Since $H^+_3 - \tau_0 - \tau_1$ is planar, it follows that $cr_2(H^+_3) = 0$. It remains to show that $cr_1(H^+_3) = 3$. Since $cr_1(H^+_3) \leq 3$, we need only to show the reverse inequality. Consider an optimal drawing of $H^+_3$ in the torus, and suppose (for a contradiction) that it has fewer than three crossings. If the first row contains a crossing, then by removing its edges, we obtain a drawing of a subdivision of $H_3$ in the torus with at most one crossing—a contradiction. Thus, the first row must not have a crossing, and by a similar argument, the third row must not have a crossing. Now, we again remove the first row. This leaves us with a drawing of a subdivision of $H_3$ in the torus with at most two crossings, and with the added property that one row has no crossings. By Lemma 3.4 this must be a drawing as in Figure 7. A routine check of these drawings shows that none of them can be extended to a drawing of $H^+_3$ with fewer than 3 crossings.

Next, we shall consider $H_{n,k}$. Lemmas 3.1 and 3.2 show that $cr_0(H_{n,k}) = n + k$ and $cr_2(H_{n,k}) = 0$. We can draw $H_{n,k}$ in the torus with $n - 1$ crossings by adding a handle to the drawing from Figure 5. Moreover, any drawing of $H_{n,k}$ yields a drawing of $H_{n,0}$ (in the torus). By Lemma 3.4, a drawing of $H_{n,0}$ must have at least $n - 1$ crossings, unless $n = 4$. This completes the proof in all cases except when $n = 4$.

If $n = 4$, the same argument as above shows that $cr_1(H_{4,k}) \geq cr_1(H_{4,1})$; we shall prove now that $cr_1(H_{4,1}) \geq 3$. Suppose this is false, and consider a drawing of $H_{4,1}$ in the torus with at most two crossings. By removing the added column, we obtain a drawing of $H_4$ in the torus with at most two crossings. It follows from Lemma 3.4 that this drawing is equivalent to that in Figure 8. Since this drawing does not extend to a drawing of $H_{4,1}$ with $\leq 2$ crossings, this gives us a contradiction.

Thus $H_{n,k}, (n, k) \neq (4, 0)$, has crossing sequence $(n + k, n - 1, 0)$ as claimed. □

We are ready to prove the main result of this paper: For every $a > b > 0$, there is a graph with crossing sequence $(a, b, 0)$.

Proof of Theorem 1.3: We let $a > b > 0$ be integers, and we shall apply our earlier results to demonstrate the existence of a graph with crossing sequence $(a, b, 0)$. 

17
For \( b = 1 \) it is quite easy to construct such a graph. For instance, let \( G_1 \) be a copy of \( K_5 \), let \( G_2 \) be the graph obtained from a copy of \( K_5 \) by replacing each edge, except for one of them, with \( a - 1 \) parallel edges joining the same pair of vertices. Let \( G \) be the disjoint union of \( G_1 \) and \( G_2 \). It is immediate that \( cr_0(G) = a \), \( cr_2(G) = 0 \), and \( cr_1(G) \geq 1 \). A drawing of \( G \) in \( S_1 \) with this crossing number is easy to obtain by embedding \( G_2 \) in the torus, and then drawing \( G_1 \) disjoint from \( G_2 \) with one crossing.

For \( b \geq 2 \) we will use the graph \( H_{b+1,a-b-1} \). Unless \( b = 3 \), \( a = 4 \), it has crossing sequence \((a, b, 0)\) as verified by Lemma 3.5. For the remaining case \((b = 3 \text{ and } a = 4)\), the graph \( H_3^+ \) has the desired crossing sequence by Lemma 3.5.

\[ \square \]

References


