SMALL SEPARATIONS IN SYMMETRIC GRAPHS

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Abstract. We prove a rough structure theorem for small separations in symmetric graphs. Let $G = (V, E)$ be a vertex transitive graph, let $A \subseteq V$ be finite with $|A| \leq \frac{|V|}{2}$ and set $k = |\{v \in V \setminus A : u \sim v \text{ for some } u \in A\}|$. We show that whenever the diameter of G is at least $31(k+1)^2$, either $|A| \leq 2k^3$, or G has a (bounded) ring-like structure and A is efficiently contained in an interval. This theorem may be viewed as a rough analogue of an earlier result of Tindell, and has applications to the study of product sets and expansion in groups.

1. Overview

The study of expansion in vertex transitive graphs and in groups divides naturally into the study of local expansion, or connectivity, and the study of global expansion, or growth. The expansion properties of a group are those of its Cayley graphs, so vertex transitive graphs are the more general setting. Our main theorem concerns local expansion in vertex transitive graphs, but it is also meaningful for groups, and it has some asymptotic applications. Accordingly, we begin here with a tour of some of the important theorems in expansion.

Local Expansion in Groups. This is the study of small sum sets or small product sets. Throughout this section, we let $\mathcal{G}$ be a (multiplicative) group and let $A, B \subseteq \mathcal{G}$. The main questions of interest here are finding lower bounds on $|AB|$, and in the case when $|AB|$ is small, finding the structure of A, B. The first important result in this area was proved by Cauchy and (independently) Davenport. For every positive integer n, we let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Theorem 1.1 (Cauchy [6], Davenport [7]). *Let p be prime and let $A, B \subseteq \mathbb{Z}_p$ be nonempty. Then $|A + B| \geq \min\{p, |A| + |B| - 1\}$.*

This theorem was later refined by Vosper who found the structure of all $A, B \subseteq \mathbb{Z}_p$ (p prime) for which $|A + B| < |A| + |B|$. Before stating his theorem, note that in any finite group $\mathcal{G}$ we must have $AB = \mathcal{G}$ whenever $|A|+|B| > |\mathcal{G}|$ by the following pigeon hole argument: $\{a^{-1}g : a \in A\} \cap B \neq \emptyset$ for every $g \in \mathcal{G}$. For simplicity, we have avoided this uninteresting case below.

Theorem 1.2 (Vosper [24]). *Let p be a prime and let $A, B \subseteq \mathbb{Z}_p$ be nonempty. If $|A + B| < |A| + |B| \leq p$, then one of the following holds:*

(i) $|A| = 1$ *or* $|B| = 1$.
(ii) *There exists* $g \in \mathbb{Z}_p$ *so that* $B = \{g - a : a \in \mathbb{Z}_p \setminus A\}$, *and* $A + B = \mathbb{Z}_p \setminus \{g\}$.
(iii) A *and* B *are arithmetic progressions with a common difference.*

Analogues of the Cauchy-Davenport theorem and Vosper's theorem for abelian and general groups were found by Kneser, Kempermann, and DeVos. In abelian groups, we have powerful theorems which yield rough structural information when a finite subset $A \subseteq \mathcal{G}$ satisfies $|A + A| \leq c|A|$ for a fixed constant c. Freiman achieved such a theorem when $\mathcal{G} = \mathbb{Z}$ and this has recently been extended to all abelian groups by Green and Ruzsa. Despite this progress, there is still relatively little known in terms of rough structure of sets with small product in general groups. The following corollary of our theorem is a small step in this direction.

Corollary 1.3. *Let* $\mathcal{G}$ *be an infinite group, let* $A \subseteq \mathcal{G}$ *be a finite generating set, and let* $B \subseteq \mathcal{G}$ *be finite. If* $|AB| < |B| + \frac{1}{2}|B|^{\frac{1}{3}}$, *then* $\mathcal{G}$ *has a finite normal subgroup* N *so that* $\mathcal{G}/N$ *is either cyclic or dihedral. Furthermore,* $|N| \leq 2|A|$ *????.*

Our corollary also applies to finite groups, but for this it requires an assumption which is more natural in the context of graphs.

Local Expansion in Graphs. Before we begin our discussion of expansion in graphs, we will need to introduce some notation. If G is a graph and $X \subseteq V(G)$, we let $\delta(X) = \{uv \in E(G) : u \in X \text{ and } v \notin X\}$ and we call any set of edges of this form an *edge cut*. We let $\partial(X) = \{v \in V(G) \setminus X : uv \in E(G) \text{ for some } u \in X\}$ and we call $\partial(X)$ the *boundary* of X. Similarly, if $\vec{G}$ is a directed graph and $X \subseteq V(\vec{G})$ we let $\delta^+(X) = \{(u, v) \in E(\vec{G}) : u \in X \text{ and } v \notin x\}$ and $\delta^-(X) = \delta^+(V(G) \setminus X)$, and we let $\partial^+(X) = \{v \in V(G) \setminus X : (u, v) \in E(G) \text{ for some } u \in X\}$ and $\partial^-(X) = \{v \in V(G) \setminus X : (v, u) \in E(G) \text{ for some } u \in X\}$. Expansion in graphs is the study of the behavior of these parameters. We will be particularly interested in the case when these parameters are small. Next we introduce the types of graphs we will be most interested in.

Again we let $\mathcal{G}$ denote a multiplicative group. For every $A \subseteq \mathcal{G}$, we define the Cayley digraph $Cayley(\mathcal{G}, A)$ to be the directed graph (without multiple edges) with vertex set $\mathcal{G}$ and (x, y) an arc if $y \in Ax$. Using this definition, the group $\mathcal{G}$ has a natural (right) transitive action on $V(G)$ which preserves incidence. If $1 \in A$, and $B \subseteq \mathcal{G}$, then AB is the disjoint union of B and $\partial^+(B)$. This observation allows us to rephrase problems about small product sets in groups as problems concerning sets with small boundary in Cayley digraphs.

Next we give three bounds on the boundary of finite sets in vertex transitive graphs. These theorems are usually stated only for finite graphs, but the version below follows from the same argument.

Theorem 1.4. *Let $G = (V, E)$ be a connected d-regular vertex transitive graph, let $\emptyset \neq A \subseteq V$ be finite, and assume that $|A| \leq \frac{|V|}{2}$.*

(1) $|\delta A| \geq d$ (Mader [19]).
(2) $|\partial A| \geq \min\{|V \setminus A|, \frac{2}{3}(d+1)\}$ (Mader [20], Watkins [25]).
(3) $|\partial A| \geq \frac{|A|}{diam(A)+1}$ (Babai, Szegedy [4]) .
(4) *If* $|A| \geq 2$ *and* $|\delta A| = d$, *then* (i) *or* (ii) *holds* (Tindell [22])
 (i) *There is a block of imprimitivity which is a clique of size d.*
 (ii) $d = 2$ *(so G is a cycle).*
(5) *If* $|A| \geq \frac{1}{3}(d+1)$ *and* $|\delta A| \leq \frac{2}{9}(d+1)^2$, *then G has a block of imprimitivity of size* $\leq \frac{2}{9}(d+1)^2$ (van den Heuvel, Jackson [14]).

Theorem 1.5 (Mader [19]). *If G is a connected d-regular vertex transitive graph and $\emptyset \neq A \subset V(G)$ is finite, then $|\delta A| \geq d$.*

Theorem 1.6 (Mader [20], Watkins [25]). *If G is a connected d-regular vertex transitive graph and $\emptyset \neq A \subset V(G)$ is finite and satisfies $A \cup \partial A \neq V(G)$, then $|\partial A| \geq \frac{2}{3}(d+1)$.*

Theorem 1.7 (Hamidoune [13]). *If G is a connected vertex transitive directed graph with outdegree d and $\emptyset \neq A \subset V(G)$ is finite and satisfies $A \cup \partial^+(A) \neq V(G)$, then $|\partial^+(A)| \geq \frac{d+1}{2}$.*

The next theorem may be viewed as a refinement of Mader's theorem which gives a structural result for graphs which have small edge-cuts.

Theorem 1.8 (Tindell [22]). *Let G be a finite connected d-regular vertex transitive graph. If there exists $X \subseteq V(G)$ with $|X|, |V(G) \setminus X| \geq 2$ so that $|\delta X| = d$, then one of the following holds:*

(i) *There is a block of imprimitivity which is a clique of size d.*
(ii) $d = 2$ *(so G is a cycle).*

The following recent theorem of van den Heuvel and Jackson gives a rough analogue of Tindell's result under the assumption that G has a fairly small edge-cut with sufficiently many vertices on either side.

Theorem 1.9 (van den Heuvel, Jackson [14]). *If G is a finite connected d-regular vertex transitive graph and there exists a set $S \subseteq V(G)$ with $\frac{1}{3}(d+1) \leq |S| \leq \frac{|V(G)|}{2}$ and $|\delta S| < \frac{2}{9}(d+1)^2$, then there is a block of imprimitivity with size $< \frac{2}{9}(d+1)^2$.*

Our main theorem may also be viewed as a rough analogue of Tindell's result, but without any assumptions relating to the degree of the graph and with an added assumption that the diameter of the graph is large. Before introducing our theorem we will require some further definitions. If $A \subseteq V(G)$, then the *diameter* of A, denoted $diam(A)$, is the supremum of $dist(x, y)$ over all $x, y \in A$. The *diameter* of G, denoted $diam(G)$, is defined to be $diam(V(G))$. If A is a proper subset of $V(G)$, then the *depth* of a vertex v in A is $dist(v, V(G) \setminus A$. The *depth* of A is the supremum over all vertices

v in A of the depth of v in A. If S is a set, a *cyclic order on S* is a symmetric relation $\sim$ so that the corresponding graph is either a circuit, or a two way infinite path. The *distance* between two elements in S is defined to be the distance in the corresponding graph, and an *interval* of S is a finite subset $\{s_1, s_2, \dots, s_m\} \subseteq S$ with $s_i \sim s_{i+1}$ for every $1 \le i \le m-1$. A *cyclic system* $\vec{\sigma}$ on a graph G is a system of imprimitivity **system of imprimitivity undefined** σ on $V(G)$ equipped with a cyclic ordering (indicated by the arrow) which is preserved by the automorphism group of G. If (s,t) is a pair of positive integers, we say that G is (s,t)-ring-like with respect to the cyclic system $\vec{\sigma}$ if every block of σ has size s and any two adjacent vertices are in blocks which are distance $\le t$ (in the cyclic order).

Theorem 1.10 (DeVos, Mohar)**.** *Let G be a vertex transitive graph, let $A \subseteq V(G)$ be finite with $|A| \le \frac{|V(G)|}{2}$ and $G[A]$ connected. Set $k = |\partial(A)|$ and assume that $diam(G) \ge 31(k+1)^2$. Then one of the following holds:*

(i) *$depth(A) \le k$ and $|A| \le 2k^3$ (and G is d-regular where $d \le \frac{3k}{2} - 1$).*

(ii) *There exist integers s, t with $st \le \frac{k}{2}$ and a cyclic system $\vec{\sigma}$ on G so that G is (s,t)-ring-like and there exists an interval J of $\vec{\sigma}$ so that the set $Q = \cup_{B \in J} B$ satisfies $A \subseteq Q$ and $|Q \setminus A| \le \frac{1}{2}k^3 + k^2$*

This is a rough structure theorem in the sense that any set A which satisfies (i) or (ii) must have $|\partial A|$ bounded as a function of k. Indeed, if A satisfies (i) then $|\partial A| \le d|A| \le 2k^3(\frac{3k}{2} - 1) \le 3k^4$ and if A satisfies (ii) then $|\partial A| \le |\partial Q| + |Q \setminus A| \le \frac{1}{2}k^3 + 2k^2 + k$.

Our theorem has an immediate consequence for separations in Eulerian digraphs. Let $\vec{G}$ be a vertex transitive digraph, and let G be the (vertex transitive) underlying unoriented graph. Let $A \subseteq V(\vec{G})$ satisfiy $|A| \le \frac{|V(G)|}{2}$ and $G[A]$ connected, and set $k = |\partial A|$. It follows from theorem 1.7 that every vertex in G has indegree and outdegree d where $d \le 2k - 1$ so we have $|\partial^-(A)| \le |\delta^-(A)| = |\delta^+(A)| \le k(2k-1)$ and we find that $|\partial A| \le 2k^2$ (in the unoriented graph G). Thus, by the preceding theorem, either $|A| \le 16k^6$ or G is (s,t)-ring-like with $st \le k^2$ and A is efficiently contained in an interval. Later in the article we derive the following corollary for infinite Eulerian digraphs which has an outcome with a different structure (here $diam^+(A)$ is the supremum over all $x \in A$ of the (digraph) distance from x to $V(G) \setminus A$)

Corollary 1.11. *Let G be an infinite connected vertex transitive Eulerian digraph, let $A \subseteq V(G)$ be finite and set $k = |\partial^+ A|$. Then one of the following holds.*

(i) *$depth^+(A) \le k$.*

(ii) *There exist integers s, t with $st \le \frac{k}{2}$ and a cyclic system $\vec{\sigma}$ on G so that G is (s,t)-ring-like and there exists an interval J of $\vec{\sigma}$ so that the set $Q = \cup_{B \in \sigma} B$ satisfies $A \subseteq Q$ and $|Q \setminus A| \le \frac{1}{2}k^3 + k^2$*

Interestingly, the same conclusion does not hold for (vertex transitive) digraphs which are not Eulerian. Let $\vec{H}$ be an orientation of the infinite

3-regular tree where every vertex has outdegree 1 and indegree 2. Then the vertex set D of a directed path has $|\partial^+ B| = 1$ but B may have arbitrarily large size.

The main parameter we use in the proof is the depth of a set. This is a convenient parameter for our purposes (usually uncrossing arguments), but leads us to make an assumption on the diameter of G (to "spread out" the graph) which is unfortunately strong. As far as we know, this theorem may be true without any such assumption. Since we work primarily with depth, the bound on $depth(A)$ in (i) is the natural consequence of our argument. To get a bound on the number of vertices in A for (i) we (rather naively) apply the following pretty theorem which relates $|A|$, $|\partial A|$ and $diam(A)$.

Theorem 1.12 (Babai, Szegedy [4]). *If G is a vertex transitive graph and $\emptyset \neq A \subseteq V(G)$ is finite with $|A| \leq \frac{|V(G)|}{2}$, then*

$$\frac{|\partial A|}{|A|} \geq \frac{1}{diam(A)+1}$$

It appears likely that our theorem should hold with a bound of the form $|A| \leq ck^2$ instead of $|A| \leq 2k^3$ in (i). This strengthening would follow from the following conjecture that $diam$ may be replaced by a constant times $depth$ in the above theorem.

Conjecture 1.13. *There exists a fixed constant c so that $\frac{|\partial A|}{|A|} \geq \frac{c}{depth(A)}$ whenever $A \subseteq V(G)$ is finite and $|A| \leq \frac{|V(G)|}{2}$.*

Asymptotic Expansion in Groups. Asymptotic expansion or growth in groups is a large an well studied topic. Here instead of looking at $|AB|$ for a pair of finite sets A, B we will consider the asymptotic behavior of $|A^n|$ when A is a generating set. The major result in this area is the following theorem of Gromov which resolved (in the affirmative) a conjecture of Milnor.

Theorem 1.14 (Gromov [11]). *Let $\mathcal{G}$ be an infinite group, let $A \subseteq \mathcal{G}$ be a finite generating set, and assume further that $1 \in A$ and $\{a^{-1} : a \in A\} = A$. Then the function $n \to |A^n|$ is bounded by a polynomial if and only if $\mathcal{G}$ has a nilpotent subgroup of finite index.*

In the special case that the growth is linear, the above theorem implies that $\mathcal{G}$ has a subgroup isomorphic to $\mathbb{Z}$ of finite index, and by a result of Freudenthal? this implies that $\mathcal{G}$ has a finite normal subgroup N so that $\mathcal{G}/N$ is either cyclic or dihedral. A clear proof of this case, which also features good explicit bounds is given by the following result due to Imrich and Seifter.

Theorem 1.15 (Imrich, Seifter [16]). *Let $\mathcal{G}$ be an infinite group, let $A \subseteq \mathcal{G}$ be a finite generating set, and assume further that $1 \in A$ and $\{a^{-1} : a \in A\} = A$. If $q = |A^k| - |A^{k-1}| \leq k$ then $\mathcal{G}$ has a cyclic subgroup of index $\leq q$.*

This result may also be obtained as a corollary of our theorem which we postpone to the next section.

Asymptotic Expansion in Graphs. Before discussing this topic, we require two more definitions. For any vertex $x \in G$ and any positive integer k, we let $B(x, k)$ denote the set of vertices of distance $\leq k$ from x. If G is vertex transitive, then $|B(x, k)| = |B(y, k)|$ for every $x, y \in V(G)$ and we denote this quantity by $b(k)$.

The study of asymptotic expansion in graphs is the study of the behavior of the function b defined above. It is easy to see that if $G = Cayley(\mathcal{G}, A)$, then $b(k) = |A^k|$, so this is a direct generalization of the study of expansion in groups. The following result is the major accomplishment in this area and gives a direct generalization of Gromov's theorem.

Theorem 1.16 (Trofimov [23])**.** *Let G be a vertex transitive graph and assume that the function b is bounded by a polynomial. Then there exists a system of imprimitivity σ with finite blocks so that $Aut(G^\sigma)$ is finitely generated, has a nilpotent subgroup of finite index, and the stabilizer of every vertex in G^σ is finite.*

As before, in the case when b is bounded by a linear function, the structure of G can be obtained by a more elementary combinatorial argument, as in the following result.

Theorem 1.17 (Imrich, Seifter [17])**.** *Let G be an infinite connected vertex transitive graph, and let $b()$ be the function defined above. Then G has two ends if and only if $b()$ is bounded by a linear function.*

Our theorem can be used to obtain a result similar to the above theorem, but it also gives an explicit lower bound on the growth of infinite vertex transitive graphs which are not ring-like.

Corollary 1.18 (DeVos, Mohar)**.** *If G is a connected infinite vertex transitive graph and $A \subseteq V(G)$ satisfies $depth(A) > |\partial A|$, then G is (s, t)-ring-like where $st \leq \frac{1}{2}|\partial A|$. In particular, $b(n) \geq n(n-1)$ if G is not ring-like.*

Structural Properties. We now turn our attention away from expansion and toward the structure of vertex transitive graphs. Next we state an important (yet unpublished) theorem of Babai which is related to our main theorem.

Theorem 1.19 (Babai [3])**.** *There exists a function f so that every finite vertex transitive graph G without K_n as a minor satisfies one of the following properties*

(i) *G is a vertex transitive map on the torus.*
(ii) *G is $(f(n), f(n))$-ring-like.*

It appears likely to us that an inexplicit version of our theorem for finite graphs might be obtained from Babai's theorem (which does not give the function f explicitly). However, at this time we do not have a proof of this. Conversely, our theorem - in particular the following corollary - might be of use in obtaining an explicit proof of Babai's theorem.

Corollary 1.20. *If G is a finite vertex transitive graph and k is a positive integer, then one of the following holds.*

(i) *G is $(,)$-ring-like.*
(ii) *G has tree-width $\leq$.*

In any case, the key tool we use to prove our main theorem is a structural lemma on vertex transitive graphs which appears to be of independent interest. Before stating this lemma, we require another definition. A finite subset $A \subseteq V(G)$ is called an (s,t)-*tube* if $G[A]$ is connected, and there is a partition of $\partial(A)$ into $\{L, R\}$ so that $dist(x,y) \leq s$ whenever $x, y \in L$ or $x, y \in R$ and $dist(x,y) \geq t$ whenever $x \in L$ and $y \in R$. Any partition satisfying this property is called a *boundary* partition.

Lemma 1.21 (Tube Lemma)**.** *If G has a $(k, 3k+6)$-tube A with boundary partition $\{L, R\}$ and $depth(V(G) \setminus (A \cup \partial A)) \geq k+2$, then there exists a pair of integers (s,t) and a cyclic system $\vec{\sigma}$ so that G is (s,t)-ring-like with respect to $\vec{\sigma}$, and $st \leq \min\{|L|, |R|\}$.*

This lemma is also meaningful for groups (although the assumptions are more natural in the context of graphs). If G is a Cayley graph for a group $\mathcal{G}$ and G has a tube which satisfies the assumptions of the Tube Lemma, then G is (s,t)-ring-like and it follows that $\mathcal{G}$ has a normal subgroup N (of size $\leq s$) so that $\mathcal{G}/N$ is either cyclic or dihedral.

2. Uncrossing

The main tool we use in the proofs of the Tube Lemma and our main theorem is a simple uncrossing argument. Indeed, this is the main tool used to prove Theorems 1.5, 1.6, and 1.7 as well. This argument is probably easiest to understood with the help of a Venn diagram, so we introduce one in Figure 1. Here it is understood that A_1, A_2 are subsets of the vertex set of a graph G, and the sets $P, Q, S, T, U, W, X, Y, Z$ are defined by the diagram (so $Q = A_2 \cap \partial A_1$, etc.). For convenience, we will frequently refer back to this diagram.

Observation 2.1 (Uncrossing)**.** *Let $A_1, A_2 \subseteq V(G)$ and let the sets P, Q, S, T, U, W, X, Y, Z be defined as in Figure 1. Then we have*

(i) *If $P, Z \neq \emptyset$ then $|\partial P| + |\partial(P \cup Q \cup S \cup T \cup X)| \leq |\partial A_1| + |\partial A_2|$*
(ii) *If $S, X \neq \emptyset$ then $|\partial S| + |\partial X| \leq |\partial A_1| + |\partial A_2|$.*
(iii) *If $|\partial A_1| = |\partial A_2| = |\partial P| = |\partial S| = |\partial X| = |\partial Z| = k$, then $|\partial A_1 \cap (A_2 \cup \partial A_2)| \geq \frac{k}{2}$.*

The proof of (1) and (2) follow from counting the sizes of the relevant sets.

I think it would be good to say the proof a little more explicitly

	A_1	∂A_1	
A_2	P	Q	S
∂A_2	T	U	W
	X	Y	Z

Figure 1. A Venn diagram

3. Two Ended Graphs

The purpose of this section is to establish a theorem which gives us some detailed structural information about vertex transitive graphs with two ends. The main tool we use is a corollary of an important theorem of Dunwoody. However, since Dunwoody's proof is rather tricky, and we have a proof of this corollary which we consider to be more transparent, we have included it here. This also has the advantage of keeping the present article entirely self-contained. Before stating the main theorem from this section, we require some further definitions.

If G is a graph, a *ray* in G is a one-way-infinite path. Two rays r, s in a graph G are *equivalent* if for any finite set of vertices X, the (unique) component of $G \setminus X$ which contains infinitely many vertices of r is also the (unique) component of $G \setminus X$ which contains infinitely many vertices of s. This relation is immediatly seen to be an equivalence relation, and the corresponding equivalence classes are called *ends*. By a theorem of Hopf [15] and Halin [12] every connected vertex transitive graph has either one, two, or infinitely many ends. We let $\kappa_\infty(G) = \inf\{S \subseteq V(G) : G \setminus S : \text{has} \geq 2 \text{ infinite components}\}$. So κ_∞ is finite only if G has at least two ends.

If G is a graph which is ring-like with respect to the cyclic system $\vec{\sigma}$, then we say that G is *q-cohesive* if any two vertices of G which are in the same block of $\vec{\sigma}$ or in adjacent blocks of $\vec{\sigma}$ can be joined by a path of length q. We are now ready to state the main result from this section.

Theorem 3.1. *Let G be a vertex transitive graph with two ends. Then there exist integers s, t and a cyclic system $\vec{\sigma}$ so that G is (s,t)-ring-like and $2st$-cohesive with respect to $\vec{\sigma}$, and $\kappa_\infty(G) = st$.*

The main tool used to establish this result is the corollary below.

Theorem 3.2 (Dunwoody [9])**.** *If there exists a finite edge-cut $\delta(X)$ of G so that both X and $V(G) \setminus X$ are infinite, then there exists such an edge-cut*

$\delta(Z)$ with the additional property that either Z or $V(G) \setminus Z$ is included in either $\phi(Z)$ or $\phi(V(G) \setminus Z)$ for every automorphsim ϕ of G.

Corollary 3.3 (Dunwoody)**.** *If G is a vertex transitive graph with two ends, then there exists a cyclic system $\vec{\sigma}$ on G with finite blocks.*

We call a subset X of vertices a *part* if both X and $G \setminus X$ are infinite but $\partial(G \setminus X)$ is finite. If X is a part, and ϵ is an end, then we say that X *captures* ϵ if every ray in ϵ has all but finitely many vertices in X. We call X a *narrow* part if $|\partial X| = \kappa_\infty(G)$.

If G is a vertex transitive graph with two ends, then every automorphism ϕ of G either maps each end to itself, or interchanges the two ends. We call automorphisms of the first type *rotations* and automorphisms of the second type *flips*. Define a map $sign : Aut(G) \to \{-1, 1\}$ by the rule that $sign(\phi) = 1$ if ϕ is a rotation and $sign(\phi) = -1$ if ϕ is a flip. It is immediate that $sign$ is a group homomorphism.

Proof. Let us denote the ends of G by $\mathcal{L}$ and $\mathcal{R}$. It follows from uncrossing that whenever P, Q are narrow parts which capture $\mathcal{L}$ ($\mathcal{R}$), then $P \cap Q$ and $P \cup Q$ are also narrow parts which capture $\mathcal{L}$ ($\mathcal{R}$). So, more generally, the set of narrow parts which capture $\mathcal{L}$ ($\mathcal{R}$) is closed under finite intersections. Note that by transitivity every vertex is contained in a narrow part which captures $\mathcal{L}$ and a narrow part which captures $\mathcal{R}$. For every $x \in V(G)$ let $L(x)$ ($R(x)$) be the intersection of all narrow parts which contain x and capture $\mathcal{L}$ ($\mathcal{R}$). Were $L(x)$ to be finite, there would exist a finite set of narrow parts whose intersection T contains $L(x)$ but no other point at distance ≤ 2 from this set, so $|\partial(T \setminus L(x))| < |\partial T|$ contradicting our assumption. Similarly, if $|\partial L(x)| > \kappa_\infty$, then there exists a finite set of narrow parts with intersection T and $|\partial T| > \kappa_\infty$, a contradiction. Thus, $L(x)$, and similarly $R(x)$, is a narrow part.

Next, define a map $\beta^R : V(G) \times V(G) \to \mathbb{Z}$ by the rule $\beta^R(x, y) = |R(x) \setminus R(y)| - |R(y) \setminus R(x)|$. Let $x, y, z \in V(G)$, and define the following values:

$$\begin{aligned} a &= |R(x) \setminus (R(y) \cup R(z))| \\ b &= |(R(x) \cap R(y)) \setminus R(z)| \\ c &= |R(y) \setminus (R(z) \cup R(x))| \\ d &= |(R(y) \cap R(z)) \setminus R(x)| \\ e &= |R(z) \setminus (R(y) \cup R(x))| \\ f &= |(R(z) \cap R(x)) \setminus R(y)| \end{aligned}$$

Now we have that $\beta^R(x, y) + \beta^R(y, z) = (a + f) - (c + d) + (b + c) - (e + f) = (a + b) - (d + e) = \beta^R(x, z)$. Define $\beta^L : V(H) \times V(H) \to \mathbb{Z}$ by the similar rule $\beta^L(x, y) = |L(x) \setminus L(y)| - |L(y) \setminus L(x)|$ and observe that $\beta^L(x, y) + \beta^L(y, z) = \beta^L(x, z)$ holds. Next, define $\beta : V(G) \times V(G) \to \mathbb{Z}$ by setting $\beta(x, y) = \beta^R(x, y) - \beta^L(x, y)$ and note again that $\beta(x, y) + \beta(y, z) =$

$\beta(x,z)$. If $\phi \in Aut(G)$ and $x \in V(G)$, then either $sign(\phi) = 1$, $\phi(L(x)) = L(\phi(x))$, and $\phi(R(x)) = R(\phi(x))$, or $sign(\phi) = -1$, $\phi(L(x)) = R(\phi(x))$, and $\phi(R(x)) = L(\phi(x))$. It follows that $\beta(x,y) = sign(\phi)\beta(\phi(x),\phi(y))$ holds for every $x, y \in V(G)$. Now, define two vertices x, y to be equivalent if $\beta(x,y) = 0$, note that this is an equivalence relation preserved by the automorphism group, and let σ be the corresponding system of imprimitivity. If $B, B' \in \sigma$, then $\beta(x,x')$ has the same value for every $x \in B$ and $x' \in B'$ and we define $\beta(B,B')$ to be this value. Next, define a relation on σ as follows. For any block $B \in \sigma$, there is a unique block B' for which $\beta(B,B')$ is minimally positive, and a unique block B'' so that $\beta(B,B'')$ is maximal subject to being negative. Include (B,B') and (B,B'') in our relation. It follows immediately that this relation is a cyclic order which is preserved by any automorphism of the graph, so we have a cyclic system $\vec{\sigma}$ as desired. **Why are the blocks finite?** □

If G is a vertex transitive graph with two ends, then we may define a relation $\sim$ on $V(G)$ by the rule $x \sim y$ if there exists a rotation $\phi \in Aut(G)$ with $\phi(x) = y$. It is immediate from the definitions that $\sim$ is an equivalence relation preserved by $Aut(G)$, and we let τ denote the corresponding system of imprimitivity. Since the product of two flips is a rotation, $|\tau| \leq 2$. We define G to be Type i if $|\tau| = i$. Graphs of Type 1 will be easiest to work with, since in this case we have a rotation taking any vertex to any other vertex. If G is a graph of Type 2, then we view τ as a (not necessarily proper) 2-colouring of the vertices. In this case, every rotation fixes both colour classes, and every flip interchanges them.

If X, Y are disjoint subsets of $V(G)$, we say that X and Y are *neighborly* if every point in X has a neighbor in Y and every point in Y has a neighbor in X. We say that G is *tightly* (s,t)-ring-like with respect to $\vec{\sigma}$ if G is (s,t)-ring-like with respect to $\vec{\sigma}$, and further, every pair of blocks in $\vec{\sigma}$ at distance t are neighborly.

Lemma 3.4. *If G is a connected vertex transitive graph with two ends, then there exists a pair of integers (s,t), and a cyclic system $\vec{\sigma}$ so that G is tightly (s,t)-ring-like with respect to $\vec{\sigma}$.*

Proof. By Corollary 3.3 we may choose a cyclic system $\vec{\sigma}$ where $\sigma = \{X_i : i \in \mathbb{Z}\}$ and the cyclic order is $\ldots, X_{-1}, X_0, X_1, \ldots$. Set s to be the size of a block of σ and t to be the largest integer so that there exist adjacent vertices which lie in blocks at distance t. Then, G is (s,t)-ring-like with respect to $\vec{\sigma}$.

First suppose that G is Type 1 and choose $i \in \mathbb{Z}$ so that $E[X_i, X_{i+t}] \neq \emptyset$. Then every point in X_i must have a neighbor in X_{i+t} since there exists a rotation taking any point in X_i to any other point in this block, and such a map must fix X_{t+i}. Similarly, every point in X_{i+t} has a neighbor in X_i, so X_i and X_{i+t} are neighborly. Now, there exists a rotation which sends X_i to X_{i+t}, so X_i and X_{i-t} are also neighborly. It follows from this that G is tightly (s,t)-ring-like with respect to σ.

Thus, we may assume that G is of Type 2, and we let $\tau = \{Y_1, Y_2\}$ be the corresponding 2-colouring. If σ is not a refinement of τ, then $\{X_i \cap Y_j : i \in \mathbb{Z} \text{ and } j \in \{1,2\}\}$ is a system of imprimitivity, and $\ldots, X_{-1} \cap Y_1, X_{-1} \cap Y_2, X_0 \cap Y_1, X_0 \cap Y_2, X_1 \cap Y_1, X_1 \cap Y_2, \ldots$ is a cyclic ordering of this system which is preserved by $Aut(G)$. Thus, by possibly adjusting $\vec{\sigma}$ and s and t, we may assume that σ is a refinement of τ. In particular, for every $x, y \in X_i$ there exists an automorphism sending x to y which fixes every block of $\vec{\sigma}$ (since every rotation sending x to y has this property). So X_i and X_j are neighborly whenever $E[X_i, X_j] \neq \emptyset$.

Note that we may modify the cyclic order on σ by "shifting the even blocks $2k$ steps to the right", replacing X_{2i} by X_{2i-2k} for every $i \in \mathbb{Z}$ to obtain a new cyclic ordering which is preserved by $Aut(G)$. Set $t_0 = \sup\{i \in 2\mathbb{Z} : E[X_0, X_i] \neq \emptyset\}$, set $t_1^- = \inf\{j \in 2\mathbb{Z}+1 : E[X_0, X_j] \neq \emptyset\}$ and set $t_1^+ = \sup\{j \in 2\mathbb{Z}+1 : E[X_0, X_j] \neq \emptyset\}$. Since there must be a block with odd index joined to X_0 we have that t_1^+ and t_1^- are both finite. By shifting, we may further assume that either $t_1^+ = -t_1^-$ or $t_1^+ = 2 - t_1^-$. If $t_0 \geq t_1^+$, then $t = t_0$ and $E[X_i, X_{i+t}] \neq \emptyset$ for every $i \in \mathbb{Z}$ so X_i and X_{i+t} are neighborly for every $i \in \mathbb{Z}$ and we are done. Similarly, if $t_1^+ = -t_1^- > t_0$, then $t = t_1^+$ and X_i and X_{i+t} are neighborly for every $i \in \mathbb{Z}$ and we are done. The only remaining possibility is that $t = t_1^+ = 2 - t_1^- > t_0$. In this case, set $\sigma' = \{X_{2i} \cup X_{2i+1} : i \in \mathbb{Z}\}$. Then σ' is a system of imprimitivity, $\ldots, X_{-2} \cup X_{-1}, X_0 \cup X_1, X_2 \cup X_3, \ldots$ is a cyclic order preserved by $Aut(G)$, and by construction, G is tightly $(2s, \frac{t-1}{2})$-ring-like with respect to σ' and this ordering. $\square$

Lemma 3.5. *If G is a connected vertex transitive graph which is tightly (s,t)-ring-like, then $\kappa_\infty(G) = st$.*

If st is prime, must G then be a Cayley graph???

Proof. It is immediate that $\kappa_\infty(G) \leq st$ as the removal of t consecutive blocks of size s leaves a graph with ≥ 2 infinite components. Thus, it suffices to show that $st \leq \kappa_\infty(G)$. Assume that G is tightly (s,t)-ring-like with respect to $\vec{\sigma}$ where $\sigma = \{X_i : i \in \mathbb{Z}\}$ and the cyclic order is given by $\ldots, X_{-1}, X_0, X_1, \ldots$. Next, choose $A \subseteq V(G)$ with A and $V(G) \setminus A$ infinite so that

(i) $|\partial A|$ is minimum.
(ii) $T = \{y \in V(G) \setminus A : \sigma_y \cap A \neq \emptyset\}$ is minimal subject to (i).

It follows from our assumptions that $|\partial A| = \kappa_\infty(G)$ is finite. Further, since there is a fixed upper bound on the maximum distance between two vertices in the same block of σ, the set T must be finite. Suppose (for a contradiction) that there exist points x, y in the same block of σ and a rotation $\phi \in Aut(G)$ so that $x \in A$, $y \notin A$, and so that $\phi(x) = y$. Then ϕ must fix every block of σ, so the symmetric difference of A and $\phi(A)$ is finite. By uncrossing, we have $|\partial(A \cap \phi(A))| + |\partial(A \cup \phi(A))| \leq 2|\partial A|$. But then it follows from (i) that $|\partial(A \cup \phi(A))| = |\partial A|$ and we see that $A \cup \phi(A)$ contradicts our choice of A for (ii). Thus, no such x, y, ϕ can exist.

First suppose that there is a rotation taking any point in a block of σ to any other point in this block. It then follows from the above argument that both A and ∂A are unions of blocks of σ. Let $Q_k = \cup_{i \in \mathbb{Z}} X_{it+k}$ for every $0 \leq k \leq t-1$. Then $Q_k \cap \partial A$ must include a block of σ for every $0 \leq k \leq t-1$ so $\kappa_\infty(G) = |\partial A| \geq st$ as desired.

Thus, we may assume that there exist $x, y \in X_0$ so that no rotation maps x to y. So, G is Type 2, and setting $\tau = \{Y_{-1}, Y_1\}$ to be the corresponding 2-colouring, we find that $\sigma' = \{X_i \cap Y_j : i \in \mathbb{Z} \text{ and } j \in \{-1, 1\}\ \}$ is a proper refinement of σ and τ. Furthermore, it follows from our earlier analysis that both A and ∂A are unions of blocks of σ'. It follows from the assumption that G is tightly (s,t)-ring-like that either $X_0 \cap Y_i$ and $X_t \cap Y_i$ are neighborly for $i = -1, 1$ or that $X_0 \cap Y_i$ and $X_t \cap Y_j$ are neighborly whenever $\{i, j\} = \{-1, 1\}$. In the former case, setting $Q_k^j = \cup_{i \in \mathbb{Z}}(X_{it+k} \cap Y_j)$ for $0 \leq k \leq t-1$ and $j = -1, 1$ we find that $\partial A \cap Q_k^j$ includes a block of σ' for every $0 \leq k \leq t-1$ and $j = -1, 1$ so $\kappa_\infty(G) = |\partial A| \geq st$ as desired. In the latter case, setting $Q_k^j = \cup_{i \in \mathbb{Z}}(X_{it+k} \cap Y_{j(-1)^i})$ for $0 \leq k \leq t-1$ and $j = -1, 1$ we find that $\partial A \cap Q_k^j$ includes a block of σ' for every $0 \leq k \leq t-1$ and $j = -1, 1$ so $\kappa_\infty(G) = |\partial A| \geq st$ and we are finished. $\square$

Lemma 3.6. *If G is an infinite connected graph which is tightly (s,t)-ring-like with respect to $\vec{\sigma}$, then it is $2st$ cohesive with respect to $\vec{\sigma}$.*

Proof. We proceed by induction on $\kappa_\infty(G)$. Think of the edges of G as "black", and then add to G a "red" edge joining $x, y \in V(G)$ whenever x and y are in the same block of $\vec{\sigma}$, and $dist(x,y) \leq 2$ in G. Every automorphism of G must map every red edge to another red edge, so the components of the red graph (the graph obtained from G by removing the black edges) form a system of imprimitivity τ which is a refinement of σ.

First suppose that the blocks of τ have size $p > 1$ and choose q so that $pq = s$. Then G^τ is tightly (q,t)-ring-like graph for the cyclic system $\vec{\sigma'}$ where each block of σ' has union equal to a block of σ and the order is accordingly inherited from the cyclic ordering of σ. Furthermore, $\kappa_\infty(G^\tau) = qt < st = \kappa_\infty(G)$, so by induction G^τ is $2qt$-cohesive. Now, G is a regular cover of G^τ with the projection $\pi : V(G) \to V(G^\tau)$ given by $\pi(x) = \sigma_x$. Let $x, y \in V(G)$ and assume that x and y are in blocks of σ with distance ≤ 1. Then $\pi(x)$ and $\pi(y)$ are in blocks of σ' with distance ≤ 1, so we may choose a path in G^τ from $\pi(x)$ to $\pi(y)$ with distance $\leq 2qt$. This path may be lifted to a path in G from x to a vertex $y' \in \tau_y$ of length $2qt$. Thus $dist(x,y) \leq dist(x,y') + dist(y',y) \leq 2qt + 2(p-1) \leq 2pqt$ and we are done.

Thus, we may assume that the blocks of τ have size $p = 1$. It follows from this that the graph induced by any two blocks of σ at distance t is a perfect matching. Modify the graph by changing the colour of every edge which joins vertices in blocks at distance t to "blue". This colouring is preserved by the automorphism group, and the graph on the blue edges consists of st components each of which is a two way infinite path. If τ' is the system of imprimitivity formed by the components of the blue graph then G is a regular cover of $G^{\tau'}$ with projection $\pi(x) = \tau'_x$. Let $x, y \in V(G)$ and assume that x and y are in blocks of σ at distance ≤ 1. Then there exists a path in $G^{\tau'}$ of length $\leq st$ from $\pi(x)$ to $\pi(y)$ since $G^{\tau'}$ is a connected graph with st vertices. This path may be lifted to a path in G from x to a vertex $y' \in \tau'_y$ of length st which does not use any blue edges. The distance between the blocks σ_y and $\sigma_{y'}$ must then be at most $st(t-1)+1 \leq st^2$, and we then find that $dist(x, y) \leq dist(x, y') + dist(y', y) \leq 2st$ as required. $\square$

Proof of Theorem 3.1. This is an immediate consequence of Lemmas 3.4, 3.5, and 3.6.

4. The Tube Lemma

The goal of this section is to prove (a slight strengthening of) the Tube Lemma. We begin by proving a lemma similar in spirit, but for vertex cuts where all points in the cut are closed to some fixed point.

Lemma 4.1. *Let $X \subseteq V(G)$ be finite and assume that there exists a point $y \in V(G)$ so that $dist(y, x) \leq k$ for every $x \in X$. If there is a finite component H of $G \setminus X$ with $depth(V(H)) \geq k+2$, then $depth(V(H')) < k+2$ for every other component H' of $G \setminus X$.*

Proof. Let X be a minimal counterexample to the lemma. Choose a finite component of $G \setminus X$ with vertex set A so that $depth(A) \geq k+2$ and so that $|A| \leq |V(G) \setminus (A \cup X)|$. Next choose a point $z \in A$ with depth $\geq k+2$ and choose an automorphism ϕ with $\phi(y) = z$. Since A contains the ball of radius $k+1$ around z, we must have $\phi(X) \subseteq A$ and $A \setminus (\phi(A) \cup \phi(X)) \neq \emptyset$. It follows from this that $S = \phi(A) \setminus (A \cup X) \neq \emptyset$. Furthermore, by our assumption on $|A|$ we must have $T = V(G) \setminus (A \cup X \cup \phi(A) \cup \phi(X)) \neq \emptyset$. Now, set $X' = X \cap \phi(A)$ and $X'' = X \setminus \phi(A)$. Then X', X'' are vertex cuts which separate S and T (respectively) from the rest of the vertices. By assumption, one of S or T must have depth $\geq k+2$, but then either X' or X'' contradicts our choice of X. $\square$

For $i = 1, 2$ let A_i be a tube with boundary partition $\{L_i, R_i\}$ and let $P, Q, S, T, U, W, X, Y, Z$ be the sets indicated by the Venn Diagram in Figure 1. We say that A_1 and A_2 *merge* if $P, S, X, Z \neq \emptyset$, $\{Q, Y\} = \{L_1, R_1\}$ and $\{T, W\} = \{L_2, R_2\}$. The following lemma will be used to guarantee that tubes merge.

Lemma 4.2. *Let G be a connected graph, and for $i = 1, 2$ let A_i be a $(k, k+2)$-tube in G with boundary partition $\{L_i, R_i\}$ and $depth(V(G) \setminus (A_i \cup \partial A_i)) \geq k+2$. Let $P, Q, S, T, U, W, X, Y, Z$ be the sets indicated in Figure 1. If $P, S, X, Z \neq \emptyset$ and $dist(\partial A_1, \partial A_2) \geq \frac{k+1}{2}$, then A_1 and A_2 merge.*

Proof. It follows from the assumption $dist(\partial A_1, \partial A_2) \geq \frac{k+1}{2}$ that $U = \emptyset$. The sets T and Q cannot be empty since $G[A_1]$ and $G[A_2]$ are connected and $P, S, X \neq \emptyset$. Suppose (for a contradiction) that $W = \emptyset$. Then $Y \neq \emptyset$ since $Z \neq \emptyset$ and G is connected. Furthermore $dist(Q, Y) \geq dist(Q, \partial A_2) + dist(\partial A_2, Y) \geq k+1$ and it follows that $\{Q, Y\} = \{L_1, R_1\}$. But then applying Lemma 4.1 to either L_1 or R_1 gives us a contradiction. Thus $W \neq \emptyset$ and similarly $Y \neq \emptyset$. Now we have $dist(Q, Y) \geq dist(Q, \partial A_2) + dist(\partial A_2, Y) \geq k + 1$ and similarly $dist(T, W) \geq dist(T, \partial A_1) + dist(\partial A_1, W) \geq k + 1$ and it follows that $\{Q, Y\} = \{L_1, R_1\}$ and $\{T, W\} = \{L_2, R_2\}$ as required. $\square$

The key ingredient in the proof of our tube lemma for finite graphs is the construction of a certain graph cover. We then use this cover together with Corollary 3.3 to obtain the desired structure. Our construction is based on voltage assignments, and the reader is referred to Gross and Tucker?? for a good introduction to this area.

For every graph G, we define $A(G) = \{(u, v) \in V(G) \times V(G) : u \text{ and } v \text{ are adjacent}\}$ and we call the members of A *arcs*. We call a map $\mu : A(G) \to \mathbb{Z}$ a *voltage* map if $\mu(u, v) = -\mu(v, u)$ for every $(u, v) \in A(G)$ (this theory extends naturally to general groups, but we have restricted our attention to $\mathbb{Z}$ for simplicity). For every graph G and voltage map μ, we define a simple graph $\mathcal{C}(G, \mu)$ as follows: the vertex set is $V(G) \times \mathbb{Z}$, and vertices (u, i), (v, j) are adjacent if $uv \in E(G)$ and $j - i = \mu(u, v)$. The map $\pi : V(G) \times \mathbb{Z} \to V(G)$ given by $\pi(v, g) = v$ is then a covering map, so $\mathcal{C}(G, \mu)$ is a cover of G.

If μ, μ' are voltage maps on G, we say that a mapping $\Psi : V(\mathcal{C}(G, \mu)) \to V(\mathcal{C}(G, \mu'))$ *respects* π if $\pi \circ \Psi = \pi$. Note that for every integer j the map $\Psi^j : V(G) \times \mathbb{Z} \to V(G) \times \mathbb{Z}$ given by $\Psi^j(u, i) = (u, i+j)$ is an isomorphism of $\mathcal{C}(G, \mu)$ which respects π. For every $S \subseteq V(G)$ and $m \in \mathbb{Z}$ let $\delta_S^m : A(G) \to \mathbb{Z}$ be the map given by the rule

$$\delta_S^m(u, v) = \begin{cases} m & \text{if } u \in S \text{ and } v \notin S \\ -m & \text{if } u \notin S \text{ and } v \in S \\ 0 & \text{otherwise} \end{cases}$$

We say that two voltage maps $\mu, \mu' : A(G) \to \mathbb{Z}$ are *elementary equivalent* if either $\mu' = -\mu$ or $\mu' = \mu + \delta_S^m$ for some $S \subseteq V(G)$ and $m \in \mathbb{Z}$. We say that μ and μ' are *equivalent* and write $\mu \cong \mu'$ if there is a sequence $\mu = \mu_0, \mu_1, \ldots, \mu_n = \mu'$ of voltage maps on G with μ_i elementary equivalent to μ_{i+1} for every $0 \leq i \leq n-1$. It is straightforward to verifty that whenever $\mu \cong \mu'$, there exists a bijection from $\mathcal{C}(G, \mu)$ to $\mathcal{C}(G, \mu')$ which respects π. **Respects is a horrible name.. also, it would be good to define regular cover here.**

Lemma 4.3. *If G has a $(3k + 6, k)$-tube A with boundary partition $\{L, R\}$ and depth$(V(G) \setminus (A \cup \partial A)) \geq k + 2$, then there exists a pair of integers (s, t) and a cyclic system $\vec{\sigma}$ so that G is (s, t)-ring-like and $2st$-cohesive with respect to $\vec{\sigma}$, and $st \leq \min\{|L|, |R|\}$.*

Proof. Choose a $(3k+6,k)$-tube A with $depth(V(G) \setminus (A \cup \partial A)) \geq k+2$ and boundary partition $\{L, R\}$ so that

(i) $\min\{|L|, |R|\}$ is as small as possible.

(ii) $|L| + |R|$ is as small as possible subject to (i).

(iii) $|A|$ is as small as possible subject to (i) and (ii).

It suffices to show that G is (s,t)-ring-like where $st \leq \min\{|L|, |R|\}$. The proof consists of a series of three numbered claims, followed by a split into two cases, dependent on wether G is finite or infinite.

(0) Every point in ∂A has a neighbor in $V(G) \setminus (A \cup \partial A)$.

Suppose (for a contradiction) that $x \in \partial A$ has no such neighbor. If $\{x\} \neq L$ and $\{x\} \neq R$ then $A \cup \{x\}$ contradicts our choice of A for (i) or (ii). But if $\{x\} = L$ or $\{x\} = R$, then $\partial(A \cup \{x\})$ is included in either L or R and applying Lemma 4.1 to this set gives a contradiction.

(1) If G is finite, then $G \setminus E[A, L]$ and $G \setminus E[A, R]$ are connected, and $|A| \leq |V(G) \setminus (A \cup \partial(A))|$.

Suppose that $G \setminus E[A, L]$ is not connected, let B be the vertex set of a component of this graph with $B \cap A = \emptyset$ and set $L' = B \cap L$ and $B' = B \setminus L$. It is an immediate consequence of Lemma 4.1 (applied to L') that $depth(B') \leq k+1$. It then follows from the same lemma (applied to R) that $L' \neq L$. But then, $A \cup B$ is a tube which contradicts our choice of A for (i) or (ii). Thus $G \setminus E[A, L]$ is connected, and by a similar argument $G \setminus E[A, R]$ is also connected. It follows from this that there is a component of $G \setminus (A \cup L \cup R)$ with vertex set C and $L \cap \partial C \neq \emptyset \neq R \cap \partial C$. By assumption (iii), we find $|A| \leq |C|$ so $|A| \leq |V(G) \setminus (A \cup \partial A)|$ as desired.

(2) If $\phi \in Aut(G)$ satisfies $A \cap \phi(A) \neq \emptyset$ and $dist(\partial A, \partial\phi(A)) \geq \frac{k+1}{2}$, then A and $\phi(A)$ merge.

Set $A_1 = A$ and $A_2 = \phi(A)$ and define the sets $P, Q, S, T, U, W, X, Y, Z$ as in Figure 1. It follows immediately from our assumptions that $U = \emptyset$ and that $P, T \cup X, Q \cup S \neq \emptyset$. If $S = \emptyset$, then $Q \neq \emptyset$, but every point in this set has all its neighbors in $P \cup Q = A \cup \partial A$ and we have a contradiction to (0). Thus $S \neq \emptyset$ and similarly $X \neq \emptyset$. If $Z = \emptyset$ then G is finite, and by (0) we must have $W = Y = \emptyset$, but then $|V(G) \setminus A \cup \partial A| = |S| < |P| + |Q| + |S| = |A_2| = |A|$ contradicting (1). Thus $Z \neq \emptyset$. Now by Lemma 4.2 we have that A and $\phi(A)$ merge as desired.

Choose a shortest path D in $G[A \cup \partial A]$ from L to R. Let v_{-2} be the end of D in L, let v_2 be the end of D in R, and choose $r \in \mathbb{Z}$ so that the length of D is either $2r$ or $2r+1$. Note that by our assumptions, $r \geq \lceil \frac{3k+5}{2} \rceil$. Choose a vertex v_0 on D with distance $r \leq dist(v_0, v_i) \leq r+1$ for $i = -2, 2$ and choose vertices $v_{-1}, v_1 \in V(D)$ at distance $\lceil \frac{3k+5}{2} \rceil$ from v_0 so that v_{-1} lies on the subpath of D from v_0 to v_{-2} and v_1 lies on the subpath from v_0 to v_2.

(3) Let $\phi \in Aut(G)$ and assume that either $dist(\phi(v_0), v_{-1}) \leq 1$ or $dist(\phi(v_0), v_1) \leq 1$. Then $dist(\partial A, \partial\phi(A)) \geq \frac{k+1}{2}$, so A and $\phi(A)$ merge.

We give the proof in the case $dist(\phi(v_0), v_{-1}) \leq 1$, the other case follows by a similar argument. Let $y \in L$ and $x \in \partial\phi(A)$. Then we have

$$\begin{aligned} dist(y,x) &\geq dist(\phi(v_0), x) - dist(\phi(v_0), v_{-2}) - dist(v_{-2}, y) \\ &\geq t - (t - \tfrac{3k+5}{2} + 2) - k \\ &= \tfrac{k+1}{2}. \end{aligned}$$

Next let $y \in R$. Then any path from $\phi(v_0)$ to y which does not contain a point in L has length $\geq t + \frac{3k+5}{2} - 1 = t + \frac{3k+3}{2}$ and any such path which does contain a point in L has length $\geq dist(\phi(v_0), L) + dist(L, R) \geq (t - \frac{3k+6}{2} - 1) + (3k+6) \geq t + \frac{3k+4}{2}$. It follows that $dist(\phi(v_0), y) \geq t + \frac{3k+3}{2}$. Thus, for every $x \in \partial\phi(A)$ we have

$$\begin{aligned} dist(x,y) &\geq dist(\phi(v_0), y) - dist(\phi(v_0), x) \\ &\geq (t + \tfrac{3k+3}{2}) - (t + 1 + k) \\ &= \tfrac{k+1}{2} \end{aligned}$$

Thus we have that $dist(\partial A, \partial\phi(A)) \geq \frac{k}{2}$ and by (2) we find that A and $\phi(A)$ merge.

Case 1: G is infinite.

We shall construct a sequence (ϕ_i, S_i) where $\phi_i \in Aut(G)$ and $S_i \in \{\phi_i(L), \phi_i(R)\}$ recursively by the following procedure. Set $(\phi_{-1}, S_{-1}) = (1, L)$ and $(\phi_0, S_0) = (1, R)$. For $i \geq 1$ choose $\phi_i \in Aut(G)$ and $S_i \in \{\phi(L), \phi(R)\}$ so that the following properties are satisfied (it follows from (3) that such a choice is possible).

(i) $\phi_i(A)$ merges with $\phi_{i-2}(A)$.

(ii) $S_{i-2} \subseteq \phi_i(A)$.

(iii) $S_i \cap \phi_{i-2}(A) = \emptyset$.

(iv) $dist(\partial\phi_i(A), \partial\phi_{i-2}(A)) \geq \frac{k}{2}$

For every $i \geq 0$, define $X_i = \cup_{j=0}^{i} \phi_i(A)$. We now prove by induction that X_i is a tube with boundary partition $\{S_{i-1}, S_i\}$ for every $i \geq 0$. It follows immediately from our definitions that this is true for $i = 0$. For the inductive step, suppose that this holds for all values less than m. If $dist(S_{m-1}, S_{m-2}) \leq k+1$ or $dist(S_m, S_{m-1}) \leq k+1$, then there is a point at distance $\leq \lceil \frac{3k+1}{2} \rceil$ from every point in ∂X_{m-1} or ∂X_m and since $depth(X_m) \geq depth(X_{m-1}) \geq depth(X_0) \geq \lceil \frac{3k+5}{2} \rceil$ we have a contradiction to Lemma 4.1. But then X_{m-1} is a $(k, k+2)$-tube and $dist(\partial\phi_m(A), \partial X_{m-1}) \geq \frac{k+1}{2}$, so by applying Lemma 4.2 to the tubes X_{m-1} and $\phi_m(A)$ we find that they merge, and it follows that X_m is a tube with boundary partition $\{S_{m-1}, S_m\}$.

A straightforward inductive argument now shows that the graph G has two ends. Furthermore, L and R are vertex cuts which separate the vertex set into two sets of infinite size. Thus, by Theorem 3.1 we find that G is

(s,t)-ring like and $2st$-cohesive with respect to some cyclic system $\vec{\sigma}$ where $st \leq \min\{|L|, |R|\}$ as desired.

Case 2: G is finite.

By possibly switching L and R, we may assume that $|L| \leq |R|$. For every $\phi \in Aut(G)$, define the mapping $\mu_\phi : A(G) \to \mathbb{Z}$ by the following rule:

$$\mu_\phi(u,v) \quad = \quad \begin{cases} 1 & \text{if } u \in \phi(L) \text{ and } v \in \phi(A) \\ -1 & \text{if } v \in \phi(L) \text{ and } u \in \phi(A) \\ 0 & \text{otherwise} \end{cases}$$

Let $\phi_1, \phi_2 \in Aut(G)$ satisfy $dist(\phi_1(v_0), \phi_2(v_0)) \leq 2$. Then choose $u \in V(G)$ so that $dist(u, \phi_i(v_0)) \leq 1$ for $i = 1, 2$ and choose $\psi \in Aut(G)$ so that $\psi(v_L) = u$. It follows from (3) that $\psi(A)$ and $\phi_i(A)$ merge for $i = 1, 2$. It follows from this that $\mu_{\phi_1} \cong \mu_\psi \cong \mu_{\phi_2}$. We conclude that $\mu_\phi \cong \mu_1$ for every $\phi \in Aut(G)$.

Let $\tilde{G} = \mathcal{C}(G, \mu_1)$ and for every $i \in \mathbb{Z}$ let $A_i = \{(v,i) : v \in V(G)\}$. By (1) we have that $G \setminus E[A, L]$ is connected, and it follows that $\{G[A_i] : i \in \mathbb{Z}\}$ is the set of components of $\tilde{G} \setminus \{uv \in E(\tilde{G} : \pi(u) \in L \text{ and } \pi(v) \in A\}$. It follows that $\tilde{G}$ has two ends, and that $\kappa_\infty(\tilde{G}) \leq |L|$.

Let $\mathcal{G} = Aut(G)$ and let $\tilde{\mathcal{G}} = \{\phi \in Aut(\tilde{G}) : \phi \text{ preserves fibers}\}$. Then define the map $\nu : \tilde{\mathcal{G}} \to \mathcal{G}$ by the rule $\nu(\psi)v = \pi(\psi(v,0))$ ($\nu(\psi)$ is simply the natural projection of ψ). The following diagram now shows the actions of $\mathcal{G}$ on G and $\tilde{\mathcal{G}}$ on $\tilde{G} = \mathcal{C}(G, \mu_1)$.

$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[r] \arrow[d, "\nu"'] & \tilde{G} \arrow[d, "\pi"'] \\ \mathcal{G} \arrow[r] & G \end{tikzcd}$$

Next we shall prove that the map ν is onto. Let $\phi \in \mathcal{G}$ and define $\tilde{\phi} : V(G) \times \mathbb{Z} \to V(G) \times \mathbb{Z}$ by the rule $\tilde{\phi}(v,i) = (\phi(v), i)$. Then $\tilde{\phi}$ is an isomorphism from $\mathcal{C}(G, \mu_1)$ to $\mathcal{C}(G, \mu_{\phi^{-1}}$. Since $\mu_1 \cong \mu_{\phi^{-1}}$ we may choose an isomorphism $\psi : \mathcal{C}(G, \mu_{\phi^{-1}}) \to \mathcal{C}(G, \mu_1)$ which respects π. With these definitions in place, we now have the following commuting diagram.

$$\begin{tikzcd} \mathcal{C}(G,\mu_1) \arrow[r, "\hat{\phi}"] \arrow[d, "\pi"'] & \mathcal{C}(G,\mu_{\phi^{-1}}) \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathcal{C}(G,\mu_1) \arrow[d, "\pi"'] \\ G \arrow[r, "\phi"] & G \arrow[r, "1"] & G \end{tikzcd}$$

Thus, we have that $\tilde{\phi} \circ \psi \in \tilde{\mathcal{G}}$ so ν is onto. Thus $\tilde{G}$ is vertex transitive, and by Theorem 3.1 we have that $\tilde{G}$ is (s,t)-ring-like and $2st$ cohesive with respect to some cyclic system $\vec{\sigma}$ where $st = \kappa_\infty(\tilde{G}) \leq |L|$. Since $\tilde{G}$ is a regular cover, we must have that $\tau = \{\pi(X) : X \in \tilde{\sigma}\}$ is a partition of G. Since ν is onto, we find further that τ must be a system of imprimitivity on G. Now, τ inherits a cyclic ordering $\vec{\tau}$ from $\vec{\sigma}$, and it follows immediately that G is (s,t)-ring-like and $2st$-cohesive with respect to $\vec{\tau}$. Since $st \le |L|$, this completes the proof. □

5. Main Theorem

The purpose of this section is to prove the main theorem. We begin by establishing a lemma on the structure of separations in ring-like graphs.

Lemma 5.1. *Let G be a graph which is (s,t)-ring-like and $2st$-cohesive with respect to $\vec{\sigma}$. Let $A \subseteq V(G)$ and assume that $|A| \le \frac{|V(G)|}{2}$ and that $|\partial A| = k$. Then there exists an interval J of $\vec{\sigma}$ so that the set $Q = \cup_{B\in J} B$ satisfies $A \subseteq Q$ and $|Q \setminus A| \le 2s^2t^2k + 2stk$.*

Proof. Let J_0 be the set of all $B \in \vec{\sigma}$ with the property that $\emptyset \ne A \cap (B \cup B') \ne B \cup B'$ for some $B' \in \vec{\sigma}$ with $B \sim B'$. Choose a maximal set of blocks $T \subseteq J_0$ with pairwise distance $> 2st^2$. It follows from the assumption that G is $2st$-cohesive, that whenever $B \in J_0$, there is a point in ∂A contained in a block at distance $\le st^2$ from B. Thus, we find that $|T| \le k$. Further, setting $J_1 = \{B \in \vec{\sigma} : B \text{ is distance} \le 2st^2 \text{ from some } B' \in T\}$ we have that $J_0 \subseteq J_1$ and $|J_1| \le k(2st^2 + 1)$. Note further, that any interval of blocks disjoint from J_1 must either all be contained in A or all be disjoint from A. Next, modify the set J_1 to form J_2 by adding every block B with the property that the maximal interval of $\vec{\sigma} \setminus J_1$ containing B has length $\le t$. By construction, $|J_2| \le (2st^2+1)k + tk$ and every maximal interval disjoint from J_2 has length $\ge t+1$. Finally, modify the set J_2 to form J_3 by adding every block B with $B \subseteq A$. It follows from the assumption that $G[A]$ is connected that J_3 is an interval of $\vec{\sigma}$. Furthermore, setting $Q = \cup_{B\in J_3} B$ we have $A \subseteq Q$ and $|Q \setminus A| \le s|J_2| \le (2s^2t^2 + s)k + stk \le 2s^2t^2k + 2stk$ as desired. □

Observation 5.2. *Let G be a connected vertex transitive graph and let $A \subset V(G)$ be finite. Then we have*

- (i) *If $G[A]$ is connected, then $diam(A) \le 2|\partial A| \cdot depth(A)$.*
- (ii) *Let d, k, ℓ, m be nonnegative integers and assume that $|\partial A| \le k$ and $diam(G) \ge mk(2d+1) + \ell + 1$. If $|A| \ge \frac{|V(G)|-\ell}{m}$, then $depth(A) \ge d+1$.*

Proof. Part (i) follows immediately from the observation that A is contained in the union of the $|\partial A|$ balls of radius $depth(A)$ which are centered at a point in ∂A (together with the assumption that $G[A]$ is connected). Part (ii) follows from this and the observation that there exists a collection of pairwise disjoint balls in G with one of radius $\ell+1$ and mk of radius d. □

Lemma 5.3. *Let G be a connected vertex transitive graph, let $A \subseteq V(G)$ be finite and set $k = |\partial A|$. If $diam(G) \ge 31(k+1)^2$ and $depth(A)$, $depth(V(G)\setminus$*

$(A \cup \partial A)) \geq k+1$ *then there exist integers* s, t *with* $st \leq \frac{k}{2}$ *and a cyclic system* $\vec{\sigma}$ *so that* G *is* (s,t)*-ring-like and* $2st$*-cohesive with respect to* $\vec{\sigma}$*.*

Proof. We may assume that A is a set which satisfies the assumptions of the lemma, and further, is chosen so that

(i) $|\partial A|$ is minimum

(ii) $|A|$ is minimum subject to (i).

Note that (ii) implies that $G[A]$ is connected and that $|A| \leq |V(G) \setminus (A \cup \partial A)|$. We proceed with two claims.

(1) $depth(A) = k+1$

Suppose (for a contradiction) that $depth(A) > k+1$. Choose a point $x \in A$ with depth $k+2$, a point $y \in A$ with depth $k+1$, and an automorphism ϕ with $\phi(x) = y$. Set $A_1 = A$ and $A_2 = \phi(A)$ and let the sets $P, Q, S, T, U, W, X, Y, Z$ be as given in Figure 1. If $|S|, |X| \geq \frac{|V(G)|-3k}{4}$ then it follows from part (ii) of the previous observation that $depth(S), depth(X) \geq k+1$ and by uncrossing that $|\partial S| + |\partial X| \leq 2k$. But then either S or X contradicts our choice of A. Thus, we may assume that either $|S|$ or $|X|$ is less than $\frac{|V(G)|-3k}{4}$. If G is finite and $|S| \leq \frac{|V(G)|-3k}{4}$, then $|S \cup Z| + k \geq |S \cup W \cup Z| \geq \frac{|V(G)|-k}{2}$ so $|Z| \geq \frac{|V(G)|-3k}{4}$. The same conclusion holds under the assumption that $|X| \leq \frac{|V(G)|-3k}{4}$, so by the previous observation we conclude that Z has depth $\geq k+1$. It follows from our construction that P has depth $\geq k+1$. Now, by uncrossing, either $|\partial(P \cup Q \cup S \cup T \cup X)| < k$ and this set contradicts the choice of A for (i), or $|\partial P| \leq k$ and this set contradicts the choice of A for (i) or (ii).

(2) For every $x \in V(G)$ and $k \leq n \leq diam(G) - 2k^2 - 2k - 1$ there exists a set D with $|\partial D| = k$ and $B(x,n) \subseteq D \subseteq B(x, n + 2k^2 + k)$

Note first that this claim holds for $n = k$ by (1) and part (i) of Observation 5.2. Suppose (for a contradiction) that such a set does not exist for n and x, and choose a maximal set $C \subseteq V(G)$ with $|\partial C| = k$ and $B(x,k) \subseteq C \subseteq B(x, n + 2k^2 + k)$ (such a set always exists by our observation). Choose a shortest path P from x to ∂C, let x, y be the ends of P and choose a point z on P at distance k from y. Using the fact that (2) holds when $n = k$, choose a set $D \subseteq V(G)$ with $|\partial D| = k$ so that $B(y,k) \subseteq D \subseteq B(y, 2k^2 + 2k)$. Now $V(G) \setminus (C \cup D)$ contains a ball of radius $\geq k+1$ so $depth(V(G) \setminus (C \cup \partial C \cup D \cup \partial D) \geq k+1$. If $|\partial(C \cap D)| < k$ then $B(y, k-1) \subseteq C \cap D$ by construction so $C \cap D$ has depth $\geq k$ and contradicts our choice of A for (i). Otherwise, it follows from uncrossing that $|\partial(C \cup D)| \leq k$ but $C \cup D \subseteq B(x, n + 2k^2 + k)$ so this contradicts our choice of C.

Set $q = 12k^2 + 8k + 5$ and $h = 6k^2 + 8k + 2$. Let $v_{-q}, v_{-q+1}, \ldots, v_{q+k^2+h+2}$ be the vertex sequence of a shortest path in G. Next, apply (2) to choose sets $C, D^-, D^+ \subseteq V(G)$ so that the following hold

$$\begin{aligned} B(v_0, q-k^2-k) \subseteq &\quad C &\subseteq B(v_0, q+k^2) \\ B(v_{-q}, k^2+3k) \subseteq &\quad D^- &\subseteq B(v_{-q}, 3k^2+4k) \\ B(v_q, k^2+3k) \subseteq &\quad D^+ &\subseteq B(v_q, 3k^2+4k) \end{aligned}$$

Then $B(v_{q-k^2-2k}, k) \subseteq C \cap D^+$ and $B(v_{q+k^2+2k}, k) \subseteq D^+ \setminus C$ so these sets have depth $\geq k+1$. Furthermore, $b(v_0, k) \subseteq C \setminus D^+$ and $B(v_{q+k^2+h,k} \subseteq V(G) \setminus (C \cup D^+)$ so these sets have depth $\geq k+1$. It now follows from our choice of A and uncrossing, that $|\partial(C \cap D^+)| = |\partial(C \setminus D^+)| = |\partial(D^+ \setminus C)| = |\partial(C \cup D^+)| = k$. But then by part (iii) of Observation 2.1 we find that $R = (D^+ \cup \partial D^+) \cap \partial C$ satisfies $|R| \geq \frac{k}{2}$. By a similar argument, we find that $L = (D^- \cup \partial D^-) \cap \partial C$ satisfies $|L| \geq \frac{k}{2}$. Thus $\{L, R\}$ is a partition of ∂C. If $x, y \in R$, then $x, y \in B(v_q, 3k^2+4k+1)$ so $dist(x,y) \leq 6k^2+8k+2 = h$. Similarly if $x, y \in L$, then $dist(x,y) \leq h$. If $x \in L$ and $y \in R$, then $dist(x,y) \geq dist(v_{-q}, v_q) - dist(v_{-q}, x) - dist(y, v_q) \geq 2q - 6k^2 + 8k + 2 = 3h+6$. Thus C is a $(3h+6, h)$-tube with boundary partition $\{L, R\}$ and $B(v_{q+k^2+h+2}, h+2)$ is disjoint from C so $depth(V(G) \setminus (C \cup \partial C)) \geq h+2$. Applying the tube lemma to this yields the desired conclusion. □

We are now ready to complete the proof of our main theorem.

Proof of Theorem 1.10. Let $A \subseteq V(G)$ satisfy the assumptions of the theorem. If G is finite, then by assumption $|V(G) \setminus (A \cup \partial A)| \geq \frac{1}{2}(|V(G)| - k)$ so by (ii) of Observation 5.2 we have that $depth(V(G) \setminus (A \cup \partial A)) \geq k+1$. The same conclusion holds trivially if G is infinite. If $depth(A) \leq k$ then by (i) of Observation 5.2 and Theorem 1.12 we find that $|A| \leq 2k^3$ so case (i) holds (the parenthetical comment here is a direct application of Theorem 1.6). Otherwise it follows from Lemma 5.1 and Lemma 5.3 that (ii) holds. □

References

[1] L. Babai, Local expansion of vertex-transitive graphs and random generation in finite groups. In *Proc. 23rd ACM Symposium on Theory of Computing* (1991) 164-174.

[2] L. Babai, Vertex-transitive graphs and vertex-transitive maps, J. Graph Theory 15 (1991) 587-627.

[3] L. Babai, Automorphism groups, isomorphism, reconstruction, Chapter 27. Handbook of Combinatorics

[4] L. Babai, M. Szegedy, Local expansion of symmetrical graphs. Combin., Prob., and Computing 1 (1992) 1-11.

[5] L. Babai, M.E. Watkins, Connectivity of infinite graphs having a transitive torsion group action, Arch. Math. 34 (1980) 90-96.

[6] A.L. Cauchy, Recherches sur les nombres, J. École polytech. 9 (1813) 99-116.

[7] H. Davenport, On the addition of residue classes, J. London Math. Soc. 10 (1935) 30-32.

[8] L. van den Dries, A. J. Wilkie, An effective bound for groups of linear growth, Arch. Math. 42 (1984), 391-396.

[9] M.J. Dunwoody, Cutting up graphs, Combinatorica 2 (1982) 13-25.

[10] H. Freudenthal. Über die Enden diskreter Räume und Gruppen. Commentari Math. Helv. 17 (1945) 1-38.

[11] M. Gromov, Groups of polynomial growth and expanding maps, Publ. Math. IHES 53 (1981) 53-73.

[12] R. Halin, Über unendliche Wege in Graphen. Math. Ann. (1964) 125-137.

[13] Y.O. Hamidoune, An application of connectivity theory in graphs to factorization of elements in groups. Europ. J. Comb. 2 (1981) 349-355.

[14] J. van den Heuvel, B. Jackson, On the edge connectivity, hamiltonicity and toughness of vertex-transitive graphs, J. Combin. Theory Ser. B 77, no. 1 (1999) 138-149.

[15] H. Hopf. Enden offenere Räume und unendliche diskontinuierliche Gruppen. Comment. Math. Helv. (1944) 81-100.

[16] W. Imrich, N. Seifter, A bound for groups of linear growth, Arch. Math. Vol 48 (1987) 100-104.

[17] W. Imrich, N. Seifter, A note on the growth of transitive graphs, Discrete Math 73 (1988/89) 111-117.

[18] H.A. Jung, M.E. Watkins, Fragments and automorphisms of infinite graphs, Europ. J. Comb. 5 (1984) 149-162.

[19] W. Mader, Minimale n-fach kantenzusammenhängende Graphen. Math. Ann 191 (1971) 21-28.

[20] W. Mader, Über den Zusammenhang symmetrischer Graphen. Arch. Math 22 (1971) 333-336.

[21] J. Stallings, Group theory and Three-Dimensional Manifolds. Yale University Press, New Haven-London (1971).

[22] R. Tindell, Edge connectivity properties of symmetric graphs, Preprint, Stevens Institute of Technology, Hoboken, NJ (1982).

[23] V.I. Trofimov, Graphs with polynomial growth, Math. USSR Sbornik 51 (1985) 405-417.

[24] A. G. Vosper, The critical pairs of subsets of a group of prime order, J. London Math. Soc. 31 (1956) 200-205, Addendum 280-282.

[25] M.E. Watkins, Connectivity of transitive graphs, J. Comb. Theory 8 (1970) 23-29.

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