PROBABILISTIC METHOD
Fall 2007, Set 1

$[n]$ denotes $\{1, 2, \ldots, n\}$.

1. Determine the probability that 1 and 2 are in the same cycle of a random permutation π of $[n]$.

2. We toss a fair coin n times. What is the expected number of "runs"? (Runs are maximal sequences of consecutive tosses with the same result. For instance, the toss sequence HHHTTHTH has 5 runs, the first one containing the three H's.)

3. Show that for m large and $n > m(\ln m + 5)$, a random mapping $f\colon [n] \to [m]$ is surjective with probability at least 0.99.

4. Prove that if there is a real $p \in (0, 1)$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{l} (1-p)^{\binom{l}{2}} < 1 \,,$$

then the Ramsey number $R(k, l)$ satisfies $R(k, l) > n$.

Using this, show

$$R(4, t) \geq \Omega(t^{3/2}/(\ln t)^{3/2}) \,.$$

5. Let $G_{n,p}$ be a random graph on n vertices with edge probability p. Show that if $p \in (0, 1)$ is independent of n then $G_{n,p}$ is connected almost surely. What is the smallest function $p(n)$ for which you can still show that $G_{n,p(n)}$ is connected almost surely?

Hints: Use the basic method, the expression of a random variable as a sum of indicator variables, the linearity of expectation, and the bound for the probability of the union of events.