PROBABILISTIC METHOD
Fall 2007, Set 2

Homework is due on Monday, October 15. Problems 4, 5, 6, and just one of the problems 1, 2, 3 are to be solved. Numbers on the right indicate the grading scheme.

$[n]$ denotes $\{1, 2, \ldots, n\}$.

1. Let P be a property of graphs such that if G has P then any graph H with $V(H) = V(G)$ and $E(G) \subseteq E(H)$ also has P. Let $G(n, p)$ denote the random graph on n vertices with each possible edge present with probability p (independent of other edges). Prove that if $p_1 < p_2$ then the probability of $G(n, p_1)$ having P is not larger than the probability of $G(n, p_2)$ having P. **4**

2. (a) Let $\mathcal{S}$ be a system consisting of m subsets of an n-point set X, each of size $\geq s$. Prove that there exists $N \subseteq X$, $|N| \leq \lceil (n/s) \ln m \rceil$, that intersects each set of $\mathcal{S}$. **3**

   (b) Show that there exists a system $\mathcal{S}$ of $m = n^2$ sets of size s on an n-element set X such that any set meeting all sets of $\mathcal{S}$ has at least $t \geq c(n/s) \log n$ elements, for some constant $c > 0$, provided that $s < n/2$. **4**

3. Let $a \leq n$ be integers. Show that there is a coloring of $E(K_n)$ by two colors, so that the number of monochromatic copies of K_a is at most $\binom{n}{a} 2^{1-\binom{a}{2}}$. **1**

   For a fixed, show how to find such coloring in a time polynomial in n. **4**

4. Let A be random $n \times n$ matrix of -1s and 1s; each entry is -1 or 1 with the same probability, and the entries are mutually independent.

   (a) Show that the expected value of $\det(A)$ is 0. **1**

   (b) Calculate the expected value of $\det(A)^2$. **2**

5. For each $i \in [n]$ let $v_i = (x_i, y_i)$ be a pair of integers with absolute value at most $2^{n/2}/(100\sqrt{n})$. Prove that there are two disjoint nonempty sets $I, J \subseteq [n]$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j \,.$$

3

6. Let the van der Waerden number $W(k,r)$ be the minimum N such that whenever the numbers $1, \ldots, N$ are r-colored, there exists a monochromatic k-element arithmetic progression. Prove that (a) $W(3,r) \geq r^2/15$ [2 points], $W(k,2) \geq c2^k/k$ for some constant $c > 0$. **3**

Hints: For the first problem, either consider the random graph model, where not all edges have the same probability, or try to generate a random graph as a union of two random graphs.

In the next one use the basic method, and select a random set by choosing each element inpendently with a properly chosen probability.

In the latter problems, use second moment and the Lovász's local lemma.