PROBABILISTIC METHOD
Fall 2007, Set 3

Homework is due on Monday, November 12. Numbers on the right indicate the grading scheme.

$[n]$ denotes $\{1, 2, \ldots, n\}$.

1. (a) Let $\mathcal{S}$ be a system consisting of m subsets of an n-point set X, each of size $\geq s$. Prove that there exists $N \subseteq X$, $|N| \leq \lceil (n/s) \ln m \rceil$, that intersects each set of $\mathcal{S}$. **3**

   (b) Show that there exists a system $\mathcal{S}$ of $m = n^2$ sets of size s on an n-element set X such that any set meeting all sets of $\mathcal{S}$ has at least $t \geq c\,\frac{n}{s} \log n$ elements, for some constant $c > 0$, provided that $\log n < s < n/2$. **4**

2. For a subset T of the vertex set of a graph G, define $c(T)$ as the number of edges having exactly one endpoint in T. Show that for all sufficiently large even n, there is a graph G on n vertices such that for all $T \subseteq V(G)$ of size $\frac{n}{2}$, we have $|c(T) - n^2/8| \leq f(n)$, for a suitable function $f(n) = o(n^2)$. How small can you make $f(n)$? **3**

3. Consider $G(n, 1/2)$. Show that a.s. every vertex has degree close to $n/2$, and every two vertices have close to $n/4$ common neighbours. (Be sure to specify, what notion of "close" are you going to prove.) **3**

4. A graph G is called *k-emulsive* if there is a mapping $f\colon E(G) \to [k]$ such that for any mapping $g\colon V(G) \to [k]$ there is an edge $\{u, v\}$ with $g(u) = g(v) = f(\{u, v\})$.

   (a) Prove that the complete graph K_n is k-emulsive for $n = 100k^2 \log k$ (and k large if needed). **4**

   (b) Show that if G is k-emulsive then it has a vertex of degree at least $\frac{1}{100}k^2$. **3**

Hints: For the first problem, think about various ways how to generate a random set of size (at most) r. There are at least three natural ones:

- Choose one set out of all r-subsets of X.
- Decide about each element of X independently, take it with probability $p = r/n$.
- r-times pick a random element of X (possibly with repetition).

Each of these methods can be used in part (a) easily (although the last one is probably the easiest). Which one is best in (b)?

In the next two problems use Chernoff bound.

In the fourth problem, part (a), show that a random k-coloring of edges almost surely has edges of all colors on each subset of $\frac{n}{k}$ vertices. (Alternately, you may use the basic method directly.) In part (b) let f be fixed and let $A_{\{u,v\}}$ be the event $f(\{u,v\}) = g(u) = g(v)$ for g random. Then use LLL.