PROBABILISTIC METHOD
Fall 2007, Set 4

Homework is due on December 3. Numbers on the right indicate the grading scheme. Symbol $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

1. Suppose G is a graph, and for each $v \in V(G)$ we are given a set L_v. A *coloring of* G *from the list* L is a proper vertex coloring such that each vertex v gets a color from L_v.

   Suppose now that

   - $|L_v| \geq l$ for every vertex v and
   - for every vertex v, every color c is in at most $l/8$ sets L_u for u a neighbour of v.

   Prove that there is a coloring of G from the list L. **4**

2. (An open-ended question)

   Propose a model for random bipartite graphs on $n + n$ vertices (with $G(n, 1/2)$ in mind). Find a couple of quasirandom properties for this model and show that they are equivalent with each other. **4**

3. Let $G = (V, E)$ be the graph with vertices $V = [7]^n$ and two vertices being adjacent iff they differ in exactly one coordinate. Let $U \subseteq V$ be a set of 7^{n-1} vertices of G, let W be the set of all vertices of G at distance from U at least $(c + 2)\sqrt{n}$ ($c > 0$ a constant). Prove that $|W| \leq 7^n \cdot e^{-c^2/2}$. **3**

4. Let G be a graph with chromatic number $\chi(G) = 1000$. Supposed we pick a random subset of the set of vertices $X \subseteq V(G)$. This means pick randomly one of all possible $2^{|V|}$ subsets. Let $H = G[U]$ be the induced subgraph on X.

   (a) Prove that $\mathbb{E}[\chi(H)] \geq 500$. **2**

   (b) Show that $\Pr[\chi(H) \leq 400] < 1/1000$. **2**

Hints: 1. Use the Local lemma. The most natural choices of "bad events" don't work. Observe, why, and design good "bad events".

2. Consider regular graphs. Eigenvalues $\lambda_1 = -\lambda_{2n}$ are forced by regularity and bipartiteness. Others being $o(n)$ is a good property to look at. Formulate some properties similar to those presented in the class, prove equivalence for some of them.

3. and 4b. Use Azuma inequality.

4a. Consider the complement of H.