Topological Graph Theory*
Lecture 1-2: Connectivity, Planar Graphs and the Jordan Curve Theorem

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Summary: These notes cover the first and second weeks of lectures. Topics included are basic notation, connectivity of graphs, planar graphs, and the Jordan Curve Theorem.

1 Basic Notation

graph: G denotes a *graph*, with no loops or multiple edges. When loops and multiple edges are present, we call G a *multigraph*.

vertex set of G: denoted $V(G)$ or V. The *order* of G is $n = |V(G)|$, or $|G|$.

edge set of G: denoted $E(G)$ or E. $q = |E(G)|$ or $\|G\|$.

subgraphs: H is a *spanning subraph* of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. H is an *induced subgraph* of G if $V(H) \subseteq V(G)$, and $E(H)$ is all edges of G with both ends in $V(H)$.

paths: P_n denotes a *path* with n vertices; also called an n-path.

cycles: C_n denotes a *cycle* with n vertices; also called an n-cycle.

union of graph: $G \cup H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

intersection of graph: $G \cap H$ has vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

2 Connectivity

We begin by covering some basic terminology.

k-connected: A graph G is *k-connected* if $|G| \geq k + 1$ and for all $S \subseteq V(G)$ with $|S| < k$, the graph $G - S$ is connected. S *separates* G if $G - S$ is not connected.

cutvertex: $v \in V(G)$ is a *cutvertex* if $G - \{v\}$ is not connected.

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blocks: We can define an equivalence relation $\sim$ on $E(G)$, such that $e_1 \sim e_2$ if and only if $e_1 = e_2$ or there exists a cycle in G containing both e_1 and e_2. Then, $\sim$ partions $E(G)$ into sets which determine subraphs of G called *blocks*.

Observe that every block is either a single edge or a 2-connected graph.

Proposition 2.1. *Let G be a graph. Then,*

1. *$e \in E(G)$ is a cutedge if and only if the ends of e belong to different components of $G - e$.*
2. *$v \in V(g)$ is a cutvertex if and only if $G - v$ has more components than G.*
3. *If u and v are vertices in a block, B, of G containing at least 2 edges, then B has a cycle containing u and v.*
4. *Two blocks of G have at most one vertex in common, and this vertex is a cutvertex of G.*

Proof. The first three claims follow from the definition of blocks. The fourth claim is proved by contradiction. Suppose B_1 and B_2 are two blocks with two vertices in common, u and v. Then, B_1 and B_2 have cycles C_1 and C_2, respectively, containing u and v. It is then possible to find a cycle containing an edge of C_1 and an edge of C_2. □

Proposition 2.2. *Let G be connected with $|G| \geq 3$. Then, the following are equivalent:*

1. *G is 2-connected.*
2. *Any two vertices of G are on a common cycle.*
3. *Any two edges lie on a common cycle.*
4. *G has no cut vertices.*
5. *For all $v \in V(G)$, $G - v$ is connected.*
6. *G has only one block.*

Definition 2.3. An *ear* of a graph G is a path with end vertices in common with G.

Proposition 2.4. *If a graph G is 2-connected, then G can be obtained from a cycle by successively adding ears.*

Proof. We provide an algorithm which contstructs G by successively adding ears to a cycle in G. If G is not a cycle, then let G' be the current graph. We construct ears to add to G' by choosing an edge $e = uv$ with $v \in V(G')$ but $u \notin V(G')$. Let Q be the shortest path in $G - v$ from u to some vertex in G'. Then Q together with e is an ear, which we add to G'. Continue this process until we obtain G. □

Proposition 2.5. *If G is a 2-connected graph with $e = v_1 v_n \in E(G)$, ($n = |G|$), then the vertices of G can be enumerated $v_1, \ldots, v_n$ so that the graphs $G[\{v_1, \ldots, v_i\}]$ and $G[\{v_i, v_{i+1}, \ldots, v_n\}]$ are both connected for $i = 1, 2, \ldots, n$.*

Proof. Let C be a cycle in G and let $P_1, P_2, \ldots, P_r$ be the paths in an ear decomposition of G. Then, it is possible to enumerate the vertices so that for all v_i, there exists a path from v_1 to v_i with increasing indices, and there exists a path from v_i to v_n with increasing indices. Then, the required graphs are connected. □

Definition 2.6. We denote an edge contraction by G/e. We denote *strong edge contraction* by $G//e$, where e is first contracted then multiple edges are replaced by a single edge.

Lemma 2.7. *For every* 3*-connected graph G of order at least 5, there exists an edge e such that $G//e$ is still* 3*-connected.*

There is an analogous result for 2-connected graphs, which we state below.

Corollary 2.8. *Let G be a* 2*-connected graph, with $|G| \geq 4$. Then, for all $e \in E(G)$, $G - e$ or G/e is* 2*-connected.*

Definition 2.9. A multigraph, G, is 2-connected if it has no loops, and the underlying graph is 2-connected. G is k-connected if it has no multiple edges and the underlying graph is k-connected.

Theorem 2.10. *If G is a* 3*-connected graph distinct from a wheel, then G contains an edge e such that either $G - e$ or G/e is* 3*-connected.*

Theorem 2.11. *If G is a* 3*-connected graph distinct from K_4, then G contains an edge e such that $G//e$ is* 3*-connected.*

Proof. We prove this theorem by contradiction. Let G be a 3-connected graph such that for all $e = xy \in E(G)$, $G//e$ is not 3-connected. Then, the vertex xy and another vertex z separate $G//e$, so $\{x, y, z\}$ separates G. We can choose $e = xy$ and z such that the largest component, H, of $G - \{x, y, z\}$ is maximal in the number of vertices. □

Note that the proof of Thomassen's theorem is algorithmic, in that it allows us to find an edge e such that $G//e$ is 3-connected.

Proposition 2.12. *If G is a* 3*-connected graph distinct from K_4, and $e_0 = x_0y_0$ is any edge, then there exists e such that $G//e$ is* 3*-connected and e is not incident with x_0 or y_0.*

Theorem 2.13. *Let G be a graph and $s, t \in V(G)$ such that $s \neq t$, and $k \in \mathbb{N}$. Then, G does not contain k internally disjoint st-paths if and only if there exists $S \subseteq V(G) \backslash \{s, t\}$ with $|S| \leq k - 1$ such that S separates s and t.*

Theorem 2.14. *Let G be a graph with $|G| \geq k + 1$. Then, the following are equivalent:*

1. *G is k-connected.*

2. *For all $x, y \in V(G)$ that are nonadjacent, there exists k internally disjoint xy-paths.*

3. *For all $x, y \in V(G)$ there exists k internally disjoint xy-paths.*

4. *For all $k = t_1 + t_2 + \cdots + t_p = s_1 + s_2 + \cdots + s_q$ with $p, q \geq 1, t_i, s_j \geq 1$ and for all $A = \{a_1, \ldots, a_p\}, B = \{b_1, \ldots, b_p\} \subseteq V(G)$, there exist k AB-paths such that t_i paths start at a_i ($1 \leq i \leq p$) and s_j paths end at b_j ($1 \leq j \leq q$).*

3 Planar Graphs and the Jordan Curve Theorem

Definition 3.1. A *curve* or *arc* in $\mathbb{R}^2$, is the image of a continuous mapping $f : I \to \mathbb{R}^2$, where $I = [0, 1]$. An arc is *simple* if f is injective, i.e. the arc has no self-intersections, and an arc is *closed* if $f(0) = f(1)$. Moreover, if $J \subseteq I$ is a connected interval, then $f(J)$ is a *segment* of $f(I)$.

Claim 3.2. *Let $F_1, F_2 \subseteq \mathbb{R}^2$ be disjoint and closed. If A is an arc from F_1 to F_2, then A contains a segment (or subarc) A' such that A' connects F_1 and F_2, and the interior of A' is disjoint from $F_1 \cup F_2$.*

Definition 3.3. Let G be a graph, and X a topological space. G is *embedded* in X if $V(G) \subseteq X$, every edge of G is a simple arc in X connecting its end vertices, and the arcs corresponding to the edges are pairwise disjoint, except for their common end vertices. (i.e. no edges intersect). Moreover, an *embedding* in X of a graph G is an isomorphism with a graph G' embedded in X. G' is called a *representation* of G in X.

Definition 3.4. A *polygonal arc* is an arc composed of a finite number of straight line segments in $\mathbb{R}^2$.

Lemma 3.5. *If a graph G admits an embedding in $\mathbb{R}^2$, then it has an embedding in $\mathbb{R}^2$ in which all edges are polygonal arcs.*

Proof. $V(G) \subseteq \mathbb{R}^2$. For all $v \in V(G)$, let D_v be a disc around $v \in \mathbb{R}^2$ such that no edge of $G - v$ intersects D_v, and $D_v \cap D_u = \emptyset$ for $u \neq v$. For all $e = uv$, there is a segment A_{uv} of e joining D_u and D_v, but internally disjoint from $D_u \cup D_v$. Since A_{uv} is compact, there exists a finite subcover $S_1, S_2, \ldots, S_m$, such that each S_i is a disc around $i \in A_{uv}$ where no other edge intersects S_i. Drawing line segments through the centres of $S_1, \ldots, S_m$, we obtain a polygonal arc. $\square$

Definition 3.6. $D \subseteq \mathbb{R}^2$ is *arcwise connected* if for all $x, y \in D$, $x \neq y$, there exists an arc in D from x to y.

Theorem 3.7 (The Jordan Curve Theorem for Polygonal Arcs)**.** *If C is a simple closed polygonal curve in $\mathbb{R}^2$, then $\mathbb{R} \backslash C$ consists of precisely two arcwise connected components, each of which has C as the boundary.*

Proof. Let $C = P_1 P_2 \cdots P_n$, the line segments of C, and assume that no P_i is horizontal. For all $z \in \mathbb{R}^2$, let $ray(z)$ denote the horizontal right ray from z, and let

$$\pi(z) = |\{i \mid ray(z) \text{ intersects } P_i \text{ in a point which is not the top end of } P_i\}|.$$

Let $\overline{\pi}(z) = \pi(z) \bmod 2$. Then, we claim that $\overline{\pi}(z)$ is constant on every arcwise conneted component of $\mathbb{R}^2$. Thus, we can show that $\mathbb{R}^2 \backslash C$ has at least two arcwise connected components.

To show that there are not more than two arcwise connected components, consider $a, b, c \in \mathbb{R}^2 \backslash C$, and a disc D around a point in C. Then, there exist arcs A_a, A_b, A_c from a, b, c to D, two of which can be joined in D. Therefore, there are at most two arcwise connected components in $\mathbb{R}^2 \backslash C$. $\square$

Corollary 3.8. *Let C be a simple closed polygonal curve in $\mathbb{R}^2$, and $p, q \in C$. Let P be a polygonal arc from p to q such that $P \cap C = \{p, q\}$. Then, $P \cup C$ has precisely three arcwise connected components (faces), whose boundaries are $C, S_1 \cup P, S_2 \cup P$ where $S_1 \cup S_2 = C$.*

Proof. We apply 3.7 to see that $C, P \cup S_1, P \cup S_2$ each have two faces. Assume that $P \in \overline{int}(C)$. Then, each face of $C \cup P$ is both a face of $P \cup S_1$ and $P \cup S_2$. Let X_1, X_2 be the bounded face of $P \cup S_1$, $P \cup S_2$ respectively, and let Y_1, Y_2 be the unbounded face of $P \cup S_1$, $P \cup S_2$ respectively. Then, $Y_1 \cap Y_2 = Y$, the unbounded face of $C \cup P$, $X_1 \cap Y_2 = X_1$ is a face of $C \cup P$, and $X_2 \cap Y_1 = X_2$ is a face of $C \cup P$. We have now counted all the faces of $C \cup P$. □

Theorem 3.9. *If G is a 2-connected plane graph, whose edges are polygonal arcs, then the number of faces in G is precisely*

$$\|G\| - |G| + 2$$

and each of these faces is bounded by a cycle of G.

Proof. Let G have ear decomposition $C, P_1, P_2, \ldots, P_r$. Let $G_0 = C$, and $G_i = C \cup G_{i-1}$ for $i = 1, \ldots, r$, where each G_i is 2-connected in the plane. We perform induction on r.

If $r = 0$, we apply the Jordan Curve Theorem, and the claim holds. Otherwise, for $r \geq 1$, the embedding G_{r-1} has the required number of faces, which are bounded by cycles. P_r is a path in one of the faces, bounded by a cycle C' of G. By 3.8, G_e has one more face than G_{r-1}. Counting the appropriate number of vertices end edges for G_r, we see that the result holds. □

Proposition 3.10. *Let G be a plane graph, ebedded with polygonal edges such that $|G| = n \geq 4$. Then,*

1. *$\|G\| \leq 3n - 6$, with equality if and only if every face of G is bounded by a 3-cycle (including the outer face).*
2. *If G has no cycle of length 3, then $\|G\| \leq 2n - 4$, with equality if and only if all faces are bounded by 4 cycles.*

Proof. Assume G is 2-connected, and let $f = 2 - |G| + \|G\|$, the number of faces of G. Then, double count the pairs (e, F) where $e \in E$ and F is a face containing e on the boundary, so the number of pairs is $2\|G\|$. Since every F contains at least 3 edges on its boundary, the number of pairs is at least $3f$. By applying 3.9, we see that $\|G\| \leq 3n - 6$.

To show the second part, we apply the same method but observe that the number of pairs is at least $4f$. If G is not 2 connected, we apply the same method to blocks of G. □

Corollary 3.11. *K_5 and $K_{3,3}$ are not planar.*

Proof. If K_5 has a planar representation, then there exists a representation with polygonal edges. Applying 3.10 we see $\|K_5\| \leq 3 \cdot 5 - 6 = 9$, but $\|K_5\| = 10$. Hence, K_5 cannot be planar.

To show that $K_{3,3}$ is not planar, we use the same technique but apply the second inequality in 3.10. □

Theorem 3.12 (The Jordan Curve Theorem)**.** *If C is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \backslash C$ consists of two arcwise connected components (whose boundary is C).*

Proof. We first prove that there are at least 2 faces in $\mathbb{R} \backslash C$. Let p, q be the leftmost and rightmost points on C. Then $C \backslash \{p, q\}$ divides C into disjoint segments S_1, S_2. Let P be a polygonal arc in $ext(C)$ from p to q, and let L be a line segment intersecting S_1 and S_2. Then, there exists $Q \in L$ which joins S_1 and S_2, and let x be the midpoint of Q, which is disjoint from C. If there is only one face in $\mathbb{R}^2\backslash C$, then there exists a simple curve in $\mathbb{R}^2\backslash C$ from x to $y \in P\backslash C$, which is disjoint from $Q \cup P\backslash\{x, y\}$. However, we have just constructed an embedding of $K_{3,3}$ in the plane, which is not possible.

Now we shall prove that there are at most 2 faces in $\mathbb{R}\backslash C$. We claim that if P is a simple (non-closed) arc in $\mathbb{R}^2$, then $\mathbb{R}^2\backslash P$ is arcwise connected. Thus, it follows that a point $y \in C$ is accessible from x if there is a polygonal curve from x to y disjoint from $C\backslash\{y\}$. Suppose that $\mathbb{R}^2\backslash C$ has 3 distinct faces, $\Omega_1, \Omega_2, \Omega_3$. Let $p_1, p_2, p_3 \in C$ and $q_j \in \Omega_j$ $(j = 1, 2, 3)$. Then, we can draw polygonal arcs from q_j to each p_i, where arcs $q_1p_i \subseteq \Omega_1, q_2p_i \subseteq \Omega_2, q_3p_i \subseteq \Omega_3$. Thus, we obtain an embedding of $K_{3,3}$ in $\mathbb{R}^2$. $\square$

Theorem 3.13 (Jordan-Schönflies)**.** *If $\phi\colon C \to C'$ is a homeomorphism of two simple closed curves $C, C' \in \mathbb{R}^2$, then there exists $\overline{\phi}\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\overline{\phi}$ is a homeomorphism and extends ϕ.*