plane such that the outer cycle of the first graph is mapped to the outer cycle of the image, can be extended to a homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$, and is therefore also a plane-isomorphism.

Induced nonseparating cycles are important also for nonplanar graphs. For example, Tutte [**Tu63**] proved that the induced nonseparating cycles generate the cycle space of an arbitrary 3-connected graph. In fact, Tutte proved the following stronger result.

THEOREM 2.5.2 (Tutte [**Tu63**]). *Let G be a 3-connected graph. Then every edge e of G is contained in two induced nonseparating cycles having only e and its ends in common. Moreover, the induced nonseparating cycles of G generate the cycle space $Z(G)$ of G.*

Following [**Th80a**], we shall derive Theorem 2.5.2 from Proposition 1.4.6 combined with Lemma 2.5.3 below.

LEMMA 2.5.3. *Let G be a 3-connected graph with at least 5 vertices. Let $e = xy \in E(G)$ be an edge of G, and let $G' = G/\!\!/e$. Let C' be an induced nonseparating cycle of G'. If the vertex $z \in V(G')$ corresponding to the contracted edge e is not in C', then C' is also an induced nonseparating cycle in G. Otherwise, there is an induced nonseparating cycle C of G containing the edges of C' not incident with z, and the other edges of C are either adjacent to e, or equal to e.*

PROOF. If $z \notin V(C')$, then C' is clearly an induced nonseparating cycle of G. Assume now that $z \in V(C')$ and denote by a, b the two neighbors of z on C'. If both x and y are adjacent to each of a, b, then either x or y has a neighbor in $G' - C'$. (Otherwise $\{a, b\}$ would be a separating set since G has at least 5 vertices. But this is a contradiction to the 3-connectedness of G.) Suppose this vertex is y. Then G contains a cycle using $C' - z$ and the edges ax, xb. This cycle is induced and nonseparating. Another possibility is that x but not y is joined in G to both a and b. Then G contains a cycle C with vertex set $(V(C')\backslash\{z\}) \cup \{x\}$. This cycle is clearly induced. Since C' is induced and nonseparating and G is 3-connected, C is also nonseparating. If none of x and y is adjacent in G to both a and b, then C' gives rise to a unique cycle in G containing the edge e. This cycle is induced and nonseparating. □

PROOF OF THEOREM 2.5.2. We will use induction on $n = |V(G)|$. If $n \leq 4$ then $n = 4$ and $G = K_4$. The assertion of the theorem is easily verified in this case. Assume now that $n \geq 5$. By Proposition 1.4.6, G contains an edge e' having no ends in common with e such that $G' = G/\!\!/e'$ is 3-connected. Clearly $E(G') \subseteq E(G)\backslash\{e'\}$. By the induction hypothesis, the edge e belongs to two induced nonseparating cycles C_1', C_2' in G' having only e and its endvertices in common. By Lemma 2.5.3 there are induced nonseparating cycles C_1, C_2 in G which differ from C_1', C_2' only at the vertex z from the lemma. Since e and e' are nonadjacent edges of G, z

belongs to at most one of C_1', C_2'. Therefore C_1, C_2 satisfy the conclusion of Theorem 2.5.2.

To prove that the induced nonseparating cycles in G generate $Z(G)$, let C be any cycle in G. We shall show that G contains induced nonseparating cycles $C_1, \ldots, C_m$ such that $E(C) = E(C_1) + \cdots + E(C_m)$. As the induced cycles generate $Z(G)$, we may assume that C is induced. By the induction hypothesis, G' (as defined above) has a collection of induced nonseparating cycles $C_1', \ldots, C_m'$ such that $E(C_1') + \cdots + E(C_m') = E(C)$ or $E(C/\!\!/e')$. Let C_i be the induced nonseparating cycle of G such that $E(C_i) = E(C_i')$ or $E(C_i) = E(C_i') \cup \{e'\}$ for $i = 1, \ldots, m$. Then $E(C)$ is the sum of $E(C_1) + \cdots + E(C_m)$ and triangles containing e'. To see this, we observe that $E(C) + E(C_1) + \cdots + E(C_m)$ is Eulerian and contains only edges incident with the ends of e'. Hence, $E(C) + E(C_1) + \cdots + E(C_m)$ is is the sum of triangles containing e'. Note that since $G/\!\!/e'$ is 3-connected, all triangles containing e' are nonseparating. This completes the proof. $\square$

We obtain from Theorems 2.5.1, 2.5.2, and 2.4.5 the following characterization of planar graphs, due to Tutte.

COROLLARY 2.5.4 (Tutte [**Tu63**]). *A 3-connected graph is planar if and only if every edge is contained in precisely two induced nonseparating cycles.*

A special case of Kuratowski's theorem for cubic graphs was first discovered by Menger (see [**Kö36**]). In this case, only $K_{3,3}$ is needed. Hall [**Ha43**] and Wagner [**Wa37b**] showed that a 3-connected graph distinct from K_5 is planar if and only if it does not contain a $K_{3,3}$-subdivision. This follows immediately from the following observation:

LEMMA 2.5.5. *Let G be a 3-connected graph of order six or more. If G contains a subdivision of K_5, then it also contains a subdivision of $K_{3,3}$.*

PROOF. Let K be a K_5-subdivision in G. Since G is 3-connected and distinct from K_5, there is a K-bridge B in G whose vertices of attachment are not all in just one path of K corresponding to an edge of K_5. It is easy to see that $K \cup B$ contains a subdivision of $K_{3,3}$. $\square$

Kelmans [**Ke84a, Ke84b**] and Thomassen [**Th84b**] independently proved a stronger result.

THEOREM 2.5.6 (Kelmans [**Ke84a, Ke84b**], Thomassen [**Th84b**]). *Every 3-connected nonplanar graph G distinct from K_5 contains a cycle with three chords which together form a subgraph of G homeomorphic to $K_{3,3}$.*

Note that this result easily implies Theorem 2.4.4.

Kelmans [**Ke97**] proved that a 3-connected nonplanar graph without 3-cycles in which every separating set of 3 vertices separates a single vertex