Topological Graph Theory*
Lecture 15-16 : Cycles of Embedded Graphs

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Summary: These notes cover the eighth week of classes. We present a theorem on the additivity of graph genera. Then we move on to cycles of embedded graphs, the three-path property, and an algorithm for finding a shortest cycle in a family of cycles of a graph.

1 Additivity of Genus

Theorem 1.1 (Additivity of Genus)**.** *Let G be a connected graph, and let $B_1, B_2, \ldots, B_r$ be the blocks of G. Then the genus of G is,*

$$g(G) = \sum_{i=1}^{r} g(B_i)$$

and the Euler genus of G is,

$$eg(G) = \sum_{i=1}^{r} eg(B_i).$$

Observation 1.2. *In order to prove the theorem it suffices to prove that if $G = G_1 \cup G_2$, where $G_1 \cap G_2 = \{v\}$, then $g(G) = g(G_1) + g(G_2)$ and $eg(G) = eg(G_1) + eg(G_2)$.*

Proof. We set

$$g(G) = min\{g(\Pi) \mid \Pi \text{ is an orientable embedding of } G\}$$
$$g(G_1) = min\{g(\Pi_1) \mid \Pi_1 \text{ is an orientable embedding of } G_1\}$$
$$g(G_2) = min\{g(\Pi_2) \mid \Pi_2 \text{ is an orientable embedding of } G_2\}$$

Let v_1, v_2 be vertices in G_1, G_2 respectively, distinct from v. Note that the local rotations of v_1, v_2 are independent, but the local rotation of v is not. So if we take Π an embedding of G and split it along v we obtain Π_1, Π_2 embeddings of G_1, G_2 respectively. Facial walks in G that do not pass from G_1 to G_2 remain unchanged in Π_1, Π_2, however if a facial walk does pass from G_1 to G_2 then when we split we form an extra face. But we also have an extra vertex. So,

$$g(\Pi) = g(\Pi_1) + g(\Pi_2).$$

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Claim: The *min* value of $g(G)$ is attained at an embedding where the local rotation at the cut vertex v groups the edges from G_1 into a single block.
Assume that this is not the case. Then each facial walk must use "mixed angles" in pairs (by mixed angle we refer to the angle formed at v between two edges, one from G_1 and one from G_2). By considering the local rotations we see that we must traverse an odd number of negative signatures from v to v on any walk using mixed angles. Thus we have a 1-sided cycle in our embedding. However this is a contradiction as we assumed that Π was orientable. This proves the claim. It also proves that

$$g(G) = g(G_1) + g(G_2).$$

This proof can be repeated with minor changes to prove that,

$$eg(G) = eg(G_1) + eg(G_2).$$

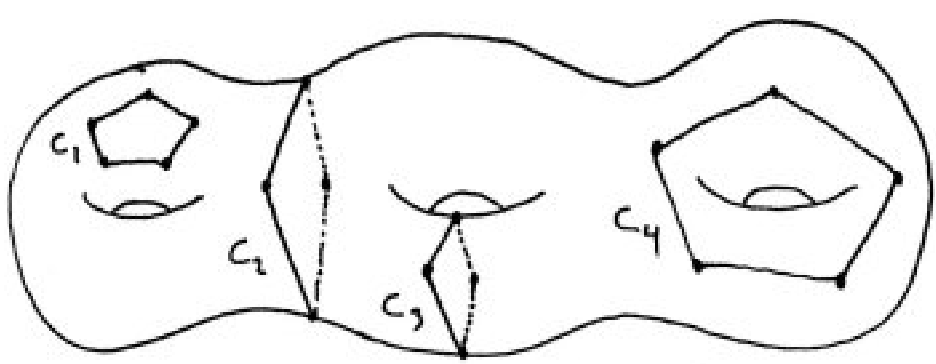
This completes the proof. □

2 Induced Embeddings

Given Π an embedding of G we wish to develop the notion of a corresponding embedding Π' of H a subgraph of G. Suppose that $e \in E(G)$ is not a cut-edge of G, then we can consider deleting e from G. Without loss of generality, we may assume that $\lambda(e) = +1$. If e is in two distinct facial walks, then we can see that $G - e$ embeds in the same surface as G, this follows as $G - e$ will have the same Euler characteristic as G. If e appears twice on one face, then $G - e$ will embed in a simpler surface (the Euler genus will decrease by either 1 or 2). From this we can see that in the specific case of multigraphs, the embedding of the underlying simple graph cannot have larger genus, and never changes from orientable to non-orientable.

3 Cycles of Embedded Graphs

We start by giving an intuitive explanation of the classes of cycles we will consider. For example, say we are considering the surface $\mathbb{S}_3$, pictured below.



- C_1 bounds a disk on the surface. Such cycles are called contractible cycles.
- Cutting along C_2 separates the surface. Such cycles are called surface separating cycles.

- C_3 and C_4 are called surface non-separating cycles.
- Note that contractible cycles are also surface separating.

Now we formally define these notions. Let C be a two-sided cycle of a Π-embedded graph G. We may assume that $\lambda(e) = +1$ for all edges $e \in E(C)$. We can arbitrarily choose a "perspective" from which to view C. Now since C is two-sided we have a "left" and "right" side of C with respect to our perspective. This allows us to define:

$$\begin{aligned} E_L &:= \text{ the set of all edges incident to a vertex of } C \text{ embedded on the "left" of } C \\ E_R &:= \text{ the set of all edges incident to a vertex of } C \text{ embedded on the "right" of } C \\ G_L(C) &:= \text{ the subgraph of } G \text{ consisting of } E_L \text{ and all vertices and edges in } G - C \text{ reachable from } E_L \\ G_R(C) &:= \text{ the subgraph of } G \text{ consisting of } E_R \text{ and all vertices and edges in } G - C \text{ reachable from } E_R \end{aligned}$$

Definition 3.1. For a two-sided cycle C, if $G_L(C) \cap G_R(C) \subseteq C$, then we say that C is a *surface separating cycle*.

Proposition 3.2. *Let C be a surface separation cycle of a Pi-embedded graph G. Consider the induced embeddings of $G'_L := G_L(C) \cup C$ and $G'_R := G_R(C) \cup C$. The sum of the Euler genera of the embeddings of G'_L and G'_R is $eg(\Pi)$.*

Proof. Claim #1: Every facial walk of G is either a facial walk of G'_L or a facial walk of G'_R.
If a facial walk never reaches C, the claim holds trivially. Suppose we have a facial walk F tat intersects C. Since we take all signatures on C to be positive and all rotations to be equal, we can see that each time F enters C on a right edge, it will leave on a right edge (and vice versa). This proves the claim.
Claim #2: C is facial in G'_L and G'_R.
This holds for the same reason as Claim #1.
Thus the number of faces of G is two less than the sum of the number of faces in G'_L and G'_R. Now we simply apply Euler's formula to G, G'_L and G'_R to prove the proposition. ☐

Definition 3.3. A cycle C of a Π-embedded graph G is Π-*contractible* if it is surface separating and the Euler genus of either the induced embedding of $G_L(C) \cup C$ or $G_R(C) \cup C$ is zero.

Note that this is equivalent to C bounding a disk on the surface.

Definition 3.4. If C is contractible and $G_L(C) \cup C$ has genus zero, then

$$\begin{aligned} int(C) = int(C, \Pi) &:= G_L(C) \\ Int(C) = Int(C, \Pi) &:= G_L(C) \cup C \end{aligned}$$

Now we can classify cycles of embeddings as:

$$\begin{tikzcd}[row sep=small]
 & & \text{contractible} \\
 & \text{surface separating} \arrow[ur] \arrow[dr] & \\
 & & \text{non-contractible} \\
\text{cycles} \arrow[uur] \arrow[ddr] & & \\
 & & \text{two-sided} \\
 & \text{surface non-separating} \arrow[ur] \arrow[dr] & \\
 & & \text{one-sided}
\end{tikzcd}$$

4 Cutting Surfaces Along Cycles

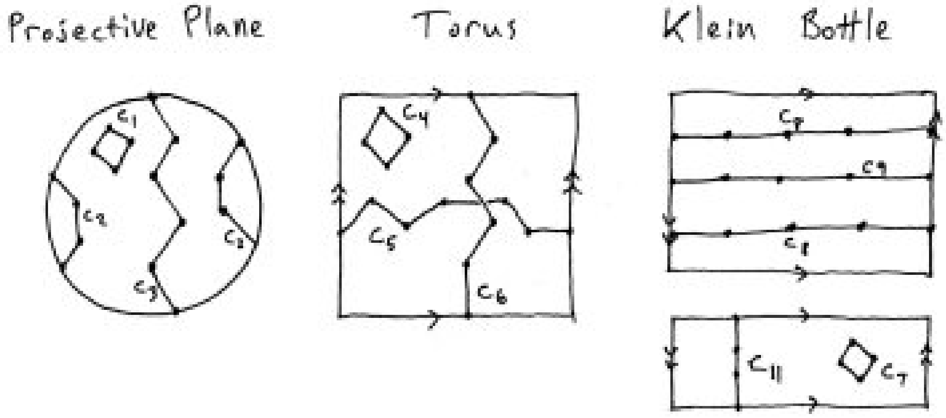
Cutting a surface along a cycle C gives rise to a graph in which C is replaced by 2 cycles, C' and C'' (both are copies of C). The edges on the left of C (with respect to some perspective) are incident with the corresponding vertices of C', the vertices on the right of C are incident with the corresponding vertices of C''. The graph that we obtain be performing this operation is isomorphic to $G_L(C) \cup G_R(C) \cup C' \cup C''$. Embeddings of G induce embeddings of C' and C'' as one might expect.

Proposition 4.1. *If C is surface separating, then cutting along C gives two graphs isomorphic to G'_L and G'_R respectively, and $eg(G'_L, \Pi) + eg(G'_R, \Pi) = eg(G, \Pi)$.*
If C is two-sided, but not surface separating, then the graph obtained after cutting along C. G', is connected, and $eg(G', \Pi) = eg(G, \Pi) - 2$.
If C is one-sided, then the graph G', obtained after cutting along C, is connected, and $eg(G', \Pi) = eg(G, \Pi) - 1$.

Note that in proposition 4.1, the orientability may change from non-orientable to orientable in the last two cases.

In the following drawings of the Projective Plane, Torus, and Klein Bottle we have the following:

- C_1, C_2, C_4 and C_7 are contractible
- C_3, C_9 and C_{10} are one-sided
- C_5 and C_6 are non-contractible
- C_8 is surface separating and non-contractible
- C_{11} is two-sided and non-separating



Note that the only surface separating cycles on the Torus are contractible, as the Torus is orientable. Also note that cutting along $C_1 1$ is the inverse operation of adding a twisted handle.

5 The Three-Path Property

Given vertices x, y of a graph G and internally disjoint xy-paths $P_1, P_2, \ldots, P_r$, we denote the cycle formed by paths P_i, P_j as C_{ij}.

Definition 5.1. Let $\mathcal{C}$ be a family of cycles in G. $\mathcal{C}$ has the *three-path property* if:
$\forall_{x,y \in V(G)}$, $\forall_{P_1,P_2,P_3 \text{ internally disjoint } xy\text{-paths}}$, if $C_{12} \notin \mathcal{C}$ and $C_{23} \notin \mathcal{C}$, then $C_{13} \notin \mathcal{C}$.

Example 5.2. The following are examples of families with the three-path property:

1. $\mathcal{C} = \{\mathcal{C} \mid \text{the length of } C \text{ is odd}\}$
2. $\mathcal{C} = \{\text{cycles with and odd number of edges in } E'\}$, where $E' \subseteq E(G)$
3. $\mathcal{C} = \{\text{one-sided cycles of } \Pi\}$
4. $\mathcal{C} = \{\text{non-contractible cycles of } \Pi\}$

We now give a short proof of 4.

Proof. Take $x, y \in V(G)$ and P_1, P_2, P_3 internally disjoint xy-paths. Assume that C_{12} and C_{23} are contractible. First we alter Π so that all signatures on C_{12} are positive. If $P_3 \subseteq Int(C_{23})$ then the result is clear. Similarly, if $P_1 \subseteq Int(C_{23})$ then the result is also clear. So we need only consider the case where P_2 lies "between" P_1 and P_3. Now we have that C_{13} is surface separating and $int(C_{13}) = int(C_{23}) \cup int(C_{12}) \cup P_2$. It follows from Euler's formula that the genus of $Int(C_{13})$ is zero, and thus C_{13} is contractible. This completes the proof. □

We now present an algorithm for finding a shortest cycle in $\mathcal{C}$, where $\mathcal{C}$ has the three-path property.

Algorithm 5.3. Input: A graph G and a family of cycles $\mathcal{C}$ with the three-path property.
For all $v \in V(G)$:
  Build the breadth-first search spanning tree of G starting at v, T_v.
  For every edge $e \notin E(T_v)$:
    Let C_e be the unique cycle in $T_v + e$.
  Choose a shortest of the cycles C_e to be C_v.
Choose C to be a shortest of the cycles C_v.
Return: C is a shortest cycle in $\mathcal{C}$.

Proposition 5.4. *Algorithm 5.3 correctly finds a shortest member of $\mathcal{C}$ in time $O(nqT+nq)$, where $n =\mid G \mid$, $q =\| G \|$, and T is the time complexity of $C \in \mathcal{C}$ queries.*

Proof. For a vertex $v \in V(G)$ let T_v be the BFS tree built by Algorithm 5.3. Let $C \in \mathcal{C}$, $C = v_0, \dots, v_{k-1}$, and $v_0 \in V(C)$ be selected subject to C being shortest in $\mathcal{C}$ and then having minimum intersection with $T = T_{v_0}$. We prove that Algorithm 5.3 finds a cycle of length $\mid C \mid$.

Claim 1: *$d_G(v_0v_i) = d_C(v_0v_i)$ for $i = 0, \dots, k-1$.* Let i be smallest such that $d_G(v_0v_i) \neq d_C(v_0v_i)$ and let P be the shortest path from v_0 to v_i. P contains a subpath P' that connects two vertices $x, y \in V(C)$. Let A, B be the cycles, formed by P' and xCy, yCx. A and B are shorter than C, thus none of them is in $\mathcal{C}$. By the three-path-property, neither is C. This contradiction establishes the claim.

Claim 2: *There exists $e \in E(C)$, incident with v_t, $t = \left\lfloor \frac{k}{2} \right\rfloor$, such that $C - e \subseteq T$.* Let i be smallest such that $v_iv_{i+1} \notin E(T)$. By symmetry we may assume $i < t$. Let P be the path from v_0 to v_{i+1} in T. The claim follows by the same argument as the previous one, using the fact that P has length $i+1$, implied by Claim 1.

Since there is only one edge of C missing in T, this cycle is examined by the algorithm, thus the cycle that is chosen at v_0 and subsequently in G has length at most $\mid C \mid$.

BFS tree can be found in time $O(q)$ and using it the length of the cycles can be compared in constant time. Thus there is at most $O(qT + q)$ time spent in the loop for each of the n vertices. The complexity follows. □

Corollary 5.5. *If $\mathcal{C}$ satisfies the three-path property and membership in $\mathcal{C}$ can be determined in polynomial time, then the above algorithm finds a shortest cycle in $\mathcal{C}$ in polynomial time.*

Note that Algorithm 5.3 can be applied to find a shortest one-sided cycle, non-contractible cycle, surface non-separating cycle, etc.