Topological Graph Theory*
Lecture 16-17 : Width of embeddings

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Summary: These notes cover the ninth week of classes. The newly studied concepts are homotopy, edge-width and face-width. We show that embeddings with large edge- or face-width have similar properties as planar embeddings.

1 Homotopy

It is beyond the purpose of these lectures to deeply study homotopy, but as we will need this concept, we mention it briefly. Two closed simple curves in a given surface are homotopic, if one can be continuously deformed into the other. For instance, on the sphere any two such curves are (freely) homotopic, whereas on the torus $S_1 \times S_1$ we have two homotopy classes: $S_1 \times \{x\}$ and $\{y\} \times S_1$. Formally:

Definition 1.1. Let $\gamma_0, \gamma_1 : I \to \Sigma$ be two closed simple curves. A homotopy from γ_0 to γ_1 is a continuous mapping $H : I \times I \to \Sigma$, such that $H(0,t) = \gamma_0(t)$ and $H(1,t) = \gamma_1(t)$.

The following proposition may be of interest:

Proposition 1.2. *Let $\mathcal{C}$ be the family of disjoint non-contractible cycles of a graph G embedded in a surface Σ of Euler genus g, such that no two cycles of $\mathcal{C}$ are homotopic. Then $|\mathcal{C}| \leq 3g-3$, for $g \geq 2$, and $|\mathcal{C}| \leq g$, for $g \leq 1$.*

2 Edge-width

Definition 2.1. Let Π be an embedding of a graph G in a surface Σ of Euler genus g. If $g \geq 1$ we define the edge-width of Π, $\mathtt{ew}(G,\Pi) = \mathtt{ew}(\Pi)$, to be the length of the shortest Π-non-contractible cycle in G. If $g = 0$, we define $\mathtt{ew}(G,\Pi) = \infty$.

In this section we study embeddings with large edge-width and show that they have similar properties as planar embeddings. In this sense edge-width measures degeneracy of the embedding – the smaller the edge-width, the larger the degeneracy. First we use a gentle homological argument to prove that the edge-width is well-defined for $g \geq 1$, i.e. that there is at least one non-contractible cycle.

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Let f be the number of faces, q the number of edges and n the number of vertices of Π. Then $f = q - n + 2 - g$. The cycle space $Z(G)$ has dimension $q - n + 1$ and its subspace $B(G)$, spanned by facial walks, has dimension $f - 1$ (one face can be expressed as sum of all the others, but otherwise there is always an edge that is covered only once by a subset of faces). Thus the quotient space $Z(G)/B(G)$ has dimension g. Since all the contractible cycles are in $B(G)$ (they are sums of faces lying in their interior), there must be at least $g \geq 1$ Π-non-contractible cycles in G.

An embedding Π is a large edge-width embedding (LEW-embedding), if $\mathtt{ew}(\Pi)$ is strictly larger than the length of any Π-facial walk. Examples of such embeddings are surface triangulations without non-contractible triangles.

Theorem 2.2. *Let Π, Π' be two embeddings of G of genera g, g', respectively. If Π is an LEW-embedding, then $g' \geq g$ and if $g = g'$, then the surfaces of Π and Π' are the same.*

Proof. Let f (f') be the number of faces of Π (Π') and let $l_1, \ldots, l_f$ $(l'_1, \ldots, l'_{f'})$ be the lengths of faces of Π (Π'). Set $m = \max_i l_i$. We prove the claim by induction on $\| G \|$.

If $g = 0$ the claim is trivial, so we assume $g \geq 1$. If G has a vertex of degree 1 we remove it and obtain G_1. The induced embedding Π'_1 is still a LEW-embedding, and the claim follows by induction.

Assume Π has a contractible cycle C, shorter than some face of Π. Then define G_1 to be the graph with the non-C vertices of the interior of C removed. Since C is short, the Π-induced embedding of G_1 is a LEW-embedding, and the claim follows by induction.

Now we may assume that every non-facial cycle of Π is longer than every facial cycle of Π. Then $\sum l_i = 2 \| G \| = \sum l'_i$ implies $f' \leq f$, since all cycles of G that are not facial walks of Π have length at least m. Thus $g' \geq g$ with equality if and only if $f = f'$ and $l_i = l'_i$ after a suitable permutation. But then the facial cycles of Π' are the facial cycles of Π and the embeddings are equivalent. $\square$

This theorem has the following corollaries that show the behavior of LEW-embeddings to be similar to planar embeddings.

Corollary 2.3. *If G is a 2-connected graph with a LEW-embedding Π, then every facial walk of Π is a cycle.*

Corollary 2.4. *If G is a 3-connected graph with a LEW-embedding Π of minimum Euler genus, then Π is the only minimum-Euler genus embedding of G (up to equivalence of embeddings).*

3 Face-width

Face-width is another measure of local planarity. Another name that was initially used for it is representativity of an embedding.

Definition 3.1. Let Π be an embedding of a graph G of Euler genus $g \geq 1$, and let k be the smallest integer such that there exist k Π-facial walks, containing a Π-non-contractible cycle in their union. We define $\mathtt{fw}(G, \Pi) = k$ to be the face-width of Π. If the Euler genus of Π is 0, we define $\mathtt{fw}(G, \Pi) = \infty$.

Let C be the shortest Π-non-contractible cycle of G. Each edge on the cycle corresponds to at most one facial walk, thus we observe the following:

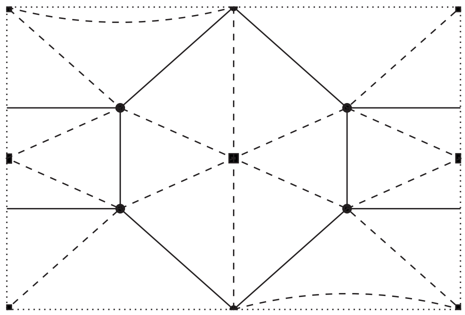

Figure 1: The vertex-face graph $\Gamma(G,\Pi)$ (dashed edges) of a graph G (solid edges) in torus.

Observation 3.2. $\mathtt{fw}(G,\Pi) \le \mathtt{ew}(G,\Pi)$.

Lemma 3.3. *Let Π be an embedding of G and let $\mathtt{eg}(\Pi) \ge 1$, $\mathtt{fw}(G,\Pi) \ge 2$. Then every facial walk W of G contains a Π-contractible cycle C and $W \subseteq \mathrm{Int}(C)$.*

Proof. By induction on r. If W is a cycle then $C = W = \mathrm{Int}(C)$. Otherwise W contains a closed subwalk $W' = vu_1 \dots u_r v$, such that $u_1, \dots, u_r$ appear only once in W'.[1]

If $r = 1$ we apply induction to the graph $G - u_1$. Otherwise, since $\mathtt{fw}(G,\Pi) \ge 2$, the subwalk W' is a contractible cycle. If $\mathrm{Int}(W')$ contains W', then W' is the cycle we seek. Otherwise we apply induction to the graph $G - \mathrm{Int}(W')$. □

The face-width can be defined in another way using the face-vertex graph, which we already defined for plane graphs (cf. Lecture notes, week 4); for general surfaces it is defined in the same way:

Definition 3.4. Let Π be an embedding of G and let F be the set of facial walks of Π. The vertex-face graph $\Gamma(G,\Pi)$ has vertices $V(\Gamma) = V(G) \cup F$, a vertex $v \in V(G)$ is incident with $f \in F$ if and only if $v \in f$, and these are all the incidences of $\Gamma(G,\Pi)$.

Example of a vertex-face graph is presented in Figure 1. The graph G, drawn with solid edges, is embedded in the torus. The vertices of $\Gamma(G,\Pi)$ corresponding to F are drawn as squares, and its edges are drawn as dashed lines.

It is obvious that $\Gamma(G,\Pi)$ is a bipartite graph, embedded in the same surface as G and that all the faces of the corresponding embedding Π_Γ have length 4: each face corresponds to an edge of G.

Proposition 3.5. . *Let G be a graph and Π its embedding. Then $\mathtt{fw}(G,\Pi) = \frac{1}{2}\mathtt{ew}(\Gamma(G,\Pi),\Pi_\Gamma)$.*

Proof. Let C be the shortest non-contractible cycle in Π_Γ. The union of facial walks, corresponding to all $|C|/2$ vertices of $C \cap F$ contain a Π-non-contractible cycle. If a subset of these cycles would contain a Π-non-contractible cycle, then there would be a shorter non-contractible cycle in Π_Γ. □

Proposition 3.5 has two interesting corollaries:

Corollary 3.6. *Face-width of an embedding can be computed in polynomial time.*

[1]Exercise: prove that there is a subwalk $W' = vu_1 \dots u_r v$, such that $u_1, \dots, u_r$ appear only once in W.

Corollary 3.7. *Let G^*, Π^* be the dual graph and embedding of G, Π. Then $\mathtt{fw}(G^*, \Pi^*) = \mathtt{fw}(G, \Pi)$.*

In the sequel we study local planarity properties of embeddings with large face-width. First we introduce two operations on an embedded graph G. If $v \in V(G)$, then $G - v$ is the graph G after removing the vertex v and all edges emanating from v. The dual of this operation is shrinking a face: if W is a facial walk in G, then G/W is the graph G after contraction of all the edges of W. The following two lemmas show that these operations have only minor impact on the face-width of an embedding. For the proof just observe that both operations correspond to contracting all edges incident with one vertex of $\Gamma(G, \Pi)$, apply Euler formula to establish the claim about the same surface and Proposition 3.5 for the inequality.

Lemma 3.8. *Suppose $\mathtt{fw}(G, \Pi) \geq 2$. Let $v \in V(G)$, $G' = G - v$, and Π' be the induced embedding of G'. Then G' is Π'-embedded in the same surface and $\mathtt{fw}(G, \Pi) - 1 \leq \mathtt{fw}(G', \Pi') \leq \mathtt{fw}(G, \Pi)$.*

Lemma 3.9. *Suppose $\mathtt{fw}(G, \Pi) \geq 2$. Let W be a facial walk of G, $G' = G/W$, and Π' be the induced embedding of G'. Then G' is Π'-embedded in the same surface and $\mathtt{fw}(G, \Pi) - 1 \leq \mathtt{fw}(G', \Pi') \leq \mathtt{fw}(G, \Pi)$.*

For a vertex $v \in V(\Gamma(G, \Pi))$ we define the graphs $B_i(v)$ inductively as follows: (i) $B_0(v) = \{v\}$ and (ii) $B_i(v)$ equals the union of all the facial walks sharing at least one vertex with $B_{i-1}(v)$.

Proposition 3.10. *If $\mathtt{fw}(G, \Pi) \geq k$, then for every vertex $v \in V(G)$ there exist disjoint cycles C_i, $i = 1, \ldots, \lfloor \frac{k-1}{2} \rfloor$, such that*

(a) every C_i is contractible and $\mathrm{Int}(C_i)$ contains $\mathrm{Int}(C_{i-1})$,

(b) $\mathrm{Int}(C_i)$ contains B_i,

(c) $C_i \subseteq B_i(v) \setminus B_{i-1}(v)$.

Proof. For $k \leq 2$ then claim is trivial. For $k = 3$ we take C_1 to be the non-contractible cycle of the facial walk of $G - v$ not present in G, which exists by 3.3. We proceed by induction using the following claim:

Claim: *Let $i \leq \lfloor \frac{k-1}{2} \rfloor$. Then every cycle in $B_i(v)$ is contractible.* Let $C = u_1, \ldots, u_t$ be a cycle in $B_i(v)$. For every $j = 1, \ldots, t$ there exists a path P_j from u_j to v using at most i facial walks and not intersecting P_{j-1}, P_{j+1}. The cycle $D_j = u_j P_j v P_{j+1} u_{j+1}$ meets at most $2i < k$ facial walks and is contractible, thus it can be in $B(G)$ expressed as the sum of the faces in $\mathrm{Int}(D_j)$. Now $C = \sum_j D_j$ and is contractible.

Let G, Π, v satisfy the assumptions. Let W be the new facial walk in $G - v$ that is not present in G, and let $G' = (G - v)/W$ with the induced embedding Π', with w the vertex corresponding to W. Then $\mathtt{fw}(G', \Pi') \geq \mathtt{fw}(G, \Pi) - 2$ by Lemmas 3.8 and 3.9 and by induction there exist Π'-contractible cycles $C_2, \ldots, C_k$ around w in G', which satisfy (a)–(c) and are disjoint from C_1 in G. Together with C_1 they establish the claim. □

We conclude with two propositions that further relate the embeddings with large face-width to planar graphs.

Proposition 3.11. *Let G be a Π-embedded graph, $\mathtt{eg}(\Pi) \geq 1$. Then all Π-facial walks are cycles, if and only if G is 2-connected and $\mathtt{fw}(G) \geq 2$.*

Proposition 3.12. *Let G be a Π-embedded graph, $\mathtt{eg}(\Pi) \geq 1$. Then all Π-facial walks are cycles and any two facial walks are either disjoint, share a vertex or share an edge, if and only if G is 3-connected and $\mathtt{fw}(G) \geq 3$.*