# Math 821 – Combinatorics
# Final Assignment
# 28/3/2007

To be handed by Thursday April 4, 2007 (5 p.m.)

**Question 1** Let $k$ and $l$ be fixed natural numbers. Let $n(k,l)$ be the maximal $n$, such that there exist sets $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ satisfying the following conditions

1. $|A_i| = k$, $|B_i| = l$ for all $i = 1, 2, \ldots, n$.
2. $A_i \cap B_i = \emptyset$ for all $i = 1, 2, \ldots, n$.
3. $A_i \cap B_j \neq \emptyset$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$.

Determine $n(k,l)$.

[*Hint:* Let $X = \bigcup_i A_i \cup \bigcup_i B_i$. Consider all permutations of $X$ and distinguish them according to the position of elements of $A_i$ and of $B_i$ for various $i$.]

**Question 2** A *blocking set* in a projective plane $X$ is a set $S$ of points of the plane, such that $S$ meets every line of $X$, but $S$ does not contain any line of $X$. Prove the following claims:

1. The Fano plane has no blocking set, but any larger plane contains a blocking set.
2. A blocking set in a plane of order $q$ contains at least $q + \sqrt{q} + 1$ points.
3. If the bound from 2 is attained, then the blocking set is a subplane of order $\sqrt{q}$. Such blocking set exists whenever $q$ is a square.
4. A blocking set in a plane of order $q$ contains at most $q^2 - \sqrt{q}$ points.
5. Let $S$ be a blocking set in a plane of order $q$ and suppose $S$ is minimal (that is, after removing any point of $S$ the resulting set is not blocking). Then $|S| \leq q\sqrt{q} + 1$.

**Question 3** Let $\Omega$ denote the set of all partitions of a set of nine elements into three triples. If $\pi$ and $\sigma$ are two of these partitions, define their product to be the partition whose cells are all possible nonempty intersections of the cells of $\pi$ with those of $\sigma$. Define two elements of $\Omega$ to be adjacent if their product contains exactly seven cells. Show that the graph on $\Omega$ with this adjacency relation is strongly regular, and determine its parameters.

**Question 4** Two Latin squares of the same size are said to be *orthogonal* to each other if for each pair of symbols $(a, b)$ there is exactly one position in which the first square has $a$ and the other $b$. A collection of Latin squares of the same size is said to be *mutually orthogonal* if every pair of squares in it is orthogonal.

Suppose that $A_1, \ldots, A_r$ is a set of $r$ mutually orthogonal Latin squares of size $n$. Let $\Omega$ be the set of $n^2$ cells in the array. For $\alpha \neq \beta$ in $\Omega$, let $(\alpha, \beta) \in R_1$ if $\alpha$ and $\beta$ are in the same row or are in the same column or have the same letter in any of $A_1, \ldots, A_r$; otherwise $(\alpha, \beta) \in R_2$.

1. Find the size of $R_1$. Hence find an upper bound on $r$.
2. Show that these definitions of $R_1$ and $R_2$ make an association scheme on $\Omega$. It is called the Latin-square type association scheme $L(r+2; n)$. When does it have only one associate class?