

SOME COMMENTS ON THE FOUR-COLOUR PROBLEM

1 Introduction

There are two ways of posing the Four Colour Problem. Usually we colour the vertices of a planar triangulation. In an older tradition we colour the faces of a cubic map M on the sphere. The two methods are related by duality.

To begin with I use the second procedure. Let $Q = \{A, B, C, D\}$ be a 4-colouring of M . Here A, B, C, D are the four "colour-classes", sets of faces of Q . Each is required to be an independent set, that is no two of its faces are to have a common edge. The four are to be disjoint and together they are to include all the faces of M . Changing the order of writing of the four colour-classes is not considered to give a new 4-colouring.

Given a face X of M we write $f(X)$ for the number of its incident edges. Given a set U of faces we write

$$f(U) = \sum_{X \in U} f(X).$$

An elementary theorem asserts that in any 4-colouring Q the four integers $f(A)$, $f(B)$, $f(C)$ and $f(D)$ have the same parity. According as this is odd or even we call Q an odd or even 4-colouring. It follows that if two 4-colourings of M have a common colour-class they have the same parity.

Two distinct colour-classes of Q constitute by their union a colour-dyad. It is easy to show that the colour-dyad AB is separated from the complementary colour-dyad CD by a set of disjoint

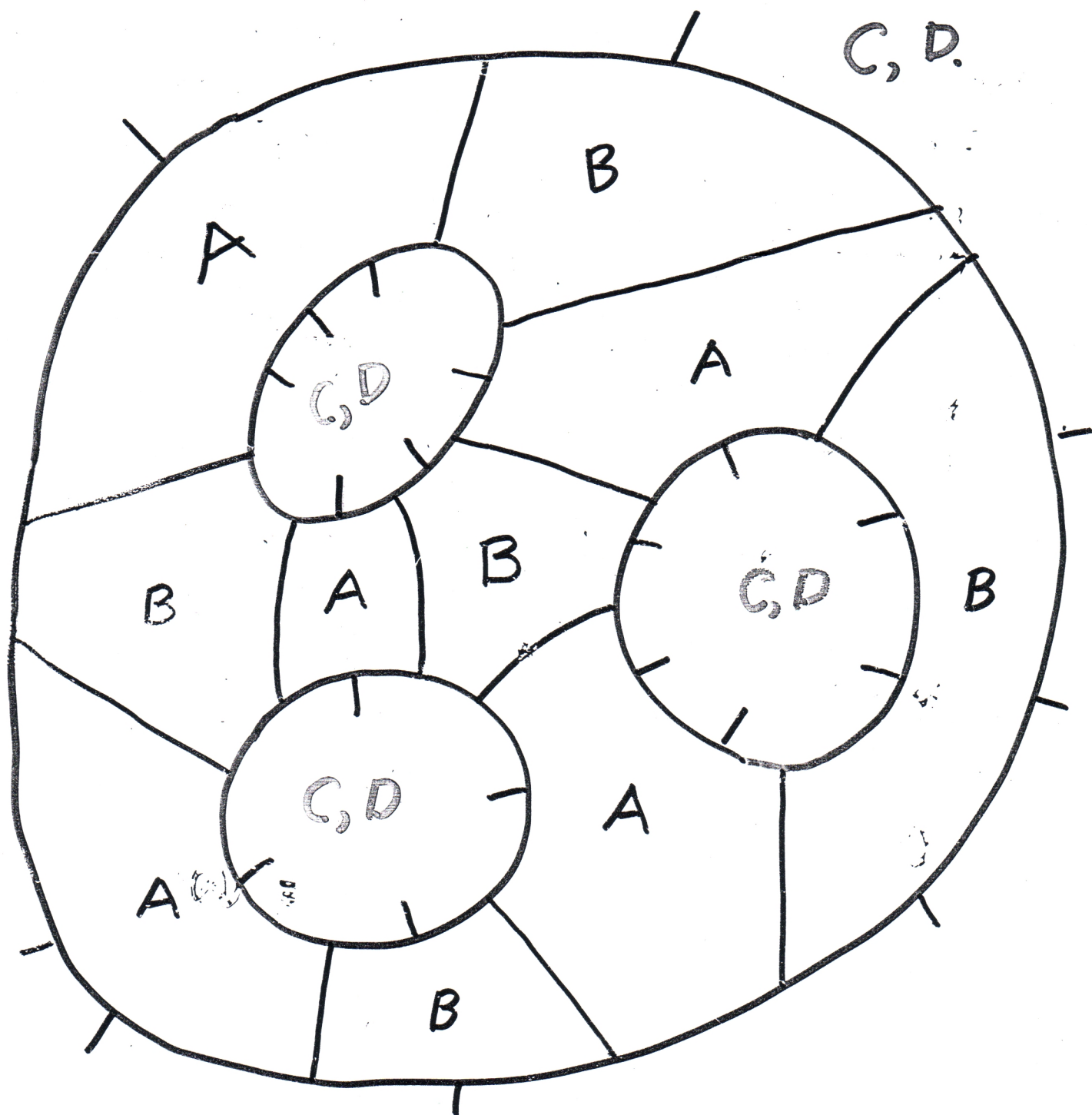


Fig 1

Circuits which together traverse all the vertices of M , and even that each of these circuits is of even length.

The number of vertices of M is an even number $2n$, and the number of edges is $3n$. The number of faces (by Euler's formula) is then $n+2$. It follows from the preceding paragraph that the total number of boundary edges, that is of edges in the bounding circuits is $2n$.

The complex constituted by a colour-dyad and its incident edges and vertices falls apart into components called Kempe chains

If it is the colour-dyad XY we denote the number of these components by $h(Q; X, Y)$

The AB Kempe chain K is bounded by $m(K)$ circuits. (See Figure 1). Each of these has a residual domain in which faces of colour C and D alternate around the circuit. A residual domain may contain other AB Kempe chains

It is well known that a 4-colouring Q can often be transformed into a different one Q' by interchanging the colours A and B in one or more AB Kempe chains. However complementary sets of AB Kempe chains give the same Q' , modulo a permutation of the four colour-classes. Counting the original Q the number of distinct 4-colourings that can be obtained from it by AB Kempe interchanges is

$$2^{h(Q; A, B)} - 1$$

2. Proof of the first Parity Theorem

Consider the colour-dyad AB .

The number of edges in its boundary is the number of vertices of the map. It is thus even ($= 2n$ say).

Let $g(A)$ be the number of boundary edges incident with A , and $h(A, B)$ the number of edges separating A from B .

$$f(A) = g(A) + h(A, B)$$

$$f(B) = g(B) + h(A, B)$$

$$\begin{aligned} \therefore f(A) + f(B) &\equiv g(A) + g(B) \pmod{2} \\ &= 2n \equiv 0 \pmod{2} \end{aligned}$$

$\therefore f(A) \equiv f(B) \pmod{2}$, and by extension

$$f(A) \equiv f(B) \equiv f(C) \equiv f(D) \pmod{2}$$

3 Colour complexes

From now on I dualize these results to vertex-colourings of triangulations.

For a given triangulation T it is possible to derive all the 4-colourings and to show how they relate one to another in a diagram. This diagram is the "colour-complex" T' of T . T' is a simplicial 3-complex. Its vertices represent the colour-classes. Its tetrahedra represent the 4-colourings of T , the four colour-classes of each colouring corresponding to the four vertices of its tetrahedron.

Here are some properties of a general colour-complex.

- (i) Two tetrahedra have at most an edge in common.
- (ii) The number of tetrahedra incident with a given edge is a power of 2.

To prove (ii) consider the tetrahedron representing $Q = \{A, B, C, D\}$ and take the edge AB . The tetrahedra on this edge represent the colourings derived from Q by Kempe interchanges of Type CD . T' is thus

$$2^{h(Q; C, D) - 1}$$

It seems convenient to denote $h(Q; C, D) - 1$ by $\text{ind}(AB)$ and call it the index of the edge AB . Let us go a step further and define the "weight" of Q and its tetrahedron as the sum of the indices of the sides

- (iii) In any component of T' the weights of the tetrahedra have all the same parity. (The Second Parity Theorem).

4 Proof of the Second Parity Theorem

Consider an AB Kempe chain (as in Fig. 1).

$m(K)$ = Number of bounding circuits

$k(X)$ = No. of faces of colour-class X .

$\tau(K)$ = No. of edges separating A, B .

We see a plane map N with

$m(K) + k(A) + k(B)$ faces,

$f(A) + f(B) - \tau(K)$ edges in all,

$f(A) + f(B) - 2\tau(K)$ boundary edges of the dyad,

$f(A) + f(B) - 2\tau(K)$ vertices.

Apply Euler's formula to it

$$\begin{aligned} m(K) + k(A) + k(B) + f(A) + f(B) - 2\tau(K) \\ = f(A) + f(B) - \tau(K) + 2, \end{aligned}$$

$$(1) \quad m(K) + k(A) + k(B) = \tau(K) + 2$$

Sum this over all AB Kempe chains
(Write $m(Q; A, B)$ for the number of bounding circuits of the dyad.
Write $\lambda(Q; A, B)$ for the number of vertices separating A, B in it.) Then

$$(2) \quad m(Q; A, B) + |A| + |B| = \lambda(Q; A, B) + 2h(Q; A, B)$$

$$(3) \quad m(Q; C, D) + |C| + |D| = \lambda(Q; C, D) + 2h(Q; A, B)$$

Proof Continued

Note however that

$$m(Q; A, B) = m(Q; C, D).$$

And the number of edges of the mah

$$= 3n = f(A) + f(B) - \lambda(Q; A, B) + \lambda(Q; C, D)$$

So by subtracting (3) from (2) we have

$$\begin{aligned} |A| + |B| - |C| - |D| &= f(A) + f(B) - 3n \\ &\quad + 2(h(Q; A, B) - h(Q; C, D)) \end{aligned}$$

There are two similar equations

$$\begin{aligned} |A| + |C| - |B| - |D| &= f(A) + f(C) - 3n \\ &\quad + 2(h(Q; A, C) - h(Q; B, D)) \end{aligned}$$

$$\begin{aligned} |A| + |D| - |B| - |C| &= f(A) + f(D) - 3n \\ &\quad + 2(h(Q; A, D) - h(Q; B, C)) \end{aligned}$$

Add.

$$\begin{aligned} 3|A| - |B| - |C| - |D| &- (3f(A) + f(B) + f(C) + f(D)) \\ &+ 9n \end{aligned}$$

$$\begin{aligned} &= 2(h(Q; A, B) + h(Q; A, C) + h(Q; A, D) \\ &\quad - h(Q; C, D) - h(Q; B, D) - h(Q; B, C)) \\ &= 2J_A(Q) \text{ say.} \end{aligned}$$

Simplify.

$$4|A| - n - 2 - 2f(A) - 2 \cdot 3n + 9n = 2J_A(Q)$$

$$2J_A(Q) = 4|A| - 2f(A) + 2n - 2$$

$$J_A(Q) = 2|A| - f(A) + n - 1$$

Proof further continued

But the parity of $J_A(Q)$ is that of the weight, or index-sum of Q .

Our equation shows that this is the parity of $f(A)$ (i.e. of Q) if n is odd, but of the opposite parity if n is even.

So all tetrahedra in any one component of the colour-complex have weights of the same parity.

5. Examples.

8

I have worked out the colour-complexes of some of the simpler triangulations and have been accustomed to get a colour-complex of two components, one even and one odd. So far I have never had more than one component of the same parity. But the data are too few to justify a Conjecture. Let me just say that if anyone knows of any case of two components of the same parity I would be glad to hear of it.

The nearest I have got to such disconnection is with the maps of my Figures I and II

I, II.

Note that though the colour-complex has a connected component with two blocks, you cannot get from one block to another by Kempe interchanges (which always retain two vertices).

An extreme case of this phenomenon is given by the regular dodecahedron. Its colour-complex is related to the 4-regular graph got by deleting a 1-factor from K_{55} .

Expand each vertex into a tetrahedron while contracting each edge into a vertex. That is the colour-complex.

No two tetrahedra have more than one vertex in common. So no 4-colouring can be converted into another by a Kempe interchange. But the colour-complex is connected.

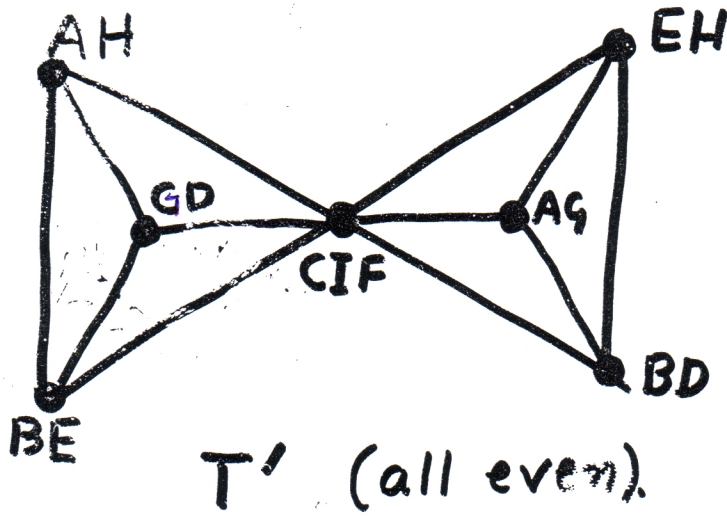
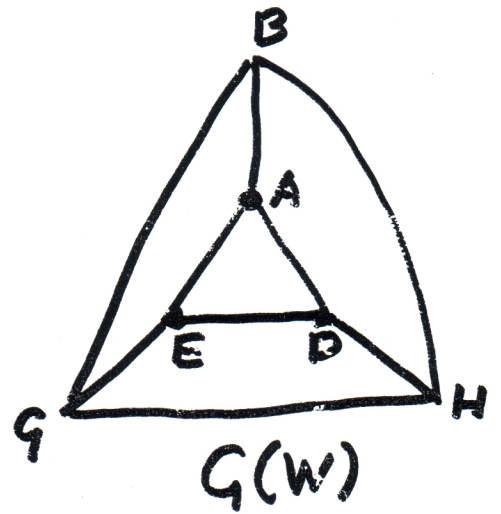
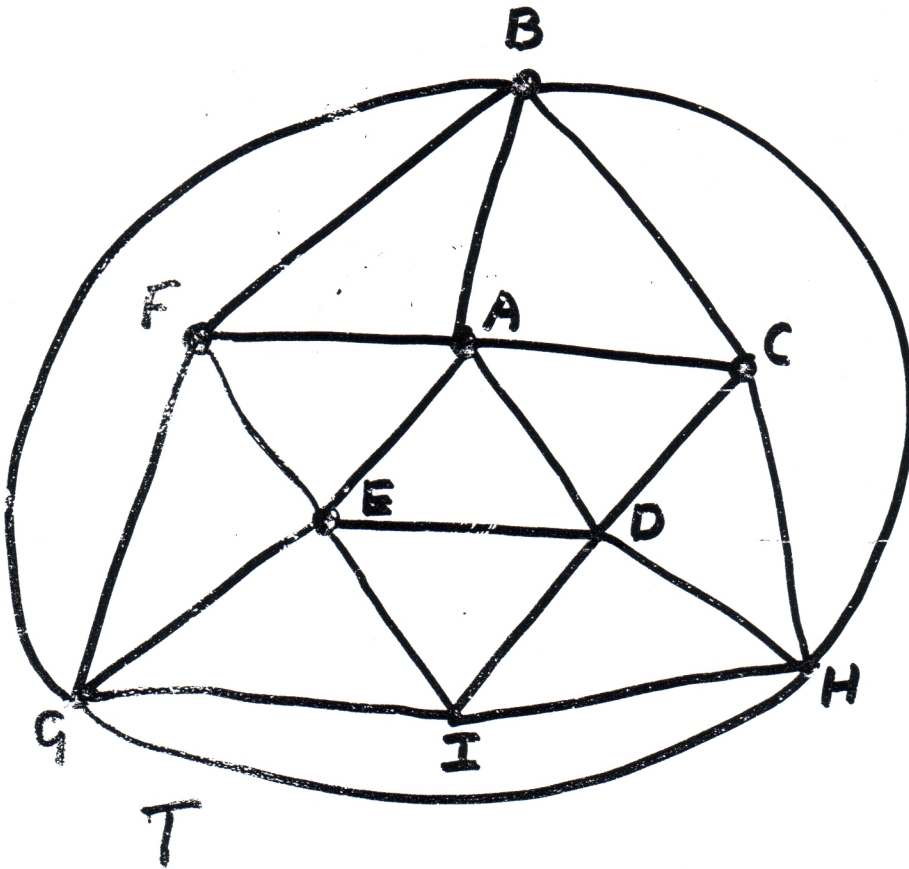


Fig I

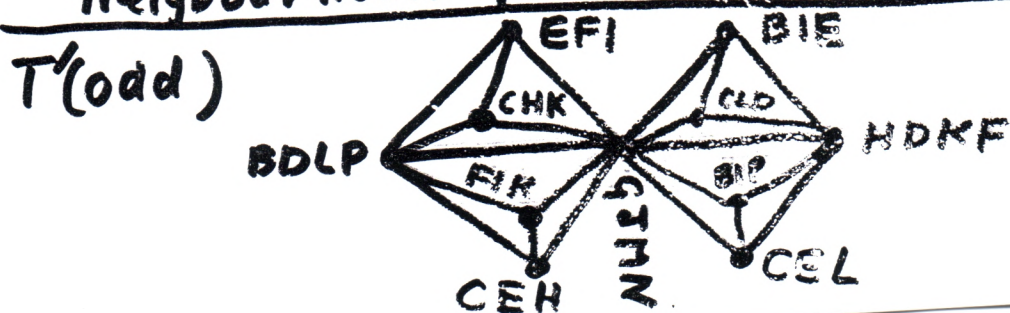
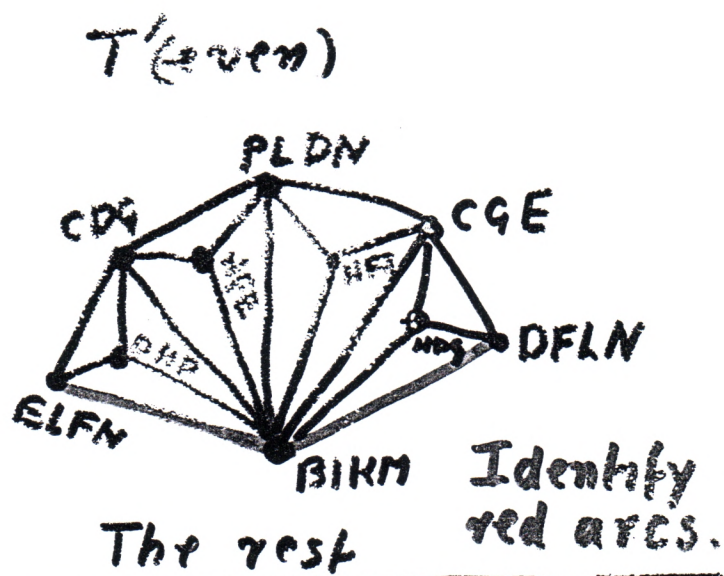
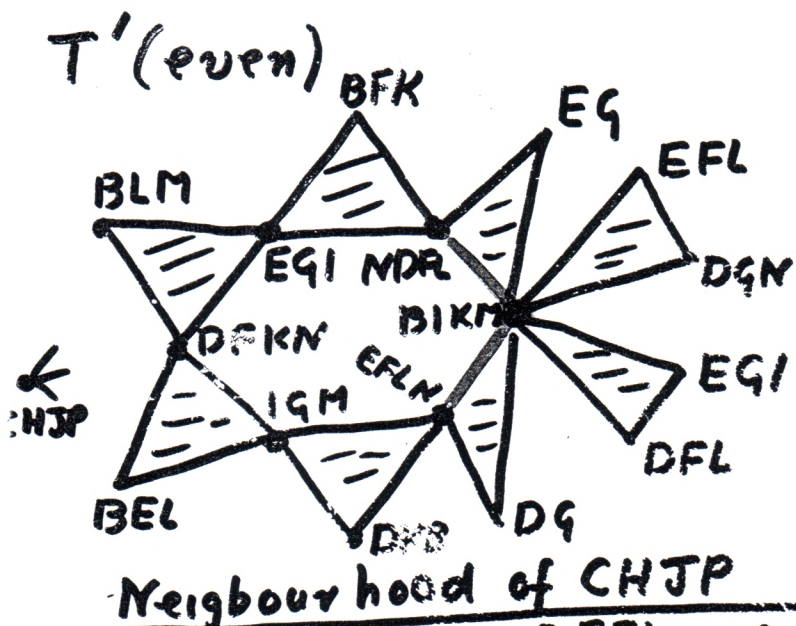
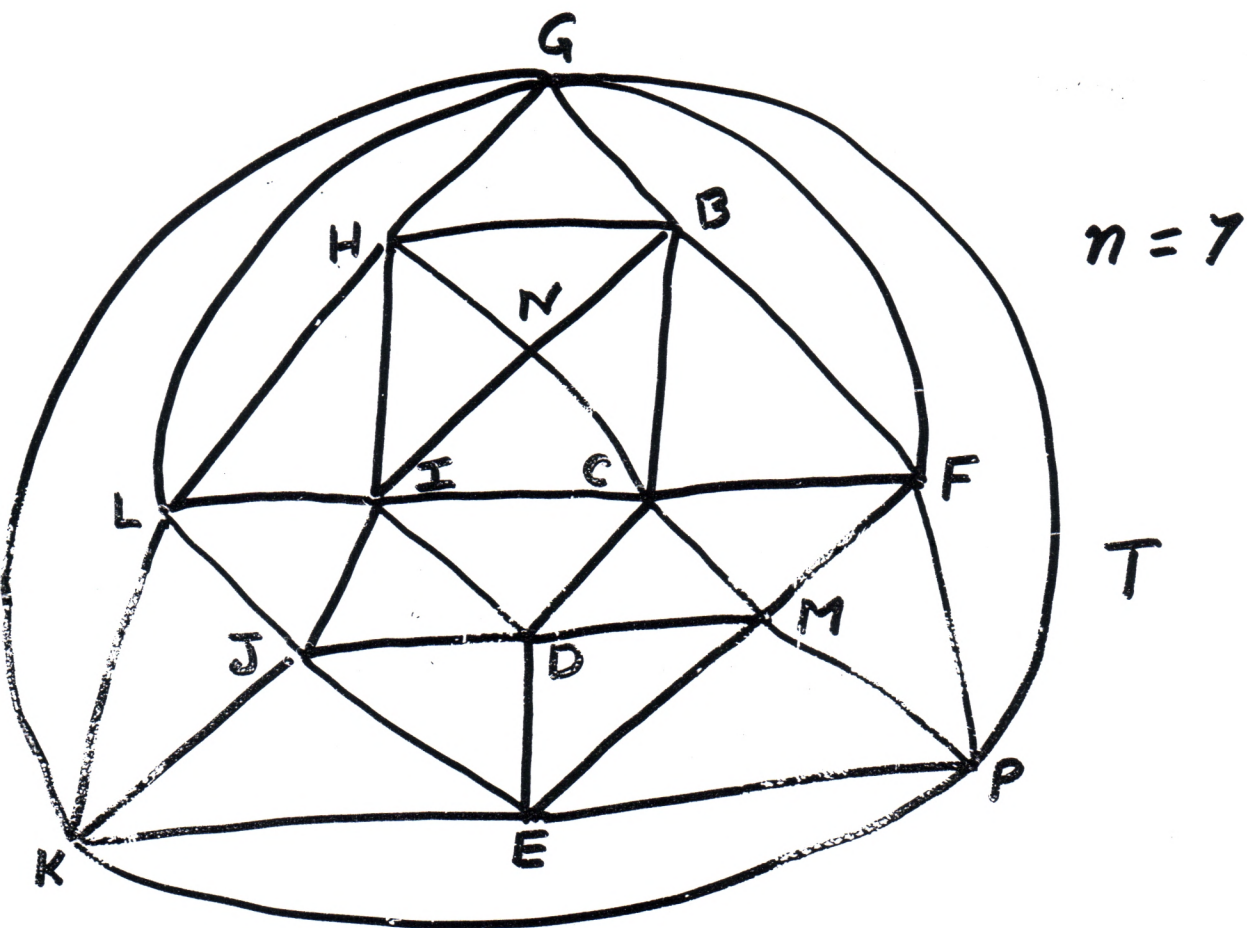


Fig. II

6 Restrictions on Parity.

In many cases I found that T' had only one component, sometimes odd and sometimes even. But in all the examples I have a simple reason can be given for the restriction.

Given T write W for its set of odd (odd-valent) vertices, and $G(W)$ for the subgraph they induce.

In an odd (even) 4-colouring of T each colour-class must contain an odd (even) number of members of W , by the first parity theorem.

So for an odd colouring W has at least 4 odd vertices. Triangulations with no odd vertices, or with only 2, can have no odd 4-colouring.

In the T of Figure I G_W is a triangular prism. If we split W into 4 odd independent sets one must have 3 members. So it must have two members in one of the two triangles of $G(W)$. But then that set is not independent. No odd 4-colouring is possible.

In Figure III suppose W has a partition into four odd independent sets. The one containing A has its other members, if any, all in the triangle CGF . Hence this set consists of A alone. Similar reasoning applies to the sets containing I , G and F . So no such partition is possible: T has no odd 4-colouring.

In Figure 4 there is only one partition of W into 4 even independent sets. It is $\{JB, IF, HD, \emptyset\}$. So in any even 4-colouring J and B receive one colour, I and F a second and H and D a third. Then A , being joined to B , D and F must receive the fourth colour. But now G , being joined to B , I , F and A , cannot be coloured. So T has no even 4-colouring.

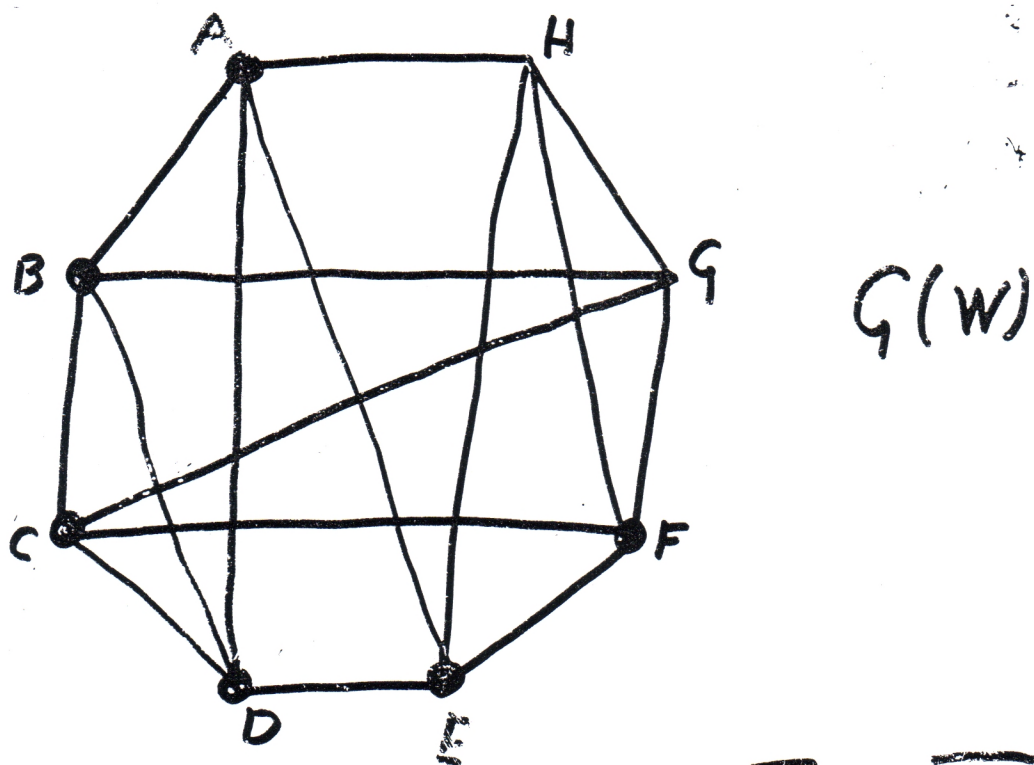
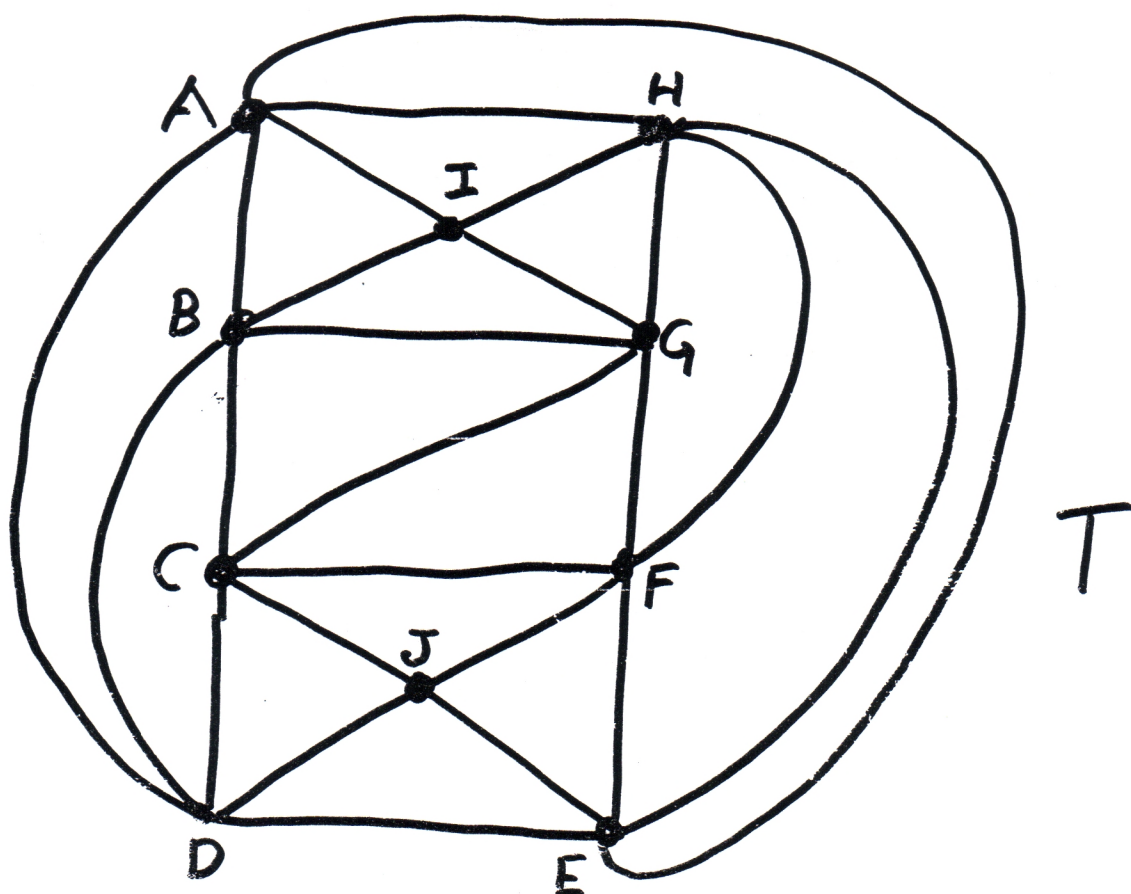


Fig III

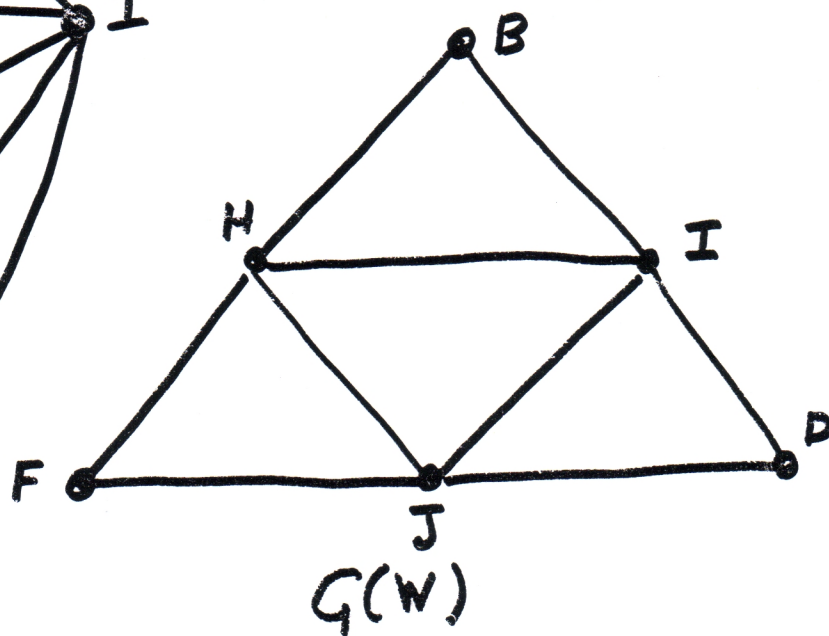
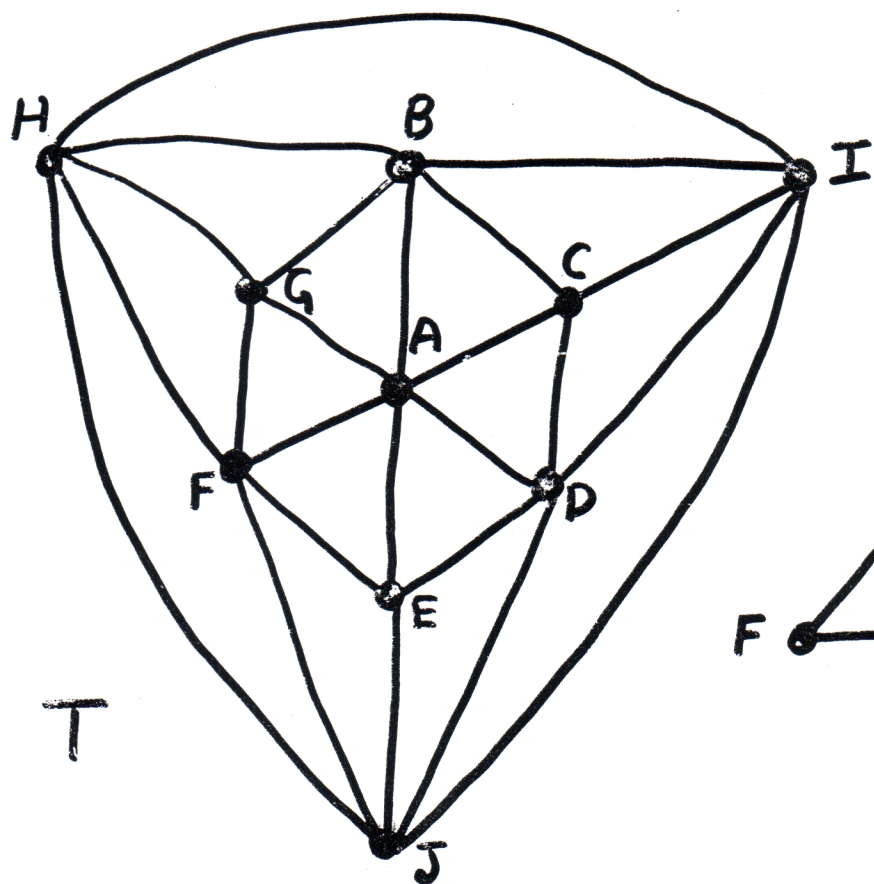
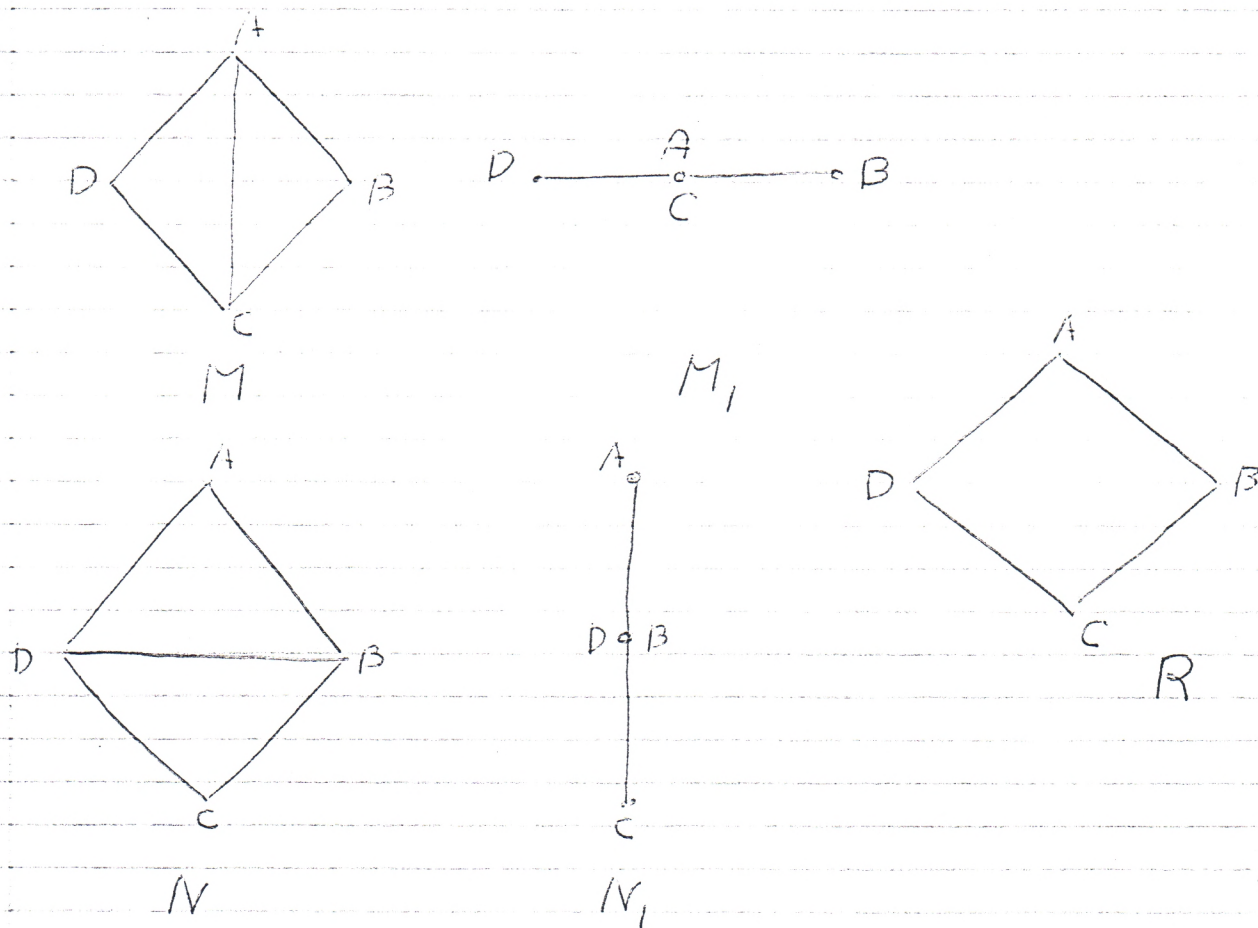


Fig IV

The Recursion formula

Two edge-adjacent faces of a triangulation M can be modified to produce new triangulations M_1, N, N_1 and a near-triangulation R .



We are familiar with a recursion formula for chromatic polynomials.

$$P(M, \lambda) + P(M_1, \lambda) = P(R, \lambda) \\ = P(N, \lambda) + P(N_1, \lambda)$$

We can use it here with $\lambda = 4$

Moreover we can generalize it from a relation between mere numbers of 4-colourings to a construction with colour-classes. Starting with those of M and M_1 ,

we can determine those of N and N_1 .
This is an advance; with chromatic
polynomials we need to know three
to calculate the fourth.

We denote the colour-complex of
a map X by X' .

Given M' we can partition it
into two parts $M'(B \neq D)$ and $M'(B = D)$.

Similarly partition M_1' into
 $M_1'(B \neq D)$ and $M_1'(B = D)$. (Here the
equality sign means "is of the same
colour as".)

Now put together $M'(B \neq D)$
and $M_1'(B \neq D)$. The first complex
is made up of those colourings of R
in which $A \neq C$ and $B \neq D$. The
second is made up of those in which
 $A = C$ and $B \neq D$. The combination
is the set of all colourings of R in
which $B \neq D$, that is it is N' .

Similarly, putting together $M'(B = D)$
and $M_1'(B = D)$ we get N_1' .