Contemporary Mathematics
Volume 111, 1990

A Note on Hamilton Cycles in Block-Intersection Graphs

BRIAN ALSPACH, KATHERINE HEINRICH, AND BOJAN MOHAR

ABSTRACT. We show that the block-intersection graph of a pairwise balanced design has a Hamilton cycle provided that the cardinality of a largest block in the design is no more than twice the cardinality of a smallest block.

A set $V = \{1, 2, ..., n\}$ together with a collection $\mathbf{B} = \{B_1, ..., B_m\}$ of subsets of V is called an *incidence structure*. The elements of $\mathbf{B}$ are called *blocks* and the *size* of a block is the number of elements in it. Let $\mathbf{B}$ be an incidence structure. The *block intersection graph* of $\mathbf{B}$, denoted $\mathbf{G}(\mathbf{B})$, has the blocks of $\mathbf{B}$ for its vertices and two vertices are adjacent if and only if the blocks have non-empty intersection.

EXAMPLES: (i) Let $\mathbf{G}$ be a graph. Letting the vertices of $\mathbf{G}$ be the elements of V and the edges of $\mathbf{G}$ be the blocks, $\mathbf{G}$ determines an incidence structure $\mathbf{B}$. The block intersection graph of $\mathbf{B}$ is then the ordinary line graph of $\mathbf{G}$. C. Thomassen has conjectured that every 4-connected line graph has a Hamilton cycle.

(ii) Let $\mathbf{B}$ be a Steiner triple system. At a regional meeting of the American Mathematical Society in March 1987, R. L. Graham asked if the block-intersection graph of a Steiner triple system has a Hamilton cycle. This question is answered in the affirmative in [1] and is a corollary of the main result in this paper.

Let $\mathbf{B}$ be an incidence structure on a set V. Define $\mathbf{M}(\mathbf{B})$ to be the multigraph with vertex-set V such that for each pair of distinct vertices a and b, there is an edge between a and b of color B_i if $\{a, b\}$ is a subset of B_i for $B_i \in \mathbf{B}$.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05C38, 05C45; Secondary 05B05, 05B07.

This research was partially supported by the Natural Sciences and Engineering Research Council of Canada under Grants A-4792 and A-7829.

This paper is in final form and no version of it will be submitted for publication elsewhere.

A *trail* in a multigraph is a walk in which no edge appears twice. A trail in $\mathbf{M}(\mathbf{B})$ is said to be *block-dominating* if the vertices of the trail intersect every block of $\mathbf{B}$.

LEMMA. *Let* $\mathbf{B}$ *be an incidence structure. The block intersection graph of* $\mathbf{B}$ *is hamiltonian if and only if* $\mathbf{M}(\mathbf{B})$ *has a block-dominating closed trail* W *with no two edges of* W *having the same color.*

PROOF. Let $H = B_1, B_2, B_3, \ldots, B_m, B_1$ be a Hamilton cycle in $\mathbf{G}(\mathbf{B})$. Let a_i be an element of $B_i \cap B_{i+1}$, $i = 1, 2, \ldots, m$, where B_{m+1} is taken to be B_1. Consider the sequence $a_1, a_2, \ldots, a_m, a_1$ of elements of V. Replace each maximal length constant subsequence of consecutive elements by the single element making up the subsequence. For example, if $a_1 = a_2 = \ldots = a_j$, then replace this subsequence by a_1. The resulting closed trail is certainly block-dominating and each of its edges has a different color. (A single point is viewed as a closed trail.)

Let $W = a_1, e_1, a_2, e_2, \ldots, a_k, e_k, a_1$ be a closed block-dominating trail in $\mathbf{M}(\mathbf{B})$ such that each of its edges has a different color. Let B_i be a block containing the edge e_i. We construct a Hamilton cycle in $G(B)$ as follows. Let $C_1, C_2, \ldots, C_t$ be all the blocks of $\mathbf{B}$, other than $B_1, \ldots, B_k$, that contain the element a_1. Then $C_1, C_2, \ldots, C_t, B_1$ is a path in $\mathbf{G}(\mathbf{B})$. Now let $C_{t+1}, \ldots, C_{t+r}$ be all the blocks of $\mathbf{B}$, other than $B_2, \ldots, B_k$ and those already appearing in the path, that contain the element a_2. Then $C_1, C_2, \ldots, C_t, B_1, C_{t+1}, \ldots, C_{t+r}, B_2$ is a path in $\mathbf{G}(\mathbf{B})$ because B_1 also contains a_2. We continue in this way and produce a Hamilton cycle because W is block-dominating. This completes the proof.

A *pairwise balanced design* is an incidence structure $\mathbf{B}$ in which each unordered pair of distinct elements of V occurs in precisely λ blocks of $\mathbf{B}$.

THEOREM. *Let* B *be a pairwise balanced design with* $\lambda = 1$, M *the maximum size of a block of* $\mathbf{B}$ *and* m *the minimum size of a block of* $\mathbf{B}$. *If* $M \leq 2m$, *then the block intersection graph* $\mathbf{G}(\mathbf{B})$ *of* $\mathbf{B}$ *has a Hamilton cycle.*

PROOF. We construct a block-dominating closed trail in $\mathbf{M}(\mathbf{B})$. In fact, it will be a cycle in $\mathbf{M}(\mathbf{B})$. If $\mathbf{B}$ is the trivial design (that is, has only one block), then $\mathbf{G}(\mathbf{B})$ is K_1 which has a trivial Hamilton cycle. Thus, we assume that $\mathbf{B}$ is a non-trivial pairwise balanced design.

Since $\mathbf{B}$ is non-trivial, there are three points a, b, c in V not all in the same block. Then $C = a, b, c, a$ is a 3-cycle in $\mathbf{M}(\mathbf{B})$ all of whose edges have a different color. If C is block-dominating, we are finished. Otherwise, we extend C to be a longer cycle with edges of different colors. Eventually, we must reach a block-dominating cycle (for example, if the cycle contains all the vertices of $\mathbf{M}(\mathbf{B})$).

Let C be a cycle of $\mathbf{M}(\mathbf{B})$ all of whose edges have different colors and let B be a block of $\mathbf{B}$ that does not intersect C. For simplicity, assume the consecutive vertices of C are 1, 2, ..., k and $B = \{x_1, \ldots, x_b\}$.

The edge $e_i = i, i+1$ of C is said to be *destructive* if the block B_i containing it intersects B in some element x_j. If there is an edge $i+1, x_r$ (or i, x_r) whose color is not used on C, then C may be extended to a longer cycle by replacing the edge $i, i+1$ on C with the 3-path $i, x_j, x_r, i+1$.

If there are two consecutive destructive edges $e_{i-1} = i-1, i$ and $e_i = i, i+1$, then C may be extended as follows. Let x_j be a vertex of B so that the color of $i-1, x_j$ is the same as that of $i-1, i$ and let x_r be a vertex of B so that the color of $i+1, x_r$ is the same as that of $i, i+1$. Notice that x_j and x_r must be distinct vertices since C has no two edges of the same color. Then replace the two consecutive edges $i-1, i, i+1$ of C with the 3-path $i-1, x_j, x_r, i+1$.

Assume that C cannot be extended. Thus, we may assume that the destructive edges of C form a matching and that if $i, i+1$ is a destructive edge of C, then the colors of the edges i, x_j and $i+1, x_j$ are already used on C for every x_j belonging to B. Let p be the number of destructive edges on C. If the sizes of the corresponding blocks are $b_1, b_2, \ldots, b_p$, they destroy at most

$$(b_1 - 1) + \ldots + (b_p - 1) = \sum_{i=1}^{p} b_i - p$$

edges between $V(C)$ and B. Hence, there are at least

$$bk + p - \sum_{i=1}^{p} b_i$$

edges between $V(C)$ and B whose colors do not appear on C.

The destructive edges of C have $2p$ endvertices so that all the edges between $V(C)$ and B whose colors do not appear on C appear at the remaining $k-2p$ vertices. The number of edges at these vertices is $b(k-2p)$. Hence,

$$(k - 2p)\, b \geq u \geq b\, k + p - \sum_{i=1}^{p} b_i \geq b\, k + p - pM,$$

where u denotes the number of edges between $V(C)$ and B whose colors do not already appear on C. The above implies that

$$pM - p - 2p\, b \geq 0,$$

which in turn implies that

$$M - 1 - 2b \geq 0.$$

The latter is a contradiction and completes the proof.

A *Steiner* 2-*design* is an incidence structure $\mathbf{B}$ in which each block has the same cardinality and $\lambda = 1$.

COROLLARY. (Horák and Rosa [1]). *The block intersection graph of a Steiner 2-design has a Hamilton cycle.*

REFERENCES

1. P. Horák and A. Rosa, *Decomposing Steiner triple systems into small configurations*, Ars Combinatoria, to appear.

DEPARTMENT OF MATHEMATICS AND STATISTICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, V5A 1S6 CANADA

Current address: B. Mohar, Department of Mathematics, University of Ljubljana, Jadranska 19, 61111 Ljubljana, Yugoslavia