

DISJOINT ESSENTIAL CIRCUITS IN TOROIDAL MAPS

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IMA Preprint Series # 937

April 1992

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Abstract

Necessary and sufficient conditions are given for a toroidal map to contain two disjoint essential cycles. The result is applied in the study of embeddings of planar graphs into general surfaces.

1 Introduction

Dirac [D] (cf. also [L]) proved that a 3-connected graph G contains no two disjoint cycles if and only if one of the following cases occurs: G is a wheel $K_1 * C_n$ ($n \geq 3$) with 3 or more spokes, $G = K_5$, or G has at least 6 vertices and contains vertices $x, y, z \in V(G)$ which cover all the edges of G , i.e. $G = K_{3,k}$ ($k \geq 3$) or G is a graph obtained from $K_{3,k}$ by adding 1, 2, or 3 edges between the vertices in the color class of $K_{3,k}$ containing 3 vertices. Dirac's result can be generalized to arbitrary graphs. Since the removal of vertices of degree 0 or 1, and the suppression of vertices of degree 2 in a graph do not change the number of cycles we may without loss of generality treat only the case when the minimal vertex degree of the graph is at least 3. A graph G with the minimal degree 3 or more does not contain two disjoint cycles if and only if one of the following cases occurs:

- a) G has a vertex $x \in V(G)$ such that $G - x$ is a forest,

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- b) G has a vertex $x \in V(G)$ such that $G - x$ is a simple cycle and G has no loops at x , i.e., G is a wheel with the spokes allowed to be multiple edges,
- c) $G = K_5$, or
- d) there are vertices $x, y, z \in V(G)$ such that $G - \{x, y, z\}$ is edgeless, there are no loops at x, y, z and no parallel edges between $\{x, y, z\}$ and $V(G) \setminus \{x, y, z\}$. (But parallel edges between x, y, z are allowed.)

In the study of the structure of embeddings of planar graphs on the torus [MR] we bumped into the following problem: If a planar graph embedded on the torus contains no two disjoint essential circuits, when is it possible that the embedding is a closed-cell embedding? By an *essential circuit* we mean a cycle of the graph which is not contractible on the surface.

Let G be a graph on a closed surface Σ . We say that G is embedded with *representativity* r , and denote this by $\rho(G) \geq r$, if every essential closed curve on the surface intersects G in at least r points. Cf. [RV] for more details on this invariant. Schrijver [S1] proved that a graph G embedded in the torus with $\rho(G) \geq r$ contains $\lfloor 3r/4 \rfloor$ pairwise disjoint essential cycles. In particular, if $\rho(G) \geq 3$ then there are 2 disjoint essential cycles. Although this result is best possible, it is far from a necessary and sufficient condition for a graph on the torus to contain 2 disjoint essential cycles. We solve this problem by characterizing graphs embedded in the torus (=toroidal maps) which do not admit 2 disjoint essential cycles (Theorems 3.1 and 3.2). A short passage then leads to the answer on our original question about the planar graphs on the torus. It should be pointed out that although the outcome is similar to the Dirac's result, the combinatorial obstructions to disjoint cycles are not of much help in the torus case.

Let us mention that the same type of questions can be posed for graphs on other surfaces. Moreover, we get a variety of problems which are of importance in the study of graph embeddings. For example, one may ask questions about the existence of disjoint essential cycles, pairwise homotopic or not (and the homotopy class fixed, or not), disjoint pairwise homologic essential cycles (homology class either fixed, or free), disjoint essential non-bounding cycles, etc. It should be pointed out that the problem of the existence of pairwise disjoint cycles of given homotopies has a "good characterization" [S2].

2 Basic definitions and some auxiliary lemmas

Graphs in this paper may have loops and multiple edges. Let G be a graph embedded in a closed surface Σ . Suppose that in Σ there is a closed curve $\gamma : S^1 \rightarrow \Sigma$ which bounds an open disk D . Let $\overline{D} := D \cup \gamma(S^1)$ be the closure of D . We say that \overline{D} is a k -patch if $cr(\gamma, G) := |\{z \in S^1 \mid \gamma(z) \in G\}| = k$. Having a k -patch with $k = 0$ or 1 , the deletion of $G \cap D$ from the graph is called a k -reduction. Having a 2-patch with a path in $G \cap \overline{D}$ connecting the vertices of $G \cap \partial \overline{D}$ across D , a 2-reduction is the operation replacing $G \cap D$ with an edge connecting the vertices of $G \cap \partial \overline{D}$ across D . Note that using 2-reductions we can in particular eliminate all vertices of degree 2 in G (except isolated vertices with an essential loop). Having a 3-patch with $\{x, y, z\} = G \cap \partial \overline{D}$ such that in $\overline{D} \cap G$ there are paths between each pair (x, y) , (x, z) , and (y, z) (all of them across D) the replacement of $G \cap D$ by a new vertex w joined to x, y, z (as shown on Figure 2.1) is called a 3-reduction. A k -reduction ($k \leq 3$) is *non-trivial* if the graph obtained after the reduction is not isomorphic to G . In particular, the well-known ΔY -transformation is a special case of a 3-reduction if the triangle of the transformation bounds a face.

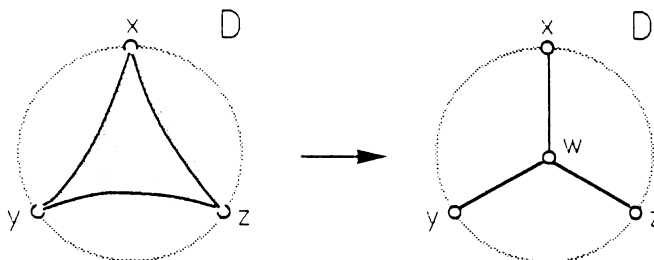


FIGURE 2.1 A 3-reduction

The following simple lemma (whose proof we omit) shows that 3-reductions preserve the maximum number of pairwise disjoint essential cycles in a graph on a surface.

Lemma 2.1 *Let G' be obtained from G by a sequence of 0-, 1-, 2-, and 3-reductions and their inverses, and let \mathcal{C} be a family of disjoint circuits of G . Then G' contains a family \mathcal{C}' of disjoint circuits which are pairwise homotopic with respective circuits in \mathcal{C} . Moreover, the representativity of G' equals the representativity of G .*

Let K be a subgraph of G . The vertices of K of degree different from 2 (in K) are called the *main vertices* of K , and the paths in K (possibly of length 1) between main vertices, where all interior vertices are of degree 2 in K , are called *branches* of K . If B_0 is a connected component of $G - V(K)$, then the subgraph B of G consisting of B_0 and all edges between B_0 and vertices in K (together with appropriate vertices of K) is called a *relative K -component*, or a *bridge* of K . Another type of relative components (bridges) of K are the edges (together with their endpoints) which are not in $E(K)$ but connect two vertices of K . If B is a relative K -component, each vertex of $V(B) \cap V(K)$ is called a *vertex of attachment* of B to K , and each edge of B adjacent to a vertex of attachment is a *foot* of B .

If C is a cycle of G and B_1, B_2 are relative C -components then B_1 and B_2 *overlap* (on C) if either they have three or more vertices of attachment in common, or there are four distinct vertices of attachment x_1, y_1 of B_1 and x_2, y_2 of B_2 whose order on C is x_1, x_2, y_1, y_2 . If $W = x_0x_1x_2 \dots x_k$ ($x_k = x_0$) is a closed walk in G and C its underlying subgraph of G , then a relative C -component is said to *overlap* with vertices x_i, x_j ($i < j$) on W if it has vertices of attachment x_p, x_q such that either $p < i < q < j$ or $i < p < j < q$. We refer to [V] for a more extensive treatment of relative components.

Let G be embedded in Σ and W as above. Assume that W bounds an open disk D . If F is a face of G contained in D then we say (with a possible slight abuse of terminology if W has some repeated vertices) that F *contains* x_i ($0 \leq i < k$) on its boundary if F contains on its boundary either the edge $x_{i-1}x_i$, the edge $x_i x_{i+1}$ (indices modulo k), or an edge in D lying between $x_{i-1}x_i$ and $x_i x_{i+1}$ according to the local rotation at x_i .

Lemma 2.2 *Let G be a graph embedded in a surface Σ and $W = x_0x_1 \dots x_k$ a closed walk in G whose underlying graph C bounds an open disk D in Σ . For $i < j$ there is a face in D containing x_i and x_j on its boundary if and only if no relative C -component embedded in $D \cup C$ overlaps with x_i and x_j on W .*

Proof. If a face F in D contains x_i and x_j then, clearly, no relative component in d overlaps with x_i, x_j . Conversely, if none of the faces F_1, F_2, \dots, F_s in D containing x_i on the boundary also contains x_j then in the union of their boundaries there is a path from a point x_p to x_q on W where without loss of generality $i < p < j < q$. This path is clearly contained in a relative component which is thus overlapping with x_i, x_j . \square

3 Disjoint essential circuits

In this section we will state our main results whose proofs are deferred until Sections 5 and 6.

Theorem 3.1 *Let G be a toroidal map with representativity $\rho(G) \leq 1$. Then G contains no 2 disjoint essential cycles if and only if the embedding of G has the structure as shown in Figures 3.1–3.2 (case $\rho(G) = 0$) or in Figures 3.3–3.5 (case $\rho(G) = 1$).*

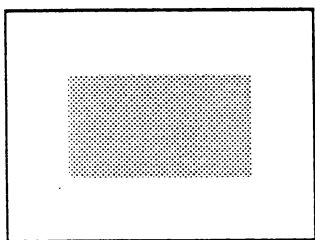


FIGURE 3.1

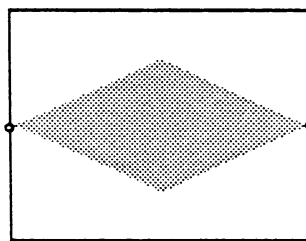


FIGURE 3.2

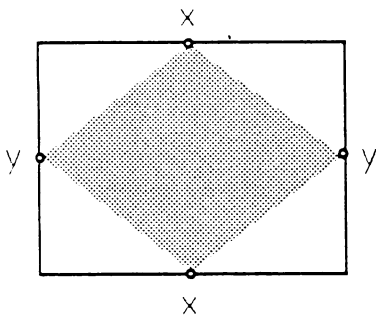


FIGURE 3.3

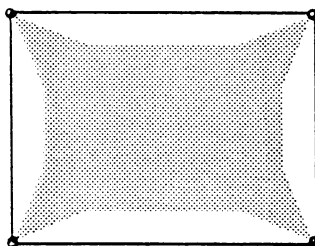


FIGURE 3.4

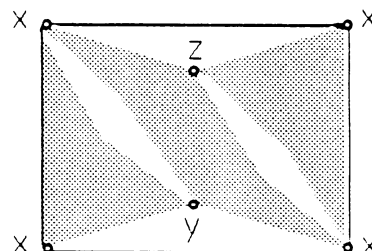


FIGURE 3.5

Note: In Figures 3.3–3.5 it may be assumed that x, y, z are distinct vertices. The exact meaning of “having the structure” is that there is a homeomorphism of the torus on the standard “flat” torus as represented in all applicable figures (with proper standard side identifications) so that any edges of G are embedded in the shaded parts. Note that Figure 3.3 and Figure 3.4 are dual to each other and that all the others are “self-dual” structures.

Theorem 3.2 *Let G be a toroidal map with representativity $\rho(G) \geq 2$. If G is 3-reduced then it contains no two disjoint essential cycles if and only*

if either $G = K_5$ embedded as in Figure 3.11, or there are distinct vertices $x, y, z \in V(G)$ which cover all the edges and so that between any two vertices $u \in \{x, y, z\}$ and $v \in V(G) \setminus \{x, y, z\}$ there are no parallel edges.

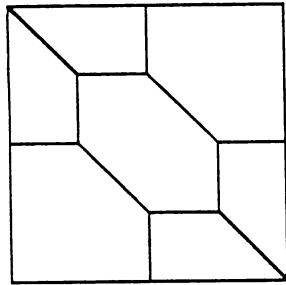


FIGURE 3.6 $K_{3,3}$

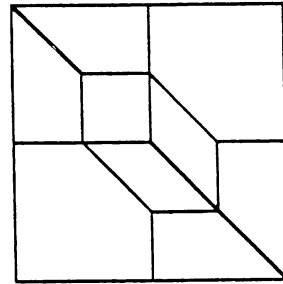


FIGURE 3.7 $K_{3,4}$

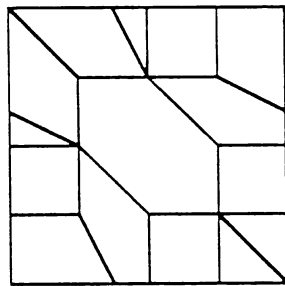


FIGURE 3.8 $K_{3,5}$

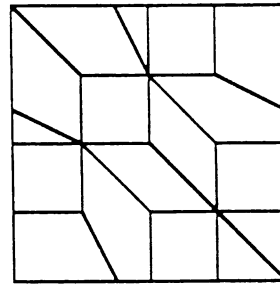


FIGURE 3.9 $K_{3,6}$

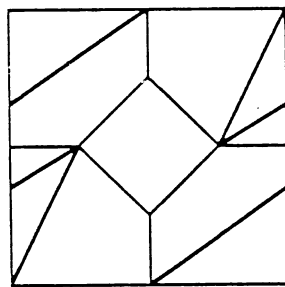


FIGURE 3.10

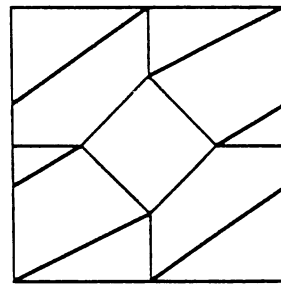


FIGURE 3.11 K_5

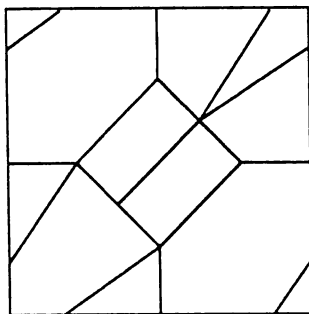


FIGURE 3.12

Note. The actual cases coming out of Theorem 3.2 are depicted in Figures 3.6–3.12. We should add all their submaps (having $\rho = 2$). Note that an edge deletion in Figures 3.6–3.9 gives rise (after a 2-reduction) to an edge between x, y, z .

It is worth mentioning the similarity of the obtained characterization with the Dirac’s graphs containing no two disjoint cycles. The case of Figure 3.1 corresponds to forests (graphs without cycles, vs. maps without essential cycles), Figures 3.2 and 3.4 imitate graphs with a vertex whose removal yields a forest, Figure 3.3 is an analogy of the wheel, in Figure 3.11 we have a K_5 , and all the other cases have the property that there is a set of 3 “vertices” whose removal leaves a “trivial graph”.

4 Planar graphs in the torus

The following corollary to the results of Section 3 is needed in [MR].

Theorem 4.1 *Let G be a planar graph embedded in the torus with representativity $\rho(G) \geq 2$. Then G contains no two disjoint essential cycles if and only if there is a sequence of 0-, 1-, 2-, and 3-reductions transforming G into the map in Figure 3.10.*

Proof. By Lemma 2.1 reductions preserve the representativity. Therefore we may use Theorem 3.2 and its proof in Section 6 which shows that the only 3-reduced map with $\rho \geq 2$ and no two disjoint essential cycles, which does not contain $K_{3,3}$ or K_5 , is the map on Figure 3.10. \square

The reductions can not increase degrees of vertices (except when introducing a new vertex of degree 3). Therefore the only non-trivial 3-reductions

leading to the map in Figure 3.10 must have been performed in discs around the vertices of degree 3. Theorem 4.1 therefore clearly describes the structure of arbitrary planar graphs in the torus with representativity 2 and without two disjoint essential cycles.

5 Proof of Theorem 3.1

Let G be a toroidal map containing no 2 disjoint essential cycles, and assume that $\rho(G) \leq 1$. The case when $\rho(G) = 0$ is easy and we leave details to the reader. So we assume now that $\rho(G) = 1$. Then there is an essential curve γ on the torus such that $cr(\gamma, G) = 1$. We may assume that γ intersects G at a vertex, say $x \in V(G)$. Cutting the torus along γ and redrawing it so that γ corresponds to the bottom and the top sides of the torus under the usual representation we get the structure as in Figure 5.1.

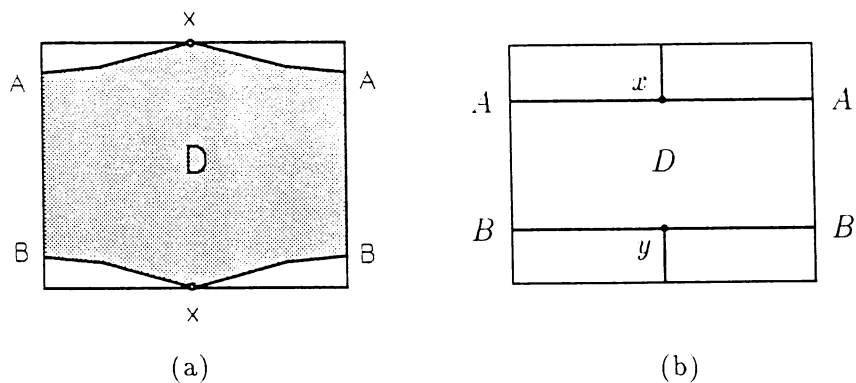


FIGURE 5.1

Let F be the face of G containing γ . Denote by A and B , respectively, the part of the facial walk of F lying in the upper and the lower part of the drawing, respectively. Since $\rho(G) > 0$, these are well-defined. It is easy to see that we may assume that G is 2-reduced. Then A and B are essential cycles of G . Consider first the case of Figure 5.1(a). If besides x they have another vertex y in common, we have the structure of Figure 3.3. Thus we may assume that $A \cap B = \{x\}$. Consider the faces in D containing x at A on the boundary. Take the modulo 2 sum of the edges of A together with the boundaries of these faces. The obtained Eulerian graph is homologous to A , so it contains an essential cycle C . If a face in D contains x at A

and at B , we have the structure of Figure 3.4 (after a suitable re-drawing). Therefore we may assume that $x \notin C$. Since $B \cap C \neq \emptyset$ there is a vertex $y \in B$ lying in a face of D together with x at A . A similar argument used from “below” shows that there is a face in D containing x at B and a vertex $z \in A$. Clearly, $y \neq z$, and we have Figure 3.5.

It remains to consider the case of Figure 5.1(b). In this case A and B have a vertex in common. Whether this vertex is equal to x or not, it turns out that G fits the structure of Figure 3.3. The details are left to the reader.

Conversely, assume that G is embedded having the structure as in one of the Figures 3.1–3.5. In the first case there are no essential cycles at all in the second and fourth, each essential cycle contains x . Having Figure 3.3 each essential cycle either contains x or y (or both) but one containing x , another y would cross. In the last case of Figure 3.5 each essential cycle either contains x , or it contains both, y and z . But the first case prevents the existence of a disjoint cycle through y, z , and vice versa. This completes the proof. \square

6 Proof of Theorem 3.2

In this section we will assume that G is a toroidal map. It is easily seen that maps satisfying the conditions of Theorem 3.2 do not have 2 disjoint essential cycles.

To prove the converse we will assume that $\rho(G) \geq 2$ and that G contains no 2 disjoint essential cycles. We will also assume that G is 3-reduced.

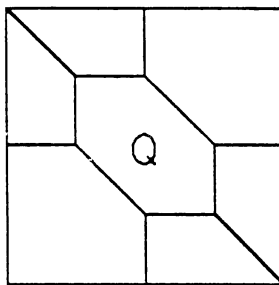


FIGURE 6.1 A $K_{3,3}$

Lemma 6.1 *If G contains a submap homeomorphic to $K_{3,3}$ as shown on Figure 6.1, then G is a $K_{3,k}$ ($3 \leq k \leq 6$) embedded as on Figures 3.6–3.9.*

Proof. Let K be the given submap of G homeomorphic to $K_{3,3}$. First we will prove that we may assume K has no *local bridges*, i.e., bridges attached to a single branch of K . We define

$$c(K) = |V(K)| + \sum_B |V(B) \setminus V(K)| \quad (1)$$

where the sum runs over all local bridges of K . Assume that among all possible choices for K we take the one with minimal $c(K)$. Suppose now that K has local bridges. Since G is 2-reduced, there must be a local bridge B which overlaps with a non-local bridge B' . Let B be attached at the branch e of K , and let p, r be its “leftmost” and the “rightmost” attachment on e . Since B' overlaps with B , it has an attachment q on e which lies between p and r . Denote by K' a submap of G obtained from K by replacing the segment from p to r on e by a path in B . Then K' is a submap homeomorphic to $K_{3,3}$. It is easy to see that $c(K') < c(K)$ since at least q does not contribute in (1) any more. This contradicts the minimality of K .

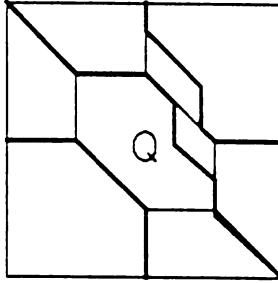


FIGURE 6.2

Let B be a bridge of K . Then the attachments of B are restricted to two adjacent branches of K (including their endvertices), or B is attached to 3 vertices of the bipartition of $K_{3,3}$. This can be seen as follows. We may assume that B is embedded in the “central” face Q . Suppose first that B is attached to an interior vertex x of a branch e of K . Then any attachment on a branch not adjacent to e gives rise to 2 disjoint essential cycles in $K \cup B \subseteq G$. If a vertex of attachment x is a main vertex then, similarly, B can not be attached to the vertex y of Q opposite x or to a vertex in an open branch at y . Since x is any vertex of attachment of B , one easily verifies that the above claim about attachments must be satisfied.

Suppose now that there is a bridge of K attached to two adjacent branches of K . Since G is 3-reduced and K contains no local bridges, there is a branch uv of K and bridges A, B of K attached as shown on Figure 6.2. Then G has 2 disjoint essential cycles (thick cycles in Figure 6.2). It follows that every bridge of K is attached to the main vertices only (to at least 3 of them) since the graph is 3-reduced. But the 3-attached bridges can be 3-reduced, each to a single vertex. If two such bridges are attached to different triples of main vertices of K one easily finds 2 disjoint essential cycles. Finally, up to symmetries there are only four possibilities for G as exhibited in Figures 3.6 – 3.9. \square

From now on we exclude the above case. We will first prove that G contains a submap K homeomorphic to K_4 shown in Figure 6.3.

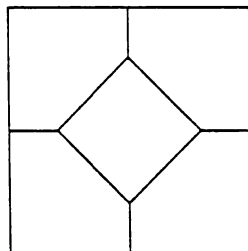


FIGURE 6.3 K_4

Lemma 6.2 *Let G be a toroidal map with representativity 2 such that there are no two disjoint essential cycles in G . If the embedding is 3-reduced then G contains a submap K homeomorphic to K_4 whose embedding is shown on Figure 6.3.*

The proof will start by a sequence of claims interlaced by introduction of notation and some small comments. In all of the claims we will assume the conditions of the Lemma and all the previous definitions and results. Moreover we will assume that G contains no submap homeomorphic to the map in Figure 6.3.

Since $\rho(G) \geq 2$ and G is 2-reduced, every face of G is bounded by a (simple) cycle of G . It follows, in particular, that there is a closed disk D in the torus which is a union of (closed) faces of G and is maximal in the sense that no other such disk properly contains D . Denote by C the boundary of

D , and let $K = G \cap D$ be the subgraph of G lying in D . Clearly, C is a cycle of G .

A (closed) face of G is an *outer face* if it is not contained in D . If F is an outer face having an edge in common with D then $F \cap D$ is not connected since otherwise $D \cup F$ would be a disk contradicting the maximality of D . Therefore C separates the boundary cycle of F into two or more paths, P_1, P_2, \dots , each of them joining two vertices on C and having no intermediate vertex on C . For each i denote by C_i a cycle obtained from P_i and a segment on C between the endpoints of P_i (there are two choices). These cycles are called *fundamental cycles* of F with respect to D .

Claim 1. Every fundamental cycle of an outer face F sharing an edge with D is essential.

Proof. If it bounds a disk D' , then this contradicts the maximality of D since $D \cup D'$ is a disk. \square

Let $e = xu, f = yv$ be distinct feet of the same bridge B of K , where $x, y \in V(C)$ are vertices of attachment. Choose a path in $G - K$ joining u and v and a segment of C joining x and y (the segment is assumed to be trivial if $x = y$), and denote by $C(e, f)$ the cycle obtained by taking the two edges e, f , the path and the segment. As in the proof of Claim 1, the maximality of D yields:

Claim 2. If $x \neq y$ then $C(e, f)$ is essential.

Claim 3. No component of $G - K$ contains an essential cycle.

Proof. Suppose that L is a component of $G - K$ containing an essential cycle S . Let F be an arbitrary outer face of G sharing an edge with D . Let C_1, C_2 be two of its fundamental cycles. Clearly, none of them can cross S since out of ∂D , C_1 and C_2 follow the boundary of a face. Therefore C_1 and C_2 are both homotopic to S . Since C_1 and C_2 are disjoint out of D , each of them can touch S only from one side, one of them from “the left”, the other from “the right” if we imagine S to be “vertical”. Now, any other outer face $F' \neq F$ of G sharing an edge with D would also touch S from both sides which is now impossible since F' either lies between C_1 and S (the part not containing C_2), or between C_2 and S (the part not containing C_1). Since $\rho(G) > 1$ there are at least two appropriate faces F, F' , and this gives a contradiction to the requirement that both of them touch S from both sides. \square

Claim 4. Let B_1, B_2 be distinct bridges of K , and let x_i, y_i be vertices of attachment of B_i , $i = 1, 2$. Then at least two of the vertices x_1, y_1, x_2, y_2 are equal.

Proof. Assume all the attachments are distinct. For $i = 1, 2$ let e_i be a foot of B_i at x_i and let f_i be a foot at y_i . If x_1, y_1, x_2, y_2 appear on C in that order then it is clear since $B_1 \neq B_2$ that $C(e_1, f_1)$ and $C(e_2, f_2)$ can be chosen to be disjoint. By Claim 2 this is not possible. Therefore we may assume that the order of attachments on C is x_1, x_2, y_1, y_2 (they interlace). However, this gives rise to a subdivision of K_4 which is clearly embedded as shown on Figure 6.3. \square

Claim 5. Every bridge of K has at least two vertices of attachment.

Proof. By Claim 3 each component L of $G - K$ is contained in an open disk D_L . We may assume that D_L contains only vertices of L and only edges of L and parts of feet of the bridge B of K containing L . We may also assume that when a foot of B leaves D_L it does not return to it any more.

Assume now that B has a single vertex of attachment. If for each pair e, f of feet of B , the cycle $C(e, f)$ is contractible, then there is a nontrivial 1-reduction which eliminates B . Otherwise there are feet e, f of B which are consecutive on $\partial\overline{D_L}$, according to how the edges leave D_L , and such that $C(e, f)$ is essential. The face of G containing the part of $\partial\overline{D_L}$ between e and f therefore contains an essential curve meeting G only at the attachment of B . However, this contradicts $\rho(G) \geq 2$.

The same arguments as above (only much simplified) resolve the case when a bridge is just an edge. \square

Claim 6. Let D be a closed disk in the torus and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be pairwise nonhomotopic essential simple closed curves on the torus. Then a pair of γ_i, γ_j ($1 \leq i < j \leq 4$) cross each other out of D .

Proof. Any two nonhomotopic essential curves in the torus must cross at least once. Assuming that the γ_i ($i = 1, 2, 3, 4$) do not cross out of D , they all cross each other in D . Contract D to a point x . This does not change the homotopies of the curves. It is easy to see that each γ_i gives rise to an essential simple closed curve γ'_i and a set of contractible loops. Consider now $\gamma''_1, \dots, \gamma''_4$ which are obtained from $\gamma'_1, \dots, \gamma'_4$ (respectively) by splitting the curves at the places where they touch to get curves disjoint apart from their common point. They divide the torus into a number of

regions. Since the curves are essential and pairwise nonhomotopic, each such region is bounded by at least 3 of them. A simple application of the Euler's formula now finishes the proof. The details are left to the reader. \square

Claim 7. K has at least two bridges.

Proof. Assume that B is the only bridge of K . Choose an outer face F sharing an edge with D and consider its fundamental cycles C_1, C_2, \dots . Denote by P_1, P_2, \dots the corresponding paths on $\partial F \setminus E(C)$ connecting vertices of C . Since ∂F is a cycle, two paths P_i, P_j ($i < j$) can only intersect if $j = i + 1$ in which case the initial vertex of P_j is the same as the terminal vertex of P_i . Consequently, if we have at least 3 fundamental cycles of F then we either get a K_4 or 2 disjoint essential cycles. (Cf. the proof of Claim 4.) Hence we have only two fundamental cycles of F and P_1, P_2 touch. See Figure 6.4. Note that C_1 and C_2 are homotopic. The face F was chosen arbitrarily. If F' is another outer face with an edge on C it also gives rise to a homotopic pair of fundamental cycles C'_1, C'_2 . Since K has only one bridge, these can not be homotopic to C_1 and C_2 .

G is 3-reduced. Therefore B has at least 4 vertices of attachment and by Figure 6.4 there are at least 4 outer faces with an edge on C . They give rise to four pairwise nonhomotopic cycles which do not cross out of D . By Claim 6 this is impossible. \square

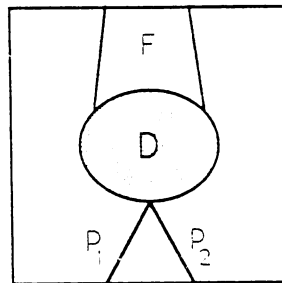


FIGURE 6.4

Claim 8. No bridge of K has more than 2 vertices of attachment.

Proof. If B has 4 or more vertices of attachment, any other bridge has two vertices of attachment (Claim 5) which are distinct from a pair of attachments of B . We are done by Claim 7 and Claim 4.

Suppose now that a bridge B of K has exactly three vertices of attachment. Since the total number of attachments is at least 4 there is a bridge B' with a new vertex of attachment, and in B' and B we can find pairs of distinct attachments. Again, we are done by Claim 4. \square

From now on we may assume that we only have bridges of K with two attachments.

Claim 9. Every bridge of K is an edge.

Proof. A bridge B with two attachments which is not an edge would give rise to a nontrivial 2-reduction unless it contains two feet e, f attached at the same vertex x on C such that $C(e, f)$ is essential. Now any bridge is attached to x since otherwise it gives rise to an essential cycle disjoint from $C(e, f)$. Also, no bridge different from B can have two feet attached at a vertex $y \in V(C)$, $y \neq x$. But since $\rho(G) \geq 2$ we have $\rho(G - x) \geq 1$ (cf. [RV]). This implies that B also has a pair of feet e', f' at the other vertex of attachment y ($y \neq x$) such that $C(e', f')$ is essential. But now, by symmetry, it follows that every bridge of K is attached at y as well. This contradicts the 2-reducibility. \square

Now we are well prepared to finish the proof of Lemma 6.2. The bridges of K are just edges. Any two of them have a vertex in common since otherwise we either get a K_4 or two disjoint essential cycles, depending whether their ends on C interlace, or not, respectively. Denote by T the graph consisting of the bridges of K (and their vertices of attachment). As explained above, T does not have a 2-matching (two edges without a common vertex) and does not have a 1-cover (a vertex whose removal leaves only isolated vertices). Therefore T is not bipartite (by the König–Egervary’s Theorem [B]). So T contains a cycle of odd length. Without a 2-matching this can only be a triangle. Finally, any edge of T adjacent to a fourth vertex can be extended to a 2-matching by one of the edges in the triangle. A contradiction. Lemma 6.2 is proved. \square

Till the rest of this section we will assume that $\rho(G) \geq 2$, G has no 2 disjoint essential cycles, but there is a submap K of G homeomorphic to K_4 as shown in Figure 6.3. Let Q denote the quadrangular face of K . We will assume that Q is as large as possible in the sense that no other subdivision K' of K_4 has its quadrangular face Q' which properly contains Q . Denote by R the other face of K . The relative K -components embedded in R will

be called *outside components* or *outside bridges* of K . Since $\rho(G) \geq 2$ there is at least one outside component. By the following lemma the attachments of outside components are quite restricted.

Lemma 6.3 *Let B be an outside component of K . If B has a vertex of attachment to a branch or a main vertex of K at the side designated by a shading in Figure 6.5 (a), (b), or (c), respectively, then every other foot of B either attaches to the same vertex or a vertex in the part designated in Figure 6.6 (a), (b), or (c), respectively.*

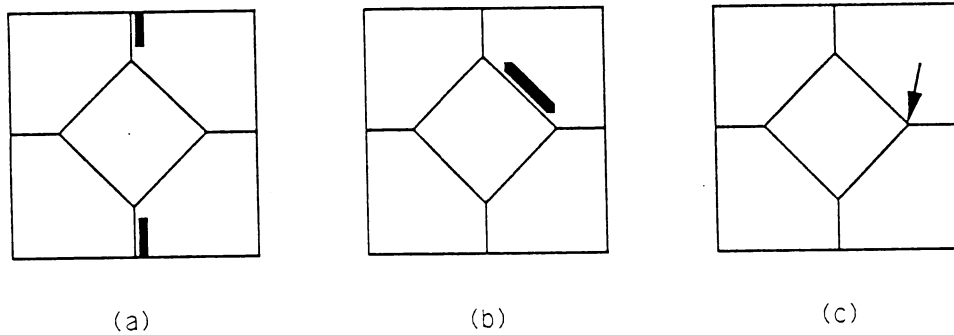


FIGURE 6.5 Attachments

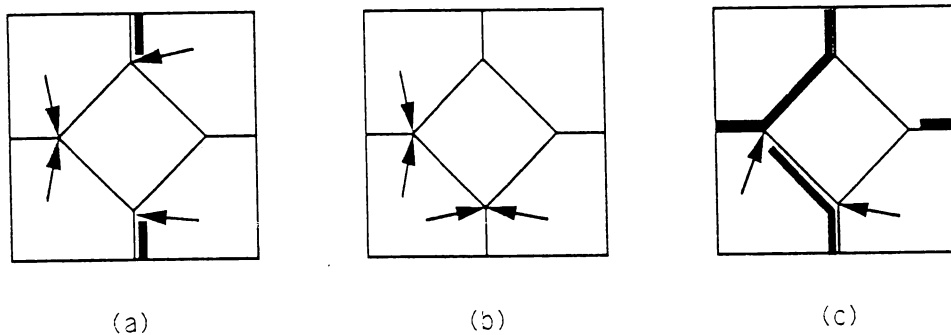


FIGURE 6.6 Other attachments

Proof. In each case particular places of possible attachments can be excluded since B being attached at that place would either give rise to the

case of Lemma 6.1, yield the existence of 2 disjoint essential cycles, or contradict the maximality of Q . \square

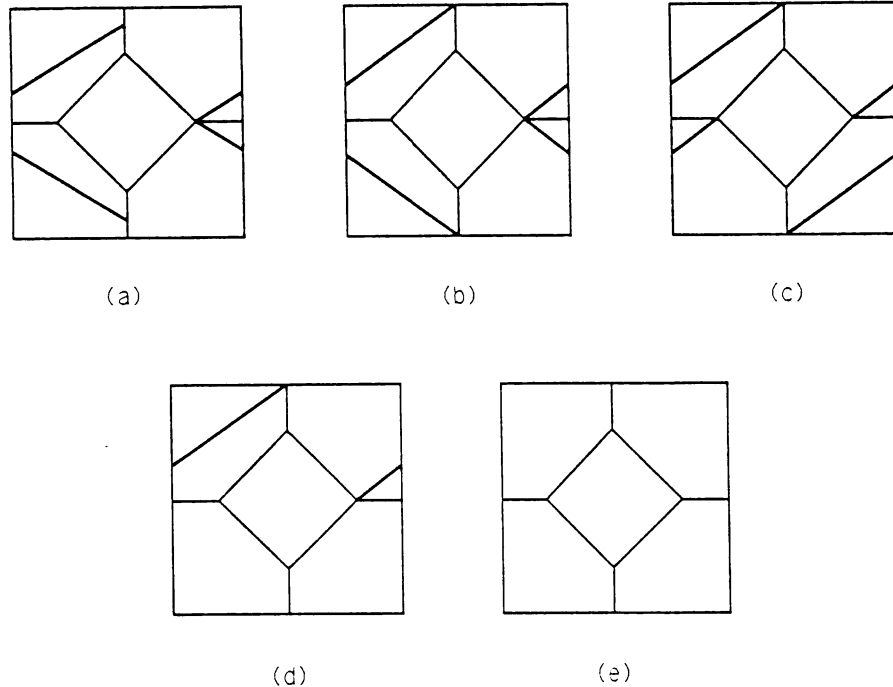


FIGURE 6.7 Subgraph K

Lemma 6.4 *Up to symmetries we may assume that G contains a subgraph K homeomorphic to one of the graphs in Figure 6.7 (a)–(e) such that:*

- a) *The quadrangle Q of K_4 is maximal,*
- b) *Every K -component is attached only to the points on the boundary of Q .*

Proof. Denote by K' the submap of G homeomorphic to K_4 as in Figure 6.3. Assume that B is its outside component which is attached at a vertex not on the boundary of Q . Denote by b the corresponding branch of K' . By Lemma 5.3 (Case (a)) B is attached only on one side of b . If B is attached only at vertices on this branch, say from the left side, then we can replace a part of b by the “leftmost” path of B . After a number of such changes we will definitely come to a case when no K' -component is

attached only to vertices of an outside branch of K' (otherwise contradicting 2-reducibility).

Assume now that B is an outside K' -component attached at an interior vertex of b . Now we know that B is attached at a vertex not on b (compare with Figure 6.7(d)). Denote this vertex by x . By Lemma 6.3 (Case (c)) B is attached only at one side of x . Therefore, if B is not an edge it is attached to b at more vertices. Let y, z be the attachments of B on b as close to each of the endpoints of b as possible. Since G is 3-reduced there is a K' -component attached to b between y and z . Now we have a subgraph of G as shown on Figure 6.8. The thick cycles in Figure 6.8 are disjoint and essential which are assumed not to exist. Therefore B is just an edge.

If two bridges are attached at different sides of b they must share the vertex on b . Otherwise there are 2 disjoint essential cycles. It is now easy to see that we must have Case (c) of Figure 6.7. If two bridges of K' attach at the same side of b then we have just two (by 3-reducibility), so we have Case (a) or (b) of Figure 6.7. Two bridges of K' attached at a point in the interior of distinct branches of K' not on ∂Q also give rise to 2 disjoint essential cycles. Therefore we are left with cases (d) or (e). \square

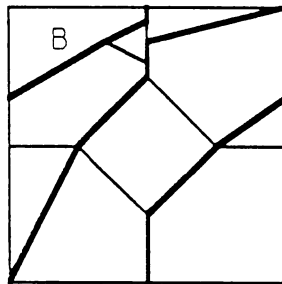


FIGURE 6.8

Lemma 6.5 *In cases (a) and (b) of Figure 6.7 we have $\rho(G) \leq 1$.*

Proof. Denote the main vertices on the boundary of Q by a, b, c, d , respectively, so that a is the vertex of degree 5 in K . Consider the face of K which has repeated vertices on its boundary. Note that a appears twice and between the two appearances of a there are only branches of K without any attached relative K -components. This implies that $\rho(G) \leq 1$. \square

Lemma 6.6 *Case (c) of Figure 6.7 gives rise to two graphs on 5 vertices shown in Figures 6.9 and 6.10.*

Proof. Denote the main vertices on the boundary of Q by a, b, c, d , respectively, so that a is on the right, b at the top. Denote by x the main vertex of K out of Q . Consider the face R of K with the facial walk $abxcdx$. Since $\rho(G) \geq 2$ there is a K -component in R . Every such component is just an edge (by the maximality of Q and 2-reducibility). If it attaches neither to a nor to d we find 2 disjoint essential cycles. Similarly we see that every such K -component must attach either to b or c . Therefore we have two possibilities which are depicted in Figures 6.9 and 6.10. In none of the cases there can be additional bridges in R . It is also easy to verify that the two 4-gons adx and $cbxa$ of K contain no bridges of K (they would give rise to a 2-reduction or 2 disjoint essential cycles).

It remains to show that no bridge of K is in Q . In case of Figure 6.9 we have the essential cycle xbd . A K -bridge in Q overlapping with bd then gives rise to a disjoint essential cycle. By Lemma 2.2 there is a face A in Q containing b and d on its boundary. Similarly we see that there is a face B containing a and c . Clearly, $A = B$. Now, by the 2-reducibility we have $A = B = Q$.

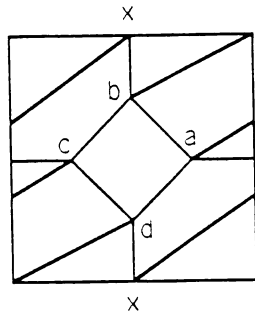


FIGURE 6.9

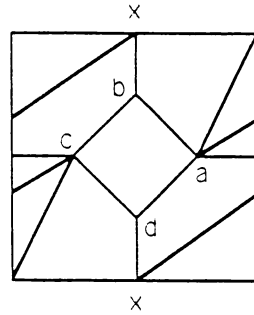


FIGURE 6.10

In case of Figure 6.10 the cycle ac (of length 2) is essential. As above this implies that there is a face in Q containing a and c . By the 3-reducibility it follows that Q contains no bridges of K . \square

The arguments used in the above proof to show that there are no bridges of K in Q will be repeatedly used later. Let us thus state this as a lemma.

Lemma 6.7 Suppose that S_1, S_2, S_3, S_4 are disjoint segments on the boundary of Q , appearing in the given order. If G contains an essential cycle C whose intersection with Q is contained in $S_1 \cup S_3$ and there is a path out of Q and disjoint from C joining a vertex of S_2 with a vertex of S_4 then in Q there is a face containing a vertex of S_1 and a vertex of S_3 .

Lemma 6.8 In Case (d) of Figure 6.7, G contains vertices p, q, r such that every bridge of $\{p, q, r\}$ is just an edge or a vertex attached with one edge to each of p, q, r . In each case G either contains a subgraph isomorphic to $K_{3,3}$, or G is the map of Figure 6.10.

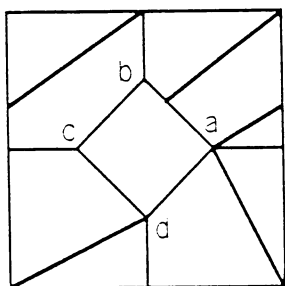


FIGURE 6.11

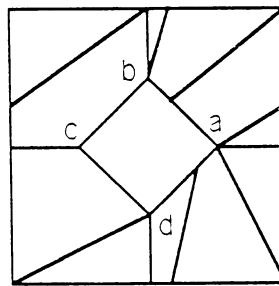


FIGURE 6.12

Proof. Under the same notation as in the proof of Lemma 6.6 consider the face R bounded by the walk $abxdacdx$. Since $\rho(G) \geq 2$ and the map is 1-reduced, the faces of G have no vertices repeated on their boundary. Therefore there must be bridges of K in R . The possible attachments in R are the branches ab, da , and cd of K . A bridge connecting cd with ad together with the cycle xba gives rise to 2 disjoint essential cycles unless the only attachment on ad is the vertex a . Similarly, a bridge from cd to ba must either have d as its only attachment on cd , or have a as the only attachment on ba . (Cf. the cycle dax). By Lemma 2.2 we have in R a bridge of K overlapping on ∂R with both appearances of a in order that $\rho(G) \geq 2$. By the restrictions obtained above, the only such possibility is a bridge attached at d on the branch cd , and attached at a vertex y on ab , where $y \neq a$. If the same bridge has another vertex of attachment, it is not on ab (by the maximality of Q), so it is the vertex a on the branch ad . If this bridge has only two vertices of attachment then there must be another bridge of K in R which overlaps with both appearances of d . The only possibility for

such a bridge is that it is attached at b on ab and between d and a on da (but not at d). Both possibilities are represented on Figures 6.11 and 6.12, respectively. In the second case we may as well assume that every outside bridge of K is just an edge.

Consider first the case of Figure 6.11. If $y \neq b$ then we have 2 disjoint essential cycles (Lemma 6.3 (b)). So $y = b$. Therefore this bridge has a, d , and y as the only attachments. It follows by the 3-reducibility that the bridge is trivial — just a vertex z together with its attachments. Consider now the subgraph H of G consisting of vertices a, b, d together with the two bridges of $\{a, b, d\}$ containing x and z , respectively. The embedding of H is cellular. Any bridge of H is attached to H at vertices a, b, d only (not necessarily all three). For each vertex $t \in \{a, b, d\}$, H contains an essential cycle which is not using t . Therefore any bridge B of H attaches to t from one side only. Since G is 3-reduced, B is either an edge joining two of a, b, d , or a vertex of degree 3 adjacent to a, b , and d . Note that $\{a, b, d\}$ have at least 3 non-edge bridges, containing z, x , and c , respectively. so $K_{3,3} \subseteq G$.

Suppose now that we have the case of Figure 6.12. Denote by y and z the attachments on the branches ab and da , respectively. If $z \neq a$ and $y \neq b$ then the boundary of Q together with the outside diagonals ac, yd , and bz give the case settled by Lemma 6.1. Therefore either $z = a$, or $y = b$.

Assume first that $z = a$, but $y \neq b$. Since the cycle axb is essential we may use Lemma 6.7 with $S_1 = \{a\}$, $S_2 = \{y\}$, $S_3 = \{b\}$, $S_4 = \{d\}$ to see that in Q there is a face containing a and b . By the 3-reducibility it turns out now that the connected component of $G - \{a, b, d\}$ containing y is just the vertex y itself and it is attached to a, b, d with 3 edges. Let H be the subgraph of G containing a, b, d together with the vertices x, y and their attachments to $\{a, b, d\}$ and together with the edge $bz = ba$. We conclude in the same way as above in case of Figure 6.11.

Next we consider the case $z \neq a$ and $y = b$. We assume that z is as close as possible to a . Then, if there is an outside bridge of K attached between z and a , its other attachment is on the branch cd . But this way we get an essential cycle disjoint from the cycle axb , unless the attachment is the vertex a . It follows that every outside bridge of K which is attached at the branch cd either goes to a or to b . We claim that the component of $G - \{a, b, d\}$ containing the vertex c is trivial and attached to a, b, d by 3 edges. This is evident because of the 3-reducibility if we show that in Q there is a face containing b and d . But the existence of such a face is guaranteed by Lemma 6.7 (cycle bdx , $S_1 = \{b\}$, $S_2 = \{c\}$, $S_3 = \{d\}$, $S_4 = \{a\}$).

Let H be the subgraph of G on the vertices a, b, d, x, c and with edges

bd and the attachments of c, x to $\{a, b, d\}$. We conclude in the same way as in the first two cases.

In the remaining case when $z = a$ and $y = b$ we may take for H the graph on vertices a, b, d, x and with edges bd, ab (the possibility going across!), ax, bx, dx . As above we see that there is a face in Q containing b and d , and thereafter we see that there are no additional bridges of K . The obtained map is equivalent to the map of Figure 6.10. The equivalence is realized by the permutation $(a)(bc)(dx)$. \square

Lemma 6.9 *In case (e) of Figure 6.7 the outside bridges of K which are attached only to the main vertices of K do not give rise to a 2-representative embedding.*

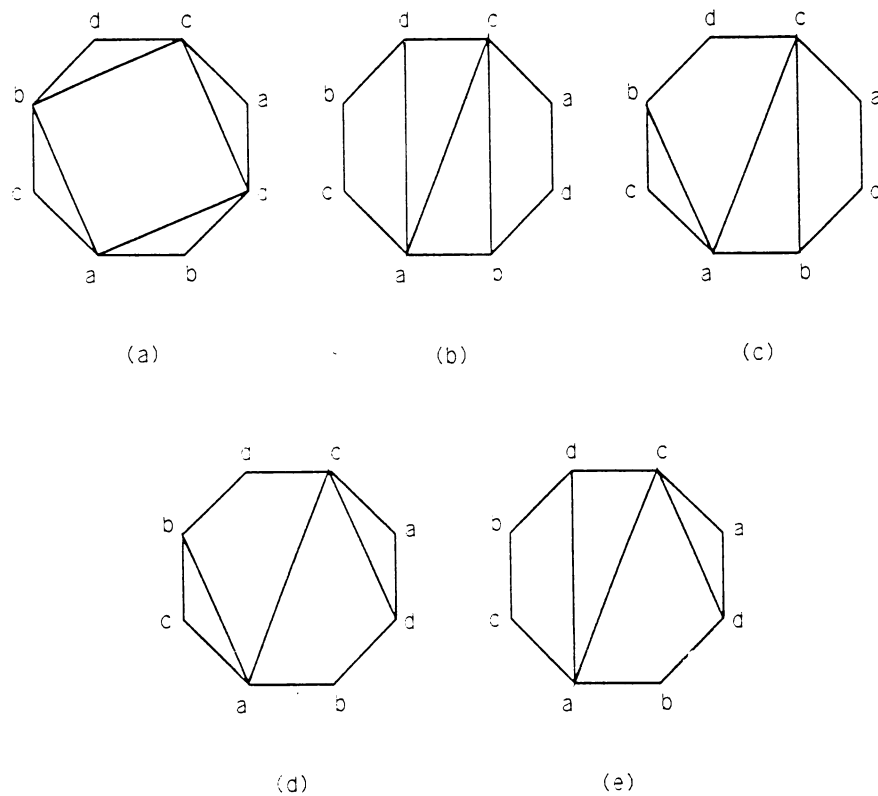


FIGURE 6.13

Proof. Consider the outer face F in Figure 6.7 (e). The outside bridges of K are just edges. In order to get $\rho(G) \geq 2$, we need for each double

occurrence of a vertex on the boundary of F an edge in F overlapping with the two occurrences (Lemma 2.2). Assume (by symmetry) that the vertex a at the branch ab has the largest number of outside bridges. It is easy to see that the possible *minimal* sets of outside bridges yielding $\rho(G) \geq 2$ are the ones shown in Figure 6.13, if we additionally assume the 3-reducibility which excludes any triangles in F which do not contain an edge from the boundary of Q . In each of the cases (a), (b), and (d) we have a pair of edges xy, wz such that $\{x, y, w, z\} = \{a, b, c, d\}$ and such a pair gives rise to 2 disjoint essential cycles.

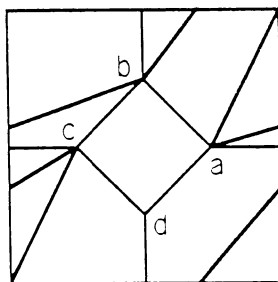


FIGURE 6.14

Case (e) is centrally symmetric to case (c). Thus it remains to consider case (c) which is exhibited in Figure 6.14. There is an essential cycle of length 2 through vertices a and c . By Lemma 6.7 ($S_1 = \{a\}$, $S_2 = \{b\}$, $S_3 = \{c\}$, $S_4 = \{d\}$) there is a face S in Q containing a and c . It follows by 3-reducibility that nothing is attached to branches ad and cd including attachments at c and a if these are coming from a face containing d . Consider now the face in Figure 6.14 bounded by the triangle abc (with the branch bc on ∂Q). It follows by the 3-reducibility that a bridge of K in Q is attached to an interior vertex of the branch bc and to a vertex $x \neq b$ on ab . Using the face S we see that there is a disk D such that $G \cap \partial D = \{a, b, c\}$ and D contains the branches ab, bc of K and the bridge obtained above. Clearly, D gives rise to a nontrivial 3-reduction which is a contradiction. \square

By Lemma 6.9, in case (e) of Figure 6.7 there is an outside bridge of K attached at an interior vertex of a branch.

Lemma 6.10 *In the remaining case only the graph $K_{3,3}$ with two additional*

edges embedded as shown on Figure 3.12 is obtained.

Proof. Till the end of this section we will assume that we have case (e) of Figure 6.7 and any minimal set of outside bridges giving $\rho \geq 2$ must contain a bridge attached to only one main vertex of K . Moreover, we know that every outside bridge of K is just an edge. Suppose that such a bridge is attached at the vertex x in the interior of the branch ab . A minimal edge-set giving $\rho \geq 2$ must separate all the double occurrences of vertices on the boundary of the outer face R of K_4 . It is easy to get all such sets by exhibiting the appropriate possibilities, and having in mind that most types of bridges in R are forbidden now. Some of the obtained configurations give rise to two disjoint essential cycles. The remaining ones are collected in Figure 6.15 (a)–(e).

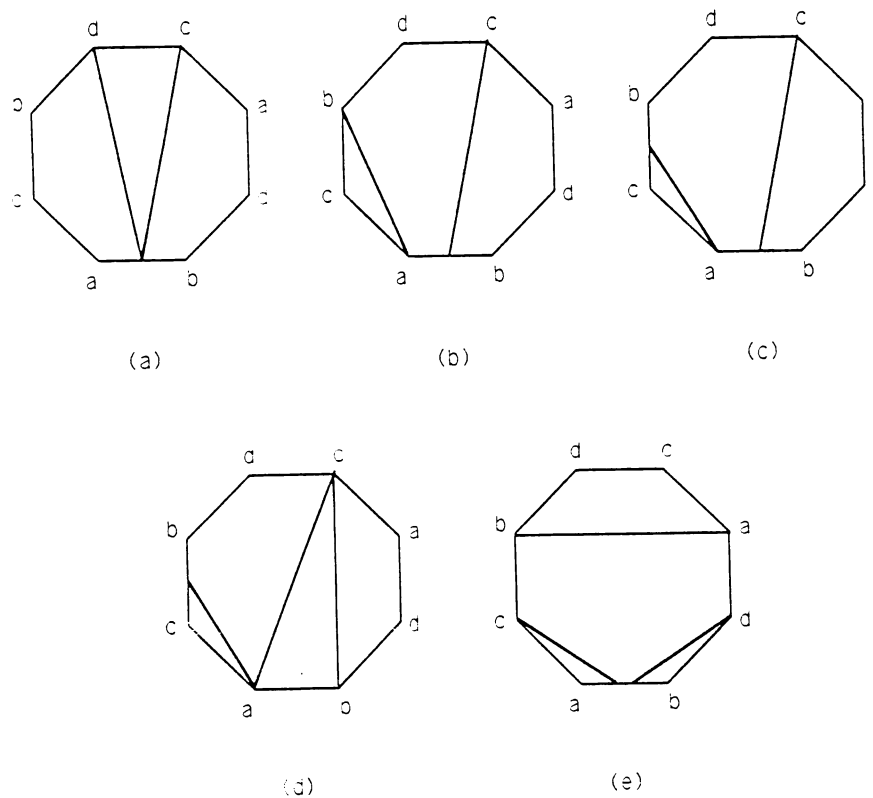


FIGURE 6.15

Cases (c), (d), and (e) are not minimal in the sense that we may delete

the edge ac or bd of K_4 and still have $\rho \geq 2$ with another copy of K_4 sitting in there. These cases therefore arise from the others (cases (a) and (b)). Case (b) can be turned into case (a) by exchanging the edge ac of K_4 with the edge xc .

It remains to consider case (a) of Figure 6.15. It is re-drawn in Figure 6.16.

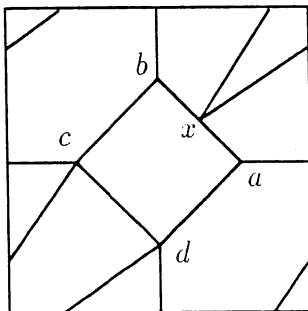


FIGURE 6.16

By Lemma 6.7 ($S_1 = [x, b]$, $S_2 = \{c\}$, $S_3 = \{d\}$, $S_4 = \{a\}$, and the cycle $bxdb$) we see that in Q there is a face F_1 of G containing d and a vertex of the segment $[x, b]$ of ∂Q . Similarly, ($S_1 = [a, x]$, $S_2 = \{b\}$, $S_3 = \{c\}$, $S_4 = \{d\}$, and the cycle $cxac$) we see that in Q there is a face F_2 of G containing c and a vertex of the segment $[a, x]$. If $F_1 = F_2$ then this face contains c and d , and there is a non-trivial 3-reduction reducing the branch cd and the edges cx, dx . Consequently, $F_1 \neq F_2$. This implies that ∂F_1 contains d and x , and ∂F_2 contains c and x . Note that no bridge of K is in the face of Figure 6.16 bounded by the triangle $cx d$ (we would have a submap on Figure 6.1 or a non-trivial 2-reduction). Since c and d are not on the boundary of the same face contained in Q (a 3-reduction of the triangle $cx d$), there is a bridge of K in Q from a vertex y in the interior of the branch cd to x . By 3-reducibility, this is trivial — the vertex y is adjacent in G to c, d , and x .

So far we have shown that we have a submap represented in Figure 3.12. We need to show that there are no additional bridges of K . It suffices to see that there are no additional outside bridges since in this case any bridges in Q give rise to a non-trivial 3-reduction. The outside bridges may only be attached to the following segments on ∂Q : $[a, x]$, $[x, b]$, $[b, c]$, or $[d, a]$. By symmetry we may consider a bridge B (if there is one) in the face bounded by $adbxc$. B is just an edge. If B is attached at the vertex $t \neq d$ on the

branch da , the other end of B can not be on the segment $[x, b]$ of ∂Q (we get two disjoint essential cycles), so the other end is c . A bridge attached at d can have c or a vertex on $[x, b]$ as the other end. By symmetry, the same restrictions apply in the other face bounded by $bcaxd$. But if there is any such bridge, we have a non-trivial 3-reduction. This completes the proof. \square

With the proof of Lemma 6.10 we exhibited all possible cases and we established the validity of Theorem 3.2.

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