CONVEX REPRESENTATIONS OF MAPS ON THE TORUS AND OTHER FLAT SURFACES

By

Bojan Mohar

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Convex representations of maps on the torus and other flat surfaces

Bojan Mohar*

University of Ljubljana
Department of Mathematics
Jadranska 19, 61111 Ljubljana
Slovenia

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Abstract

It is shown that every map on the torus satisfying the obvious necessary conditions has a convex representation on the flat torus $\boldsymbol{R}^2/\boldsymbol{Z}^2$. The same holds for the Klein bottle, and the two bordered flat surfaces – the cylinder, and the Möbius band. In each case a linear time algorithm is obtained for constructing convex representations if they exist.

PROPOSED RUNNING HEAD: Convex Maps on Flat Surfaces

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1 Flat surfaces and convex maps

By a well-known result of Stein and Tutte [S, Tu] every 3-connected planar graph has a convex embedding in the plane, i.e. there is a representation of such graph in the plane such that all bounded faces are (strictly) convex polygons and the unbounded face is a complement of a convex polygon. This result was further generalized by Thomassen [Th] who discovered necessary and sufficient conditions for the existence of convex embeddings in the plane (cf. Section 3). A result which is closely related to the Stein-Tutte's theorem is the theorem of Steinitz [St] that every 3-connected planar graph can be represented as the graph of a convex 3-polytope. There were attempts to generalize Steinitz's theorem to maps on surfaces of positive genus, e.g. [G, MGK]. But it seems that no one tried to extend the Stein-Tutte's theorem to non-simply connected surfaces. In this paper we fill in this gap by proving a corresponding result for the torus, the Klein bottle, the cylinder, and the Möbius band. These are the only flat surfaces with the boundary components being straight. The existence of convex representations of maps on the torus and the Klein bottle has some important consequences since their universal covers give rise to convex tilings of the Euclidian plane. A discussion about this can be found in [GS, p. 202].

Let S be a compact surface. A *map* on S is a pair $M = (G, S)$, where G is a connected graph embedded in S such that all faces are simply connected. Recall that a *face* of a map $M = (G, S)$ is a connected component of $S \backslash G$. Moreover, if S has non-empty boundary, $\partial S \neq \emptyset$, then we require that ∂S is covered by $E(G)$, i.e. for each point $p \in \partial S$ there is an edge of G which contains p. Note that we do not require the underlying graph of a map to be simple, i.e. we allow loops and parallel edges. A map on the torus is also called *toroidal map*. For a map $M = (G, S)$ we will use $V(M)$, $E(M)$ and other graph theory notation and terminology to denote the corresponding quantities of the graph G of the map. Two maps $M = (G, S)$ and $M' = (G', S')$ are *equivalent* if there is a homeomorphism $h : S \to S'$ mapping the graph G of the first map isomorphically to the graph G' of the second map. It is well-known (cf., e.g., [MT]) that two maps on an orientable surface without boundary and with all faces simply connected are equivalent if and only if they determine the same rotation system on the graph.

A compact Riemannian surface S (possibly with boundary) is *flat* if every point $p \in S$ has a neighbourhood which is affinely diffeomorphic to an open set in the closed upper half-plane of the Euclidian plane. This is equivalent to the condition that the curvature and torsion are identically zero, including the curvature of the boundary ∂S being zero. The special case of a flat surface is the *flat torus*, the quotient space $\boldsymbol{R}^2/\boldsymbol{Z}^2$ ($\boldsymbol{R}^2/_\sim$ where $(x,y) \sim (x',y')$ means $(x-x', y-y') \in \boldsymbol{Z}^2$). Another flat surface is the *flat Klein bottle.* This surface is the quotient of the Euclidian plane $\boldsymbol{R}^2$ corresponding to the relation $\sim$ given by $(x,y) \sim (x+n, (-1)^n y + m)$, $n, m \in \boldsymbol{Z}$. The flat torus and the flat Klein bottle are usually represented as the identification space of the unit square by identifying the top and the bottom side and then identifying the left and the right, with a previous turn by 180 degrees in case of the Klein bottle. There are two additional flat surfaces with boundary — the *flat cylinder* and the *flat Möbius band.* They are obtained from the unit square as well, by identifying only one pair of sides, the left and the right. To get the cylinder they are identified without a turn, and to get the Möbius band we perform a twist of 180 degrees of one side before the identification. It can be shown by using the Gauss-Bonnet formula (cf. [C]) that these are the only compact flat surfaces.

Let $M = (G, S)$ be a map. A face F of M is *convex* if for any two points $x, y \in F$ there is a geodesic contained in F joining x and y. The map M is *convex* if all of its faces are convex and all edges of G are geodesic segments. (It can be shown using the Gauss-Bonnet formula that on a surface with non-positive curvature an interior edge must be geodesic if the two faces containing it are convex; but this does not necessarily hold for the edges on the boundary.) On a flat surface S, a face is convex if and only if it is bounded by straight line segments and each interior angle is smaller or equal to π.

Every convex map M on a flat surface S is *reduced* (see Section 2). Roughly speaking, it has the same property as 3-connected planar maps. It is shown in this paper that being reduced is also sufficient for a toroidal map to be equivalent to a convex map on the flat torus (Theorem 4.2). The same results are then shown for maps on other flat surfaces (Theorems 5.1, 6.1, 6.2). It is worth mentioning that necessary and sufficient conditions for a weaker property, the existence of straight line representations of maps on flat surfaces, was also obtained recently [M2]. In addition to the existence of convex representations, we also present a linear time algorithm to construct such an embedding, if it exists.

2 Reduced maps

A map $M = (G, S)$ is *reducible* if it has either one of the following properties:

(**R1**) G contains a pendant edge (equivalently, G has a vertex of degree 1),

(**R2**) G contains a cycle C which bounds a disk D on S such that there are at most two vertices of C which are adjacent with an edge not in $D \cup C$, or

(**R3**) in G there are homotopic cycles C_1 and C_2 such that $C_1 \cap C_2$ is a vertex, say $x \in V(G)$, and all vertices on $C_1 \cup C_2$, except possibly x, have all their adjacent edges in $D \cup C_1 \cup C_2$, where D is the (open) disk bounded by $C_1 \cup C_2$.

A non-reducible map is said to be *reduced.* The following proposition, whose easy proof is left to the reader, shows that reduced maps are some kind of a generalization of 3-connected planar maps to maps on general surfaces.

Proposition 2.1 *Let $M = (G, S)$ be a map on a non-simply connected surface S. If G is 3-connected then M is reduced.*

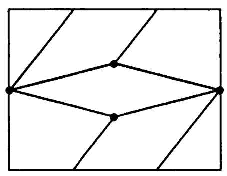


Figure 1: A reduced toroidal map whose graph is not 2-connected

On the other hand there are reduced maps which are not even 2-connected. An example of such a map on the torus is given in Figure 1.

The reason why we are interested in reduced maps is the following.

Lemma 2.2 *Let M be a map on a flat surface which has a convex representation. Then M is reduced.*

Proof. Being reduced is a topological invariant, so we may assume that M is convex. It is clear that M has no vertices of degree 1 since a face containing such a vertex can not be convex. Suppose now that G contains a cycle C satisfying (**R2**). Consider D and its interior angles. By (**R2**) it follows that all of these angles, except at most two of them are larger or equal to π. It follows by the Gauss-Bonnet Theorem (cf. [C]) that the Euler's characteristic of D is non-positive, a contradiction with D being a disk. The exclusion of the remaining possibility (**R3**) is similar. ∎

3 Convex representations in a disk

By a theorem of Stein and Tutte [S, Tu] every 3-connected planar graph has a convex representation in the plane. Moreover, Tutte established a necessary and sufficient condition for a 2-connected graph G with a given cycle C to have a convex embedding in the plane such that C bounds the unbounded face and it is a convex $|C|$-gon. The Tutte's condition is that G is a subdivision of a 2-connected graph H such that every separating set $\{u, v\}$ of H (if any) is contained in the cycle of H corresponding to C. This result was generalized by Thomassen [Th] to cover the general case where C need not be strictly convex, i.e. it could be a k-gon with $k < |C|$.

Let H be a subgraph of G. A *relative H-component* is a subgraph of G which is either an edge $e \in E(G)\backslash E(H)$ (together with its end-points) which has both end-points in H, or it is a connected component of $G - V(H)$ together with all edges (and their endpoints) between this component and H. Each edge of an H-component R having an end-point in H is a *foot* of R. The vertices of $R \cap V(H)$ are the *vertices of attachment* of R. We will use the same terminology for maps and their submaps.

Theorem 3.1 *Let C be a cycle of a simple 2-connected planar graph G, and let R be a convex k-gon ($k \leq |V(C)|$) in the plane with sides $P_1, P_2, \ldots, P_k$, occurring in the listed order on R. A given convex embedding of C into R can be extended to a convex embedding of G such that R bounds the outer face if and only if*

(a) For each vertex $v \in V(G)\backslash V(C)$ of degree at least 3, G contains three paths from v to C which pairwise have only v in common.

(b) No C-component of G has all its vertices of attachment in some P_i, $1 \leq i \leq k$.

(c) Any cycle of G which has no edge in common with C has at least three vertices of degree ≥ 3 in G.

A proof of Theorem 3.1 can be found in [Th]. It is important that the proof yields a linear time algorithm for convex drawings of planar graphs. More precisely, there is an algorithm that given G, C, R, and an embedding of C into R, extends the embedding of C to a convex embedding of G into $int(R)$, whenever possible, and uses linear time and space. See [CYN, NC] for details. In addition to this, appropriate cycles C, if any, can be found in linear time [CYN, NC].

Consider a submap L of M. A vertex of degree 3 or more in L is called a *main vertex* of L. The paths in L joining pairs of main vertices and having all internal vertices of degree 2 in L, are called *branches* of L. Each branch e gives rise to two *sides* of e which correspond to the "left" and "right" (or "upper" and "lower") side of the branch on the surface. Note that since L is a map, the sides are well-defined. For example, if L is homeomorphic to the map M_1 of Figure 2(b), it has six sides. Let Q be a relative L-component and σ a side of L. We say that Q is *attached to the side* σ if all feet of Q attach to L at the side of the branch corresponding to σ, including the possibility of the attachment at an endpoint of the branch (as far as the side is correct). In the same way we define when Q is attached to a set of sides.

As a corollary to Theorem 3.1 we get the following important lemma.

Lemma 3.2 *Let L be a submap of a reduced map M on a flat surface S. Suppose that L is convex and that no relative L-component of M is attached at a single side of the convex polygon determined by the face of L it lies in. Then L can be extended to a convex representation of M.*

Remark 3.3 Note that the number of sides of the convex polygon determined by a face F of L can be smaller than the number of branches of L on the boundary of F. See., e.g., Figure 3(a) where the facial polygon has four sides while there are six edges appearing on its boundary.

Proof. Consider a face F of L. Suppose that F is an open convex k-gon. Let M_F be the map in the disk obtained as follows. First take a convex k-gon D (in the plane) with the same angles and side lengths as there are in F. Add vertices of L on the boundary of D in exactly the same way as they appear on the boundary of F. In case when there is a multiple occurrence of a vertex on ∂F, choose a new copy of the vertex for each of the occurrences. Finally, add in D the relative L-components of M lying in F so that they are embedded in exactly the same way as in F. We claim that M_F satisfies the conditions of Theorem 3.1. (Once we will verify this claim the proof will be evident.)

By construction the outer boundary of M_F is a cycle. Denote it by C. The graph of M_F is simple since a loop or a pair of parallel edges necessarily bound a disk (also in M) and this violates (**R2**) or (**R3**) (in case when a parallel pair joins two vertices of C which represent the same vertex of L). It also follows from (**R2**) that M_F has no cutvertices and is therefore 2-connected.

It remains to check the properties (a), (b), and (c) of Theorem 3.1. Condition (b) is among the assumptions. To check (a), suppose that there is a vertex $v \in V(M)\backslash V(C)$ of degree 3 or more in M_F such that there are at most two paths from v to C. By Menger's Theorem there are vertices a, b in M_F different from v such that any path from v to C uses one of them. Let Q be the submap of M_F consisting of all edges which are contained in those paths starting at v which do not contain a or b as an interior vertex. Note that $V(Q) \cap V(C) = \{a, b\} \cap V(C)$. Q is a connected plane subgraph of M_F. Since v is of degree at least 3 in Q and Q contains at most two vertices of degree 1 (by (**R1**) the only candidates are a and b), Q contains a cycle. Let W be the facial walk corresponding to the outer face of Q. Since Q contains a cycle, there is a subwalk W' of W which is a cycle of the graph. If possible, choose W' in such a way that it does not contain both vertices a and b. In that case W' either gives rise to a contractible cycle in M contradicting (**R2**), or to a pair of homotopic cycles in M that contradict (**R3**).

It remains to check that each cycle C' in M_F such that $E(C') \cap E(C) = \emptyset$ has at least three vertices of degree greater or equal to 3. Suppose this is not true. The cycle C' bounds a disk in M_F and hence it also bounds a disk in M, although it may change there into a union of two homotopic cycles. The required property is now apparent since otherwise we would have a cycle (or two cycles) in M violating (**R2**) or (**R3**). ■

We will need another lemma concerning relative L-components attached at a single side of a branch of L. Such a relative component is said to be *local.* Those relative L-components which are not local are called *global.* A map $M' = (G', S')$ is *homeomorphic* to the map $M = (G, S)$ if there is a homeomorphism $h : S' \to S$ whose restriction to G' is a 1-1 map onto $G \subset S$ (a graph homeomorphism).

Lemma 3.4 *Let L be a submap of a reduced map M. Then there is a submap L'' of M which is homeomorphic to L and such that there are no local relative L''-components. Moreover, the homeomorphism between L and L'' is the identity on the set of main vertices of L. Given L, one can find L'' in time which is linear in the number of edges of M.*

Proof. In addition to the properties stated in the lemma, the branches of L'' are contained in the branches of L and in the local relative L-components. First we construct a submap L' as follows. Fix a side σ of L. Determine all local relative L-components attached at σ. They are partially ordered: $Q' \leq Q$ if Q together with its segment of attachment to σ (= the segment on σ between the two vertices of attachment of Q which are the most apart on σ) bounds a disk which contains Q'. For each maximal (with respect to the presented partial order) relative L-component attached at σ, replace the interval of attachment of this relative L-component by the outer path in this relative component. Repeat the procedure for one of the sides of each branch of L, and let L' be the obtained submap of M. Since M is reduced, every local relative L'-component overlaps on the other side with a global relative L'-component. It follows that for every local relative L'-component Q' there was a local relative L-component Q at the same side of the corresponding branch as Q' is, such that $Q' \supseteq Q$. Since there are no local relative L'-components on one side of each branch of L', the repetition of the above procedure with L', using the other sides of the branches, gives rise to a submap L'' of M without local relative L''-components.

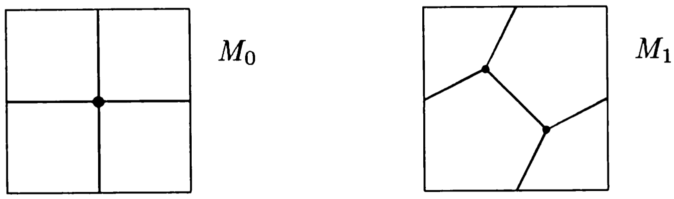
The time needed in the above algorithm for any one of the sides is linear in the number of edges in the corresponding branch together with all local relative L-components attached at this side. Therefore the total time is linear. ∎

4 The torus

Denote by M_0 and M_1 the toroidal maps represented in Figure 2(a) and (b), respectively.

Lemma 4.1 *Let M be a toroidal map. Then M contains a submap L which is homeomorphic to one of the maps M_0 or M_1 of Figure 2. One can find such a submap in linear time.*

Proof. The lemma is quite obvious. Let us just describe a way one can find the required submap L. We first find a spanning tree T of the map by using the Depth First Left Second Search. This means that in the DFS, whenever we have a choice of more than one edge to consider, we always take the one which is the leftmost untraversed according to the local rotation of the map. At the same time we make a doubly linked circular list of the edges



(a) (b)

Figure 2: Toroidal maps M_0 and M_1

in the cotree in the order as they were found during the tree construction, and such that each edge appears twice in the list, once for each of its ends. Let e and f be edges which are interlaced in this list, i.e., their order is $\ldots, e, \ldots, f, \ldots, e, \ldots, f \ldots$. (If there is no such pair then the map is not toroidal, and we stop.) Let C_e and C_f be the fundamental cycles of e and f, respectively. Then $C_e \cup C_f$ is the required map L homeomorphic to M_0 or to M_1.

It may not seem very obvious how one can choose in linear time an interlacing pair e, f. We have the doubly linked circular list of the cotree edges (so that we can go in both directions). First we traverse the list once around. At any step we check if the new edge is equal to the previous one. If so, we remove the consecutive pair from the list, return one step back and then continue the traversal. At the end we get a sublist of the original one without consecutive edge pairs. (It may happen that the list "disappears" in which case the map is not toroidal, and we stop.) Traverse the new list again and determine which edge pair is closest to each other in the list (measured in the given direction). Let e be the edge, and let f be an edge between the two occurrences of e. Then e and f interlace. ∎

Theorem 4.2 *A toroidal map is equivalent to a convex map on the flat torus if and only if it is reduced.*

Proof. The "only if" part follows by Lemma 2.2, and to prove the "if" part we use Lemmas 4.1, 3.2, and 3.4. ∎

Corollary 4.3 *Let M be a toroidal map whose graph is 3-connected. Then M has a convex representation in the flat torus.*

Proof. Obvious by Theorem 4.2 and Proposition 2.1. ■

It is important that our proof of Theorem 4.2 yields a linear time algorithm for convex drawings of toroidal maps. Our algorithm is carried out in four steps:

Algorithm *TOROIDAL CONVEX EMBEDDING*:

(1) Find a submap L homeomorphic to M_0 or M_1 (cf. Lemma 4.1 and its proof).

(2) Get rid of the local relative L-components (cf. Lemma 3.4 and its proof).

(3) Construct M_F (cf. the proof of Lemma 3.2) where F denotes the face of L. It is obvious that M_F can be constructed in linear time.

(4) Find a convex representation of M_F. This can be done in linear time as described in [CYN]. Cf. also [NC].

It is worth mentioning that instead of taking the map M_1 of Figure 2(b), one can take the map M_1' of Figure 3(a) to start with. The advantage is that the final drawing will be nicer since it will either be represented in a square (if L is homeomorphic to M_0) or in a rhombus as shown in Figure 3(b), with the obvious identifications giving rise to the flat torus.

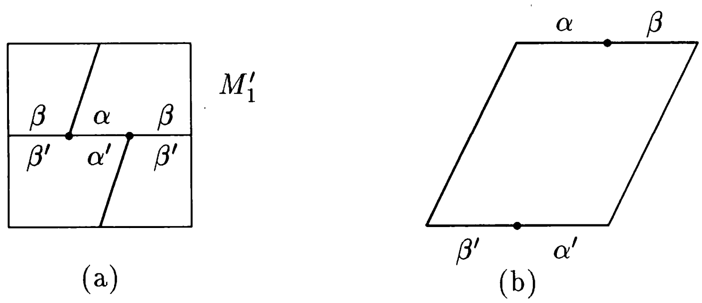


Figure 3: The map M_1' and the corresponding quadrangle

To get a realization of this form one has to show that in case we need to use the submap L homeomorphic to M_1' we can choose L so that any relative L-component attached at the sides α and β (or α' and β', respectively) is indeed local at a single one of these sides. To reach this we modify the

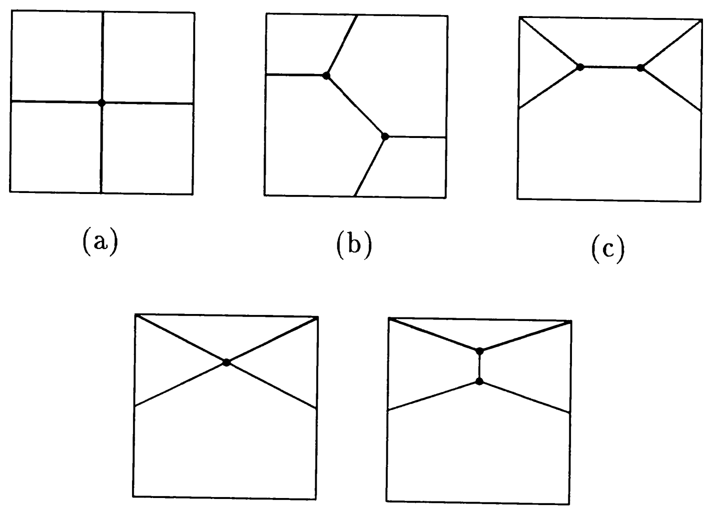
Step (2) of the above algorithm as follows. We first change the map L obtained in Step (1) so that it either becomes homeomorphic to M_0, or it is homeomorphic to M_1' and every relative L-component which is attached to α and β (or α' and β', respectively) is local. Determine all global relative L-components which are attached at the sides α and β only. Such relative L-components are nested and we find the "outermost" relative L-component Q among them. Using Q we change L by replacing the part of the branch β between the end of β and the rightmost attachment of Q with the outer path in Q. Now repeat the same step for α' and β' (in the new map). It is easy to see that after these two steps there are no more global relative components attached to α and β (or α' and β', respectively), and that these operations can be performed in linear time.

5 The Klein bottle

Theorem 5.1 *Every reduced map on the Klein bottle has a convex representation on the flat Klein bottle.*

Proof. The proof is basically the same as in case of the torus. Relying on the results of Sections 2 and 3 it suffices to find appropriate convex submap L of the given reduced map M on the Klein bottle. In order to see what are the unavoidable submaps, and how to get them (in linear time) we describe the algorithm. As in the case of the torus we use the Depth First Left Second Search to construct a spanning tree T of M. Since the Klein bottle is non-orientable the situation is a little more complex in this case. Note that the map can be given by means of local rotations at the vertices together with an assignment $\lambda : E(M) \to \{-1, 1\}$ (an *embedding scheme*, see, e.g., [M1] or [MT]). The meaning of λ is the following. Traversing an edge with $\lambda(e) = 1$ the two local rotations at the endvertices of e are consistent (as on the surface), and in case $\lambda(e) = -1$ they are non-consistent. For any vertex v we may change λ on all edges incident with v and at the same time reverse the local rotation at v, and we get an equivalent embedding scheme which represents the same map. Call this operation the *switch* at v. Constructing the DFS spanning tree we use the switch operation at the vertex v when we add the vertex v to the tree, if the edge e on which we came to v has $\lambda(e) = -1$. It follows that at the end all edges e in the obtained spanning tree T have $\lambda(e) = 1$.

During the construction of T we build the doubly linked circular list of cotree edges as in the case of the torus. Define the matrix $A = (a_{ef})$,



(a) (b) (c)

(d) (e)

Figure 4: Unavoidable submaps on the Klein bottle

$e, f \in E(M)\backslash E(T)$, indexed by the cotree edges, where

$$a_{ef} = \begin{cases} 1, & \text{if } e = f \text{ and } \lambda(e) = -1 \\ 1, & \text{if } e \neq f \text{ and } e \text{ and } f \text{ interlace in the list} \\ 0, & \text{otherwise} \end{cases}$$

It is shown in [M1] that the rank of A over the field $GF(2)$ is equal to 2. Moreover, since the surface is non-orientable, at least one diagonal entry is equal to 1. It follows that there are cotree edges e and f such that their 2×2 submatrix of A is equal to

$$A_{ef} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad (1)$$

or to

$$A_{ef} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \quad (2)$$

In the first case the union of the fundamental cycles of e and f determines a submap homeomorphic either to the map in Figure 4(a) or 4(b). In the second case we get Figure 4(c), (d), or (e).

To find appropriate edges e and f one does not need to compute the whole overlap matrix A. All the necessary information is already in the constructed doubly linked list of the cotree edges. We can find the required pair of edges e and f in linear time as follows. An edge e with $\lambda(e) = 1$ is said to be *orientation preserving* and it is *orientation reversing* if $\lambda(e) = -1$. First we try to find orientation reversing cotree edges e, f which do not interlace (the submatrix (2)). For each orientation reversing edge e in the list we determine the number of orientation reversing edges between e and the next occurrence of e in the list. Select the element in the list where this number is minimal. It is easy to see that either there is an orientation reversing edge f which does not occur in the part of the list between the selected element and its second occurrence (in which case we are done), or the orientation reversing edges interlace evenly, i.e., the sublist of orientation reversing edges is $(e_1, e_2, \ldots, e_k, e_1, e_2, \ldots, e_k)$ in which case there is no solution of this type. If this happens we have to look for an orientation reversing edge e_i which interlaces with an orientation preserving edge f (their submatrix of A is given in (1)). Find an orientation preserving edge f such that each of the two sublists between the occurrences of f contains an orientation reversing edge. It is easy to see that such an edge necessarily exists, and that it interlaces with one of the e_i $(1 \leq i \leq k)$. ■

It is worth mentioning that the above proof yields a linear time algorithm for convex drawings of reduced maps on the Klein bottle.

6 The cylinder and the Möbius band

We will represent the maps on the cylinder and the Möbius band by a unit square with the top and the bottom forming the boundary of the surface, and the left and the right side identified (after a twist for the Möbius band). To mark that the boundary is covered by edges it will be drawn thicker.

Theorem 6.1 *Every reduced map on the cylinder has a convex representation on the flat cylinder.*

Proof. Let M be a reduced map on the cylinder. It is obvious that M contains a submap homeomorphic to the map L represented in Figure 5(a). By Lemma 3.2 it suffices to see that L can be chosen such that there are no local relative L-components. But this follows by Lemma 3.4. ■

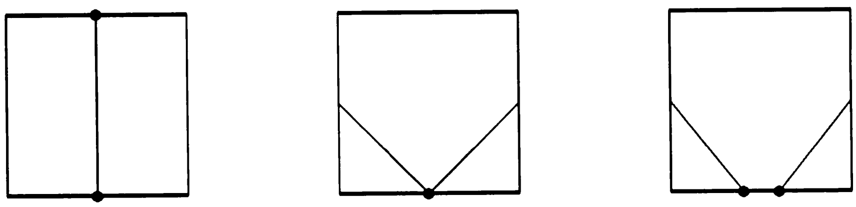


Figure 5: Unavoidable submaps on the cylinder and the Möbius band

Theorem 6.2 *Every reduced map on the Möbius band has a convex representation on the flat Möbius band.*

The proof is the same as for Theorem 6.1. The only difference is that we have three unavoidable submaps shown in Figure 5(a), (b), and (c). Note that the side identifications are meant to give the Möbius band in this case.

Linear time algorithms for the case of the cylinder and the Möbius band are similar as in the case of the torus or the Klein bottle. We omit the details.

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