COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 15 (4) (1995) 541–556

UNIQUENESS AND MINIMALITY OF LARGE FACE-WIDTH EMBEDDINGS OF GRAPHS

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Received July 15, 1994

Let G be a graph embedded in a surface of genus g. It is shown that if the face-width of the embedding is at least $c\log(g)/\log\log(g)$, then such an embedding is unique up to Whitney equivalence. If the face-width is at least $c\log(g)$, then every embedding of G which is not Whitney equivalent to our embedding has strictly smaller Euler characteristic.

1. Introduction

All graphs in this paper are undirected, finite and simple. We follow standard terminology as used, for example, in [2]. A subgraph C of a graph G is *induced* if every pair of non-adjacent vertices in C is also non-adjacent in G. It is *non-separating* if $G-V(C)$ is connected.

Embeddings of graphs in the plane are well understood thanks to the following results:

(A) (Whitney [9]) Every 3-connected planar graph has essentially unique embedding in the plane. (This means that face boundaries and local rotations are uniquely determined.)

(B) (Whitney [9]) If G is a 2-connected planar graph, then any two embeddings of G in the plane are Whitney equivalent. (One can be obtained from the other by a sequence of simple local re-embeddings. See, e.g. [4] for definition of *Whitney-equivalence*.)

(C) (Folklore) If G is a graph that is embedded in the plane, then all its face boundaries are cycles of G if and only if G is 2-connected.

(D) (Tutte [8]) If G is a 3-connected graph embedded in the plane, then the face boundaries are precisely all induced non-separating cycles of G (and

Mathematical Subject Classification (1991): 05 C 10

* Supported in part by the Ministry of Science and Technology of Slovenia, Research Project P1-0210-101-94.

conversely). In such a case, any pair of facial cycles are either disjoint or they intersect in a vertex or an edge.

These results can be generalized to graphs embedded in general surfaces by introducing the face-width of an embedding as defined below. We will consider only 2-cell embeddings in closed surfaces. They can be described in a purely combinatorial way by specifying:

(1) A *rotation system* $\pi=(\pi_v;v\in V(G))$; for each vertex v of the given graph G we have a cyclic permutation π_v of edges incident with v, representing their circular order around v on the surface.

(2) A *signature* $\lambda:E(G)\to\{-1,1\}$. Suppose that $e=uv$. Following the edge e on the surface, we see of the local rotations π_v and π_u are chosen consistently or not. If yes, then we have $\lambda(e)=1$, otherwise we have $\lambda(e)=-1$.

The reader is referred to [3] for more details. We will use this description as a definition: An *embedding* of a graph G is a pair $\Pi=(\pi,\lambda)$ where π is a rotation system and λ is a signature. Having an embedding Π of G, we say that G is Π-*embedded*. A cycle with an odd number of edges e having $\lambda(e)=-1$ is said to be Π-*onesided*. Other cycles are Π-*twosided*.

Given an embedding $\Pi=(\pi,\lambda)$, an *angle* of Π is any pair of edges $\{e,\pi_v(e)\}$ where $v\in V(G)$ and e is an edge incident to v. The cyclic sequence e, $\pi_v(e)$, $\pi_v^2(e)$, $\pi_v^3(e)$, ... is called Π-*clockwise ordering* around v. We define Π-*facial walks* as closed walks in the graph which are determined by the following process, called the *face traversal* procedure. It starts with an arbitrary angle, say $\{e_1,e_2\}$, where $e_2=\pi_v(e_1)$. Initially, we use Π-clockwise ordering around vertices when selecting the next edge on the facial walk that we traverse (like we did when selecting e_2 after e_1). Every time when we traverse an edge e with $\lambda(e)=-1$, we will change to the Π-anticlockwise ordering (or back to Π-clockwise if it was Π-anticlockwise). Starting at v with $\{e_1,e_2\}$, we first traverse the edge $e_2=vu$. Arriving to its other end u, we select the angle $\{e_2,e_3\}$ where $e_3=\pi_u(e_2)$ is the next edge in the Π-clockwise order around u if we still use the Π-clockwise ordering. Then we continue the traversal along the edge e_3. If we use Π-anticlockwise ordering, then we select the angle $\{e_2,e_3\}$, $e_3=\pi_u^{-1}(e_2)$, and proceed with the traversal along the edge e_3. Continuing the traversal in the same way, we obtain a closed walk which stops when we reach our initial angle $\{e_1,e_2\}$. This closed walk is said to be Π-facial. All other Π-facial walks are determined in the same way by starting with other angles. They correspond, bijectively, to faces of the corresponding topological embedding. Two embeddings are *equivalent* if they have same facial walks.

Let $F(\Pi,G)$ be the set of Π-facial walks. The number

$$\gamma(\Pi)=2-|V(G)|+|E(G)|-|F(\Pi,G)|$$

will be called the *characteristic* of the embedding Π. Note that it is closely related to the genus and to the negative value of the Euler characteristic of the surface of the underlying topological embedding. It is known that $\gamma(\Pi)\geq 0$ and that $\gamma(\Pi)=0$ if and only if Π corresponds to an embedding of G in the plane.

If Π is an embedding of a graph G and H is a subgraph of G, then the *induced embedding* of H, which we will denote by $\Pi|H$, is obtained from that of G by ignoring all edges in $E(G)\setminus E(H)$ and by restricting λ to $E(H)$. More precisely, if $e=uv\in E(H)$, then the successor of e in the clockwise ordering around v is the first edge in the sequence $\pi_v(e)$, $\pi_v^2(e)$, ... which is in H. It is easy to see that $\gamma(\Pi|H)\le\gamma(\Pi)$.

If G is a Π-embedded graph and C is a Π-twosided cycle of G, then we define the *left graph* and the *right graph* of C as follows. Select a vertex $v\in V(C)$, and let e and e' be the edges of C incident with v. If $e'=\pi_v^k(e)$, then all edges e, $\pi_v(e)$, $\pi_v^2(e)$, ..., $\pi_v^k(e)$ are said to be on the left side of C. As in the face tracking procedure, we will determine left edges at every vertex of C by traversing C edge by edge. After traversing an edge f of C with $\lambda(f)=-1$, we change clockwise orientation to anticlockwise, and vice versa. In particular, traversing the edge $e'=vu$ from v to u, the left edges at u are e', $\pi_u(e')$, $\pi_u^2(e')$, ..., $\pi_u^l(e')$ (where $\pi_u^l(e')\in E(C)$) if we have the clockwise orientation. On the other hand, having the anticlockwise orientation, the left edges are $\pi_u^l(e')$, $\pi_u^{l+1}(e')$, ..., e'. Since C is Π-twosided, the orientation is again clockwise when we come back to the initial vertex v after traversing the entire cycle C. An edge e which is not incident with C is said to be on the left side of C if it is connected by a path in $G-C$ to an end of an edge on the left side of C (and incident with C). Now the left graph $G_l=G_l(\Pi,C)$ is defined as the graph induced by all edges on the left of C. The right graph $G_r=G_r(\Pi,C)$ is defined analogously.

Let C be a Π-twosided cycle and G_l and G_r its left and right graph. If $G_l\cap G_r=C$, then C is said to be Π-*bounding*. An easy count shows that in such a case

$$\gamma(\Pi|G_l)+\gamma(\Pi|G_r)=\gamma(\Pi). \tag{1}$$

If $\gamma(\Pi|G_l)=0$ or $\gamma(\Pi|G_r)=0$, then C is a Π-*contractible* cycle. In particular, every Π-facial cycle is Π-contractible. If C is Π-contractible and $\gamma(\Pi|G_l)=0$, then we call the subgraph $G_l-E(C)$ the Π-*interior* of C and denote it by $\mathrm{int}(\Pi,C)$. We also write $\mathrm{Ext}(\Pi,C)=G_r$. Similarly if $\gamma(\Pi|G_r)=0$. By (1), $\mathrm{int}(\Pi,C)$ and $\mathrm{Ext}(\Pi,C)$ are well defined if $\gamma(\Pi)\neq0$.

The same notations as above can be introduced for Π-onesided cycles by defining that such a cycle C is always Π-nonbounding and Π-noncontractible.

Let G be a Π-embedded graph and let F be a Π-facial walk with angles $\{e,f\}$ at vertex u and $\{g,h\}$ at vertex v. Add the edge uv to G and extend the embedding so that uv is inserted in π_u between e and f and in π_v between g and h. If the product of signatures on a segment of F from $\{e,f\}$ to $\{g,h\}$ is 1, then we set $\lambda(uv)=1$, and otherwise $\lambda(uv)=-1$. We denote the obtained embedding of $G+uv$ again by Π. The performed operation is called a *face splitting* at u and v.

If $F_0,\dots,F_{k-1}$ $(k\ge1)$ are distinct Π-facial walks and $v_0,\dots,v_{k-1}$ are distinct vertices of G such that v_i and v_{i+1} (index modulo k) are both in F_i $(i=0,\dots,k-1)$, then we can add to G a cycle $C=v_0v_1\dots v_{k-1}$ by a sequence of face splittings at vertices v_i, v_{i+1}, $i=0,\dots,k-1$. The smallest integer $k\geq 1$ such that there are Π-facial walks $F_0,\dots,F_{k-1}$ and vertices $v_0,\dots,v_{k-1}$ for which the corresponding cycle C is Π-noncontractible is called the *face-width* (or *representativity*) of Π and denoted by $\mathrm{fw}(\Pi)$. With this notation, generalizations of (C) and (D) can be expressed as follows [5]:

(C') If G is a Π-embedded graph, then all Π-facial walks are cycles of G if and only if G is 2-connected and $\mathrm{fw}(\Pi)\geq 2$.

(D') If G is a Π-embedded graph, then the Π-facial walks are induced nonseparating cycles of G if and only if G is 3-connected and $\mathrm{fw}(\Pi)\geq 3$. In such a case, any two Π-facial cycles are either disjoint or they intersect in a vertex or an edge.

Properties (A) and (B) can also be generalized. Robertson and Vitray [5] proved that if G is a 3-connected graph embedded in a surface of genus g with face-width greater than $2g+2$, then such an embedding is unique and necessarily a minimal genus embedding (either orientable, or non-orientable). This result has been slightly improved by Mohar [4] who replaced the genus of Π in the bound by the minimal genus of an embedding of G, and who also observed that in the non-3-connected case embeddings with large face-width are unique up to Whitney equivalence. It turns out that instead of minimizing the genus, it is more convenient to minimize the characteristic of the surface. Then the distinction between the orientable and non-orientable case disappears and stronger results are obtained. We say that an embedding Π of G is *minimal* if $\gamma(\Pi)$ is minimal among all embeddings of G. Let Π be an embedding of G which satisfies certain property. Then Π is said to be *unique* embedding with this property if every embedding of G with the same property is equivalent to G.

The purpose of this paper is to improve above mentioned results by considerably weakening the assumptions on the face-width. We will show that embeddings Π whose face-width is larger than $c\log(\gamma(\Pi))/\log\log(\gamma(\Pi))$ (where c is some small constant) are unique up to Whitney equivalence (Theorem 5.4). Moreover, embeddings with $\mathrm{fw}(\Pi)\geq c\log(\gamma(\Pi))$ are also characteristic minimal (Theorem 6.1). On the other hand, examples constructed by Archdeacon [1] show that our bounds are not too far from the best possible bounds on the face-width which guarantee uniqueness.

It came to our attention that some time earlier than this paper has been completed, Seymour and Thomas [6] obtained results similar to ours. In particular, they present an improvement of our Theorem 6.1 by showing that $\mathrm{fw}(\Pi)\geq 100\log(\gamma(\Pi))/\log\log(\gamma(\Pi))$ already implies minimality of embeddings (for 3-connected graphs). This result also implies a uniqueness result in flavor of out Theorem 5.4. On the other hand, Theorem 5.4 has simpler proof and it considerably improves the constants in bounds of Seymour and Thomas.

2. Some preliminary lemmas

In this section we assume that G is a Π-embedded graph and that $\gamma(\Pi) > 0$. Our first lemma shows that we can restrict our attention to 3-connected graphs if we are interested only in embeddings with $\mathrm{fw}(\Pi) \geq 3$.

Lemma 2.1. ([5, 4]) *There exists a unique 3-connected block G_0 of G such that $\mathrm{fw}(\Pi'|G_0) = \mathrm{fw}(\Pi')$ for every embedding Π' of G with $\mathrm{fw}(\Pi') \geq 3$. If Π' and Π'' are two such embeddings which coincide on G_0, then they are Whitney equivalent. In particular, all other 3-connected blocks of G are planar.*

An immediate consequence of Lemma 2.1 is the fact that minimal embeddings of G and G_0 are the same [4, Proposition 3.1].

The next result is easy to see.

Lemma 2.2. *If C is an induced nonseparating cycle of G, then C is either Π-facial of Π-nonbounding.* ∎

Cycles C_1 and C_2 of G are Π-*crossing* if either

(i) $C_1 \cap C_2$ is a vertex v and the edges incident with v, e_1, f_1 of C_1 and e_2, f_2 of C_2, respectively, appear in π_v in the interlaced order, say e_1, e_2, f_1, f_2, or

(ii) $C_1 \cap C_2$ is an edge $e = uv$ and the following holds. Suppose that $e_i \neq e$ is the edge on C_i incident with u and that $f_i \neq e$ is the edge of C_i incident with v $(i = 1, 2)$. If the order of e_1, e_2 and e in π_u is e_1, e_2, e then the order of f_1, f_2, e in π_v is f_1, f_2, e (if $\lambda(e) = 1$), or f_2, f_1, e (if $\lambda(e) = -1$).

For further reference we state the following obvious result:

Lemma 2.3. *If C_1 and C_2 are Π-crossing cycles, then they are both Π-nonbounding. In particular, they are Π-noncontractible.*

Let $\mathscr{C}$ be a non-empty set of disjoint cycles of G and let $C \subseteq G$ be the union of all cycles from $\mathscr{C}$. Then $\mathscr{C}$ is said to be Π-*bounding* if G can be written as $G = G_l \cup G_r$ such that $G_l \cap G_r = C$ and such that every cycle from $\mathscr{C}$ is $(\Pi|G_l)$-facial and $(\Pi|G_r)$-facial. (In particular, every cycle in $\mathscr{C}$ is Π-twosided.) The next lemma is taken from [4].

Lemma 2.4. *Let $\mathscr{C}$ be a set of disjoint cycles of G. If every subset of $\mathscr{C}$ is Π-nonbounding, then*

$$\gamma(\Pi) - \gamma(\Pi|(G - \mathscr{C})) \geq 2|\mathscr{C}| - k$$

where k is the number of Π-onesided cycles in $\mathscr{C}$. In particular, $\gamma(\Pi) \geq 2|\mathscr{C}| - k$.

Lemma 2.5. *Let $\mathscr{C}$ be a set of cycles of G. If every cycle $C \in \mathscr{C}$ is Π-crossing with at most r and with at least one of the cycles from $\mathscr{C}$ and is disjoint from all other cycles in $\mathscr{C}$, then*

$$\gamma(\Pi) \geq |\mathscr{C}|/(r+1).$$

Proof. We will select pairs of cycles $\mathscr{C}_i = (C_i, C_i')$, $i = 1, 2, \ldots, k$ with the following properties:

(a) $C_1, \ldots, C_k$ are pairwise disjoint.

(b) If $1 \le i < j \le k$ then C_i' is disjoint from C_j.

(c) For $i = 1, \ldots, k$, if C_i is Π-onesided, then $C_i' = C_i$, Otherwise, C_i' is Π-crossing with C_i.

Such pairs are obtained as follows. Suppose that we have already selected $\mathscr{C}_1, \ldots, \mathscr{C}_{i-1}$, where $i \ge 1$. Denote by n_1 the number of Π-onesided cycles among $C_1, \ldots, C_{i-1}$ and let n_2 be the number of Π-twosided cycles. Then we choose for C_i an arbitrary cycle from $\mathscr{C}$ that is disjoint from $C_1, \ldots, C_{i-1}$ and from $C_1', \ldots, C_{i-1}'$. By our assumptions, at least

$$|\mathscr{C}| - 2rn_2 - (r+1)n_1 \ge |\mathscr{C}| - (r+1)(2n_2 + n_1) \tag{2}$$

cycles from $\mathscr{C}$ are at our disposal. After selecting C_i, let C_i' be an arbitrary cycle from $\mathscr{C}$ satisfying (c). It is clear that the obtained pairs $\mathscr{C}_1, \ldots, \mathscr{C}_i$ satisfy (a)–(c).

Suppose now that a subset of $\{C_1, \ldots, C_k\}$ is Π-bounding. Let C_i be a cycle in this subset with the smallest index i. Clearly, C_i is Π-twosided. By (a)–(c), C_i' and C_i are Π-crossing but C_i' is disjoint from all other cycles in our separating family. A contradiction. By selecting as many pairs $\mathscr{C}_i$ as possible, the inequality (2) shows that $2n_2 + n_1 \ge |\mathscr{C}|/(r+1)$. Consequently, an application of Lemma 2.4 completes the proof. ∎

We will use another result which implies large characteristic of an embedding.

Lemma 2.6. *Let $\mathscr{C}$ be a set of disjoint Π-noncontractible cycles of G. If the union of cycles in $\mathscr{C}$ is an induced and nonseparating subgraph of G, then*

$$\gamma(\Pi) - \gamma(\Pi|(G - \mathscr{C})) \ge 2|\mathscr{C}| - k$$

where k is the number of Π-onesided cycles in $\mathscr{C}$.

Proof. No subset of the cycles can be Π-bounding. Consequently, Lemma 2.4 applies. ∎

We will also use the following result which can be proved easily:

Lemma 2.7. *If C is a Π-contractible cycle of G, then every cycle in $C \cup \mathrm{int}(\Pi, C)$ is Π-contractible.* ∎

If $X \subseteq V(G)$, then an *X-component* is either an edge with both ends in X or a connected component L of $G - X$ together with all edges between L and X.

Lemma 2.8. *Let G be a Π-embedded graph. Suppose that X is a separating set of vertices of G such that $|X| < \mathrm{fw}(\Pi)$ and such that for any separating sets X_1, $X_2 \subseteq X$ with $X_1 \cup X_2 = X$ we have $|X_1 \cap X_2| \ge 2$. Then $G = G_1 \cup G_2$ where $G_1 \cap G_2 = X_1 \subseteq X$ such that:*

(i) *X_1 is an induced and nonseparating set in G_1, i.e., G_1 is a single X_1-component in G.*

(ii) *By face splittings one can add to G a Π-contractible cycle C such that $V(C) \subseteq X_1$ and such that $G_2 = \mathrm{int}(\Pi, C)$. In particular, G_2 contains no Π-noncontractible cycles.*

Proof. Let B be an arbitrary X-component and let $Y \subseteq X$ be the set of vertices incident with edges from B and with edges that are not from B. Each $y \in Y$ is contained in at least two Π-facial walks whose angle at y contains an edge of B and an edge from $E(G) \setminus E(B)$. (Such angles are said to be *mixed*.) On the other hand, every such Π-facial walk contains at least two vertices of Y with mixed angles. Choose such a facial walk F_1 and vertices v_1, $v_2 \in Y \cap V(F_1)$ such that one of the segments of F_1 from v_1 to v_2 is contained in B. Let F_2 be another such Π-facial walk containing v_2 and such that in the local rotation π_{v_2} only edges of B appear between the angles of F_1 and F_2. By continuing in the same way, we get a sequence of distinct Π-facial walks $F_1, F_2, \ldots, F_t$ and distinct vertices $v_1, v_2, \ldots, v_t$ $(t \geq 2)$ such that v_i and v_{i+1} are in F_i for $i = 1, \ldots, t-1$. Moreover, in F_t we have a vertex v_{t+1} which is also in some F_l, $1 \leq l < t$. We may assume that $l = 1$ and that $v_{t+1} = v_1$. By face splittings we can add a cycle $C = v_1 v_2 \ldots v_t v_1$ and since $t \leq |X| < \mathrm{fw}(\Pi)$, C is Π-contractible. Hence, $X_1 = \{v_1, \ldots, v_t\}$ is a separating set of G. By construction, only edges of B are on the left side (say) of C at every vertex v_i, $2 \leq i \leq t$. If there is an edge $e \notin E(B)$ incident to v_1 that is on the left side of C, then $X_2 = (X \setminus X_1) \cup \{v_1\}$ is a separating set of G. By our assumption on X, this is not possible, and thus B is the only X-component that is on the left side of C. If B contains a Π-noncontractible cycle, then by Lemma 2.7 we have $B \subseteq \mathrm{Ext}(\Pi, C)$. Then all other X-components are contained in $\mathrm{int}(\Pi, C)$, and their union does not contain Π-noncontractible cycles.

It remains to show that at least one of the X-components contains a Π-noncontractible cycle. It follows from assumptions on X that $|X| \geq 2$. Hence $\mathrm{fw}(\Pi) \geq 3$ and thus $\gamma(\Pi) > 0$ [5]. Therefore, G contains a Π-noncontractible cycle. We leave it to the reader to show that one of such cycles is contained in a single X-component. A similar approach as above can be used. ∎

3. Local changes

Let $\Pi = (\pi, \lambda)$ be an embedding of a graph G and suppose that $v \in V(G)$ is a vertex of degree d. If $\pi_v = (e_1 e_2 \ldots e_d)$, let F_j be the Π-facial walk containing the angle $\{e_j, e_{j+1}\}$ (index modulo d), $j = 1, \ldots, d$.

Lemma 3.1. *Let $\Pi' = (\pi', \lambda)$ be an embedding of G which differs from Π only at π_v such that $\pi'_v = (e_{s+1} \ldots e_t e_1 \ldots e_s e_{t+1} \ldots e_d)$ for some s and t, $1 \leq s < t < d$. Then $|\gamma(\Pi') - \gamma(\Pi)| \leq 2$.*

Proof. The face tracking procedure shows that the only Π-facial walks that are affected by changing Π to Π' are F_d, F_s, and F_t. The claim is then obvious since the number of faces changes by at most 2. ∎

Lemma 3.2. *Let $\Pi' = (\pi', \lambda')$ be an embedding of G which differs from Π only at v such that $\pi'_v = (e_s e_{s-1} \dots e_1 e_{s+1} \dots e_d)$, and $\lambda'(e) = -\lambda(e)$ if $e \in \{e_1, \dots, e_s\}$ and $\lambda'(e) = \lambda(e)$ otherwise. Then $|\gamma(\Pi') - \gamma(\Pi)| \leq 1$. If $F_d = F_s$, then $\gamma(\Pi') = \gamma(\Pi)$.*

Proof. From the face tracking procedure we see that only the Π-facial walks F_d and F_s are changed. The claim of then obvious. ∎

Suppose that C is a cycle of G. Define $w(\Pi, C)$ as follows. If C is Π-facial, then $w(\Pi, C) = 0$. Otherwise, $w(\Pi, C)$ is the smallest number of segments of Π-facial walks whose union is C. A simple corollary to above local re-embedding lemmas is the following result.

Corollary 3.3. *Suppose that Π is an embedding of a graph G and that C is a cycle of G. Then there is an embedding Π' of G such that C is Π'-facial and such that*

$$\gamma(\Pi') \leq \gamma(\Pi) + 2w(\Pi, C).$$

Proof. By induction on $w = w(\Pi, C)$. If $w = 0$, then $\Pi' = \Pi$ will do. For the induction step we can use Lemma 3.1 or Lemma 3.2. Appropriate application of these lemmas decreases w by 1 and increases the characteristic of the embedding by at most 2 or 1, respectively. ∎

Lemma 3.4. *Let u, $v \in V(G)$. Consider an angle $\{e, f\}$ at u and an angle $\{g, h\}$ at v. Identify u and v into a single vertex w and define an embedding $\Pi' = (\pi', \lambda)$ of the obtained graph G' so that π' coincides with π except that $\pi'_w = (ee' \dots fgg' \dots h)$ where $\pi_u = (ee' \dots f)$ and $\pi_v = (gg' \dots h)$. If the angles $\{e, f\}$ and $\{g, h\}$ are on the same Π-facial walk W, then either $\gamma(\Pi') = \gamma(\Pi)$ (if $W = ef \dots gh \dots$), or $\gamma(\Pi') = \gamma(\Pi) + 1$ (if $W = ef \dots hg \dots$). Otherwise, $\gamma(\Pi') = \gamma(\Pi) + 2$.*

Proof. If $\{e, f\}$ or $\{g, h\}$ appear on $W = efW_1ghW_2$, then W gives rise to two Π'-facial walks. If $W = efW_1hgW_2$, then $\lambda(W_1) = \lambda(W_2) = -1$ and hence W changes into the Π'-facial walk $W' = ehW_1^{-1}fgW_2^{-1}$. All other facial walks remain unchanged. Since $|V(G')| = |V(G)| - 1$, the change of the characteristic is either 0 or 1, respectively. If the angles are on distinct Π-facial walks W_1 and W_2, then they give rise to a single Π'-facial walk, and all other facial walks remain the same. The characteristic thus increases by 2. ∎

4. Comparing non-equivalent embeddings

In this section we will assume that G is a 3-connected graph with non-equivalent embeddings Π and Π' whose face-widths are $k = \mathrm{fw}(\Pi)$ and $k' = \mathrm{fw}(\Pi')$, respectively. We will also assume that $k \geq 3$.

A cycle of G is (Π,Π')-*unstable* if it is Π-facial and Π'-nonfacial. Let C be a (Π,Π')-unstable cycle and let $D_1, \ldots, D_t$ be (Π,Π')-unstable cycles that Π'-cross with C. Such cycles are called a (Π,Π')-*daisy* of size t centered at C if they are pairwise disjoint at C, i.e., $D_i \cap D_j \cap C = \emptyset$ for any $1 \leq i < j \leq t$. If $D_1, \ldots, D_t$ form a daisy, we shall implicitly assume that they are enumerated according to the (cyclic) order of their intersection with C. Since G is 3-connected and $k \geq 3$, (D') shows that this order is uniquely determined (up to cyclic shifts and reflections).

Lemma 4.1. *Let C be a (Π,Π')-unstable cycle. Then there is a (Π,Π')-daisy centered at C that is of size $w-1$, where $w = w(\Pi',C)$. If C' is a (Π,Π')-unstable cycle that Π'-crosses C, then the daisy can be chosen such that it contains C'.*

Proof. Denote by M the maximal size of a (Π,Π')-daisy centered at C. We claim that $M = w$ or $M = w-1$.

Let us first assume that there is a vertex $v \in V(C)$ such that the two edges of C incident to v are not consecutive in π'_v. Then there is a Π-facial cycle D that is Π'-crossing with C and such that $D \cap C = \{v\}$. By Lemma 2.3, D is (Π,Π')-unstable. Traverse C starting at v. The Π'-facial segments on C on one or the other side of C (as seen during this traversal) are called *left* and *right facial segments* on C, respectively. Let Q be the bipartite graph whose vertices are the left and the right facial segments of C and whose edges correspond to segments sharing an edge of C. Each edge of C is in a left and in a right facial segment and there is a bijection between $E(C)$ and $E(Q)$. Suppose that e, $f \in E(C)$ and that the corresponding edges of Q do not have common endvertices. Then there are vertices u_l, u_r between e and f on C such that G has an edge incident with u_l that is on the left of C and an edge incident with u_r that is on the right of C. This implies that there is a (Π,Π')-unstable cycle that crosses C and intersects C only between e and f. It follows that every matching R in Q determines a set of $|R|$ (Π,Π')-unstable cycles that Π'-cross C and that are pairwise disjoint at C (and *vice versa*). Similarly, a vertex cover of Q determines a set of Π'-facial segments of C which cover C (and *vice versa*). By the König–Egerváry Theorem [2] it follows that $M = w$.

Suppose now that for each $v \in V(C)$, edges of C incident to v are π'_v-consecutive. Suppose that C contains an edge $e = uv$ such that no (Π,Π')-unstable cycle which Π'-crosses C contains e. We may assume that there is a (Π,Π')-unstable cycle D through v which is Π'-crossing with C. In this case we use the same proof as above with the only difference that the edge of $D \cap C$ does not contribute to adjacency in the graph Q. (Now a cover of C with Π'-facial segments induced by a vertex cover of Q may not contain the edge of $D \cap C$. But the segment containing e

can be changed so that all of C is covered.) Similarly if C is Π'-twosided (when D is not needed at all). In all these cases we get a (Π,Π')-daisy of size w. By inserting C' and removing at most two cycles which intersect C' we get a (Π,Π')-daisy as claimed by the lemma.

It remains to consider the case when C is Π'-onesided, of odd size with consecutive edges $e_1, e_2, \ldots, e_{2r+1}$ and Π'-facial segments on C equal to e_1e_2, $e_2e_3, \ldots, e_{2r+1}e_1$. Clearly, $w = r+1$, and any of the (Π,Π')-unstable cycles at C is contained in a (Π,Π')-daisy of size r centered at C. ∎

Cycles in a daisy centered at C can have non-empty intersection out of C. We will need daisies consisting of disjoint cycles. Then the following lemma will be used.

Lemma 4.2. *Let $D_1, \ldots, D_t$ be a (Π,Π')-daisy centered at C. Suppose that k, $k' \geq 4$. Then any two distinct and (cyclically) non-consecutive cycles D_i and D_j from the daisy are disjoint. Moreover, if k, $k' \geq 5$, then D_i and D_j are at distance at least 2 in G.*

Proof. Let y be a vertex of $D_i \cap C$ and z be a vertex of $D_j \cap C$. If D_i and D_j intersect, then they share a vertex $x \notin V(C)$. Add edges xy, yz, and zx by splitting faces D_i, C and D_j, respectively. Since $k \geq 4$, the obtained triangle Δ is Π-contractible. By construction, one of the segments of C between y and z is in $\mathrm{int}(\Pi,\Delta)$ and the other one is in $\mathrm{Ext}(\Pi,\Delta)$. Therefore, the Π'-noncontractible cycles D_{i-1} and D_{i+1} are not contained in the same X-component of G, where $X = \{x,y,z\}$. By Lemma 2.8 (for the embedding Π') we get a contradiction with $k' \geq 4$.

To prove the second part, note that a possible edge xx' between D_i and D_j cannot lie on C. Now we take $X = \{x,x',y,z\}$ and conclude as above by applying Lemma 2.8. ∎

5. Uniqueness

In this section we will assume that G is a 3-connected graph with nonequivalent embeddings Π and Π' whose face-widths are $k = \mathrm{fw}(\Pi)$ and $k' = \mathrm{fw}(\Pi')$, respectively.

Theorem 5.1. *Suppose that $\mathrm{fw}(\Pi) = k \geq 4$. Then we can find, for each (Π,Π')-unstable cycle C, a (Π,Π')-daisy centered at C consisting of at least $\lfloor (k'-1)/2 \rfloor$ pairwise disjoint cycles. If $k \geq 5$ and $k' \geq 5$, then the cycles are also pairwise non-adjacent. If C' is a (Π,Π')-unstable cycle that Π'-crosses C, then the daisy can be chosen so that it contains C'.*

Proof. Note that $w(\Pi',C) \geq k'$. If $k' \leq 4$, the claim follows from Lemma 4.1. Otherwise it follows from Lemmas 4.1 and 4.2. ∎

From now on we assume that $k \geq 4$ and $k' \geq 4$. We shall construct a family $\mathscr{C}$ of (Π, Π')-unstable cycles by using the following procedure. We start by taking a (Π, Π')-unstable cycle C_0. Let $\mathscr{C}_0 = \{C_0\}$ and let $\mathscr{C}_1$ be a set of $\lfloor (k'-1)/2 \rfloor$ pairwise disjoint cycles obtained by Theorem 5.1 that form a (Π, Π')-daisy centered at C_0. Having constructed $\mathscr{C}_0, \ldots, \mathscr{C}_{i-1}$ $(i \geq 2)$, we define $\mathscr{C}_i$ to be the union of daisies $\mathscr{D}(C)$, for each $C \in \mathscr{C}_{i-1} \setminus (\mathscr{C}_0 \cup \ldots \cup \mathscr{C}_{i-2})$, where $\mathscr{D}(C)$ is a (Π, Π')-daisy centered at C that contains a cycle $C' \in \mathscr{C}_{i-2}$, and consists of at least $\lfloor (k'-1)/2 \rfloor$ pairwise disjoint cycles obtained by Theorem 5.1. On C, between any two consecutive cycles of the (Π, Π')-daisy $\mathscr{D}(C)$, there is a (Π, Π')-unstable cycle that can be added to the daisy (but may intersect with other cycles). Also, note that the cycle C' is well defined as a cycle whose daisy $\mathscr{D}(C')$ contains C. Our next lemma shows that this cycle C' is unique if i is not too large.

Lemma 5.2. *If $0 < 2i < \min\{k, k'\}$, then every cycle $C \in \mathscr{C}_i \setminus (\mathscr{C}_0 \cup \ldots \cup \mathscr{C}_{i-1})$ intersects with a unique cycle $C' \in \mathscr{C}_0 \cup \ldots \cup \mathscr{C}_{i-1}$. If $2i+1 < \min\{k, k'\}$, then no two cycles from $\mathscr{C}_i$ intersect.*

Proof. Suppose that C intersects with two cycles, $Q \in \mathscr{C}_q$ and $R \in \mathscr{C}_r$ where $q < i$ and $r < i$. For any cycle D in our sets $\mathscr{C}_j$ $(j \geq 1)$ we denote by $\varphi(D)$ a cycle in $\mathscr{C}_{j-1}$ that includes D in its daisy. Then $\varphi(Q)$, $\varphi(\varphi(Q))$, $\ldots$ and $\varphi(R)$, $\varphi(\varphi(R))$, $\ldots$ determine a sequence of at most $q+r+1 \leq 2i-1$ Π-facial cycles from Q to R such that any two consecutive cycles Π'-cross. Including C, we get a sequence of at most $2i$ Π-facial cycles that (cyclically) intersect its two neighbors in the sequence. Since $2i < k$, splitting of these Π-facial cycles gives rise to a Π-contractible curve whose interior and exterior contain (Π, Π')-unstable cycles. On the other hand, since $2i < k'$, we get a contradiction by using Lemma 2.8. The details are left to the reader.

A similar proof works for the other assertion. ∎

Let $\kappa = \min\{k, k'\}$ and $\nu = \lfloor (\kappa - 4)/2 \rfloor$, $\lambda = \lfloor (k'-3)/2 \rfloor$. The above construction gives rise to a tree-like structure $\mathscr{C}_0 \cup \ldots \cup \mathscr{C}_{\nu+1}$ of (Π, Π')-unstable cycles. By Lemma 5.2, they can be taken so that their number is

$$1 + (\lambda + 1) + (\lambda + 1)\lambda + \ldots + (\lambda + 1)\lambda^{\nu} = 1 + (\lambda + 1)\frac{\lambda^{\nu+1} - 1}{\lambda - 1} \tag{3}$$

and so that each of the cycles intersects with exactly $\lambda + 1$ of other cycles in the family (if it is in $\mathscr{C}_i$ for $i \leq \nu$) or with exactly one other cycle (if it is in $\mathscr{C}_{\nu+1}$). Now we have:

Lemma 5.3. *If $k' \geq 7$ and $k \geq 4$, then*

$$\gamma(\Pi') \geq \lambda^{\nu}. \tag{4}$$

Proof. By (3) and Lemma 2.5, we have

$$\gamma(\Pi') \geq \left(1 + (1 + \lambda)\frac{\lambda^{\nu+1} - 1}{\lambda - 1}\right) / (\lambda + 2). \tag{5}$$

For $\lambda \geq 2$ we have $(\lambda+1)\lambda > (\lambda-1)(\lambda+2)$. This inequality and (5) imply that

$$\gamma(\Pi') \geq \frac{1}{\lambda+2} + \lambda^{\nu} - \frac{1}{\lambda}. \tag{6}$$

Since $\gamma(\Pi')$ is an integer, we get (4). ∎

The main result of this section is now evident:

Theorem 5.4. *Suppose that* Π *is an embedding of* G *and that* $\mathrm{fw}(\Pi) \geq k$ *where* $k \geq 7$ *is the smallest integer such that*

$$\left\lfloor \frac{k-3}{2} \right\rfloor^{\lfloor k/2 \rfloor - 2} > \gamma(\Pi). \tag{7}$$

If Π' *is an embedding of* G *whose face-width is also greater or equal to* k*, then* Π' *is Whitney equivalent with* Π*. In particular, if* G *is 3-connected, then* Π' *is equivalent with* Π*.*

Proof. By Lemma 2.1 we may assume that G is 3-connected. By exchanging the roles of Π and Π' in Lemma 5.3, we see that

$$\gamma(\Pi) \geq \lfloor (k-3)/2 \rfloor^{\lfloor k/2 \rfloor - 2}.$$

This is a contradiction. ∎

As a corollary to Theorem 5.4 we see that embeddings Π with

$$\mathrm{fw}(\Pi) \geq 5 + \frac{2\log(\gamma(\Pi))}{\log\log(\gamma(\Pi)) - \log\log\log_2(\gamma(\Pi))} \tag{8}$$

are unique embeddings with so large face-width (up to Whitney equivalence), with possible exceptions when the right hand side of (8) is smaller than 7.

It is worth mentioning that examples due to Archdeacon [1] show that our bounds are not too far from the best possible bounds on the face-width which guarantee uniqueness.

6. Minimality

It was shown by Robertson and Vitray [5] that an embedding Π with $\mathrm{fw}(\Pi) \geq \gamma(\Pi) + 3$ is not only a unique such embedding but it is also genus minimal. This has been slightly improved by Mohar [4] so that the genus of Π is replaced by the minimal possible characteristic of an embedding of the graph. In this section we strengthen this result to get a logarithmic instead of the linear bound. The proof of uniqueness employed in Section 5 cannot be used in this case since we do not have another large face-width embedding to compare it with Π. This will result in a slightly weaker bound. Instead of the $O(\log(\gamma(\Pi))/\log\log(\gamma(\Pi)))$ lower bound of Theorem 5.4, we will obtain only a bound of order $O(\log(\gamma(\Pi)))$.

The main result of this section is:

Theorem 6.1. *Suppose that Π is an embedding of a graph G. Let $\gamma' = \gamma_{\min}(G)$ be the smallest characteristic of an embedding of G. If $\gamma' \geq 9$ and*

$$\text{fw}(\Pi) > \frac{6}{\log(9/8)} \log(\gamma' - 8) + 50 \tag{9}$$

then $\gamma(\Pi) = \gamma'$, i.e., Π is minimal. Moreover, any embedding of G of characteristic γ' is Whitney equivalent to Π.

If $\gamma' < 9$, then the bound $\text{fw}(\Pi) \geq 19$ already assures uniqueness and minimality.

A slightly better estimate (asymptotically) will also be obtained. If

$$\text{fw}(\Pi) > \frac{2\log(\gamma' - c_1)}{\log(\alpha)} - c_2 \tag{10}$$

where c_1 and c_2 are suitable constants and $\alpha > 1.04$ is the real root of the polynomial $x^3 - x/8 - 1$, then the embedding Π is minimal and it is also a unique minimal embedding up to Whitney equivalence.

The rest of this section is devoted to the proof of Theorem 6.1. By Lemma 2.1 we may assume that G is 3-connected. Let Π' be a minimal embedding of G that is not equivalent to Π. Then there is a (Π, Π')-unstable cycle C_0. It is shown in [4, Corollary 2.2] that there are cycles $C_0, C_1, \ldots, C_{\tau+1}$ where $\tau = \lfloor (k-3)/2 \rfloor$, with the following properties:

(a) $C_0, \ldots, C_{\tau+1}$ are pairwise disjoint and each of them is Π-contractible and Π'-noncontractible.

(b) For $t = 0, 1, \ldots, \tau$, $C_0 \cup \ldots \cup C_t \subseteq \text{int}(\Pi, C_{t+1})$.

(c) For $t = 0, \ldots, \tau$, C_t is an induced and nonseparating cycle of $\text{Ext}(\Pi, C_t)$.

(d) No subset of $\{C_0, \ldots, C_\tau\}$ is Π'-bounding.

Define $\beta(t)$ to be the largest number of pairwise disjoint (Π, Π')-unstable cycles contained in $\text{int}(\Pi, C_t) \cup C_t$ whose union in G is induced and nonseparating. Our goal is to show that $\beta(t)$ grows exponentially.

Let us now fix some t, $1 \leq t < \tau$. Denote by $G_t = \text{Ext}(\Pi, C_t)$ and let Π_t, Π'_t be the restrictions of Π and Π' to G_t, respectively. It is worth mentioning that property (c) implies that G_t is 3-connected up to possible vertices of degree 2 in C_t. We also select an embedding Π''_t of G_t such that $\gamma(\Pi''_t) \leq \gamma(\Pi'_t)$ and such that, according to this condition, the number of (Π_t, Π''_t)-unstable cycles of G_t is as small as possible.

By Lemma 2.6 we get

$$\gamma(\Pi''_t) \leq \gamma(\Pi'_t) \leq \gamma(\Pi') - \beta(t-1). \tag{11}$$

We will try to change Π''_t so that C_t will become a facial cycle. By Lemma 3.3, there is an embedding Π^0_t of G_t where C_t is facial and such that

$$\gamma(\Pi^0_t) \leq \gamma(\Pi''_t) + 2w(\Pi''_t, C_t) \leq \gamma(\Pi') + 2w(\Pi''_t, C_t) - \beta(t-1). \tag{12}$$

In the second inequality we have used (11). Clearly, Π_t^0 can be extended to an embedding of G of the same characteristic. By minimality of Π' and (12) we get

$$2w(\Pi_t'', C_t) \geq \beta(t-1). \tag{13}$$

By Lemma 2.2, C_t is Π_t''-noncontractible and hence (Π_t, Π_t'')-unstable. From (13) and Lemma 4.1 we get a (Π_t, Π_t'')-daisy $D_1, D_2, \ldots, D_r$ centered at C_t that has $r \geq \lceil \beta(t-1)/2 \rceil - 1$ cycles. We claim that only (cyclically) consecutive cycles in the daisy can intersect. Suppose that this is not the case. Let $y \in D_i \cap C_t$, $z \in D_j \cap C_t$, $x \in D_i \cap D_j$ be vertices as in the proof of Lemma 4.2 and suppose that i and j are not consecutive. Since $D_i \cup D_j \cup C_t \subseteq C_{t+1} \cup \operatorname{int}(\Pi, C_{t+1})$, splittings of faces D_i, D_j, and C_t determine a Π_t-contractible triangle R whose Π_t-interior $Q = \operatorname{int}(\Pi_t, R)$ contains one of the cycles from the daisy since i and j are not consecutive. Choose a vertex $w \in Q - \{x,y,z\}$ of degree in G_t at least 3, and join it in G_t by three internally disjoint paths to $\{x,y,z\}$. (This is possible since G_t is "almost" 3-connected.) Denote by T the union of the paths. The embedding of Π_t'' restricted to $(G_t \setminus Q) \cup T$ can be extended to an embedding of G_t by embedding Q in the same way as under Π_t. The characteristic of the obtained embedding of G_t is the same as the characteristic of the restricted embedding, hence at most $\gamma(\Pi_t'')$. Moreover, it has strictly fewer (Π_t, Π_t'')-unstable cycles since a (Π_t, Π_t'')-unstable cycle from the daisy contained in Q became facial. On the other hand, no cycle which is Π_t''-facial and Π_t-facial became nonfacial since, in $(G_t \setminus Q) \cup T$, only the Π_t''-facial cycles containing edges of Q can change. This contradicts our choice of Π_t''.

Next, we claim that for any two cycles D_i, D_j $(1 \leq i < j \leq r)$ from the daisy such that $4 \leq j - i \leq r - 4$, the only Π_t-facial cycle that intersects with D_i and D_j is C_t. Assuming that this is not the case, let C' be another such cycle. Denote by y, z, x, x' vertices of $D_i \cap C_t$, $D_j \cap C_t$, $D_i \cap C'$, and $D_j \cap C'$, respectively. After face splittings they determine a Π_t-contractible 4-cycle whose Π_t-interior Q contains at least three consecutive cycles from the daisy. We may assume that these cycles are D_{i+1}, D_{i+2}, D_{i+3}. We know that D_{i+2} is disjoint from D_i and D_j.

Let us first assume that D_{i+2} is Π_t''-twosided. Let $G_t' = (G_t - (Q - \{x, x', y, z\})) \cup D_i \cup D_j$. By Lemma 2.4, the embedding of G_t' induced by Π_t'' has characteristic $\gamma(\Pi_t'' | G_t') \leq \gamma(\Pi_t'') - 2$. In G_t', x and y are connected by a path in D_i whose all interior vertices are of degree 2. Therefore, x and y are on a common $(\Pi_t'' | G_t')$-facial walk. The same holds for x' and z. By applying Lemma 3.4 four times, we can get from $\Pi_t'' | G_t'$ an embedding of G_t of characteristic at most $\gamma(\Pi_t'')$ such that Q is embedded in the same way as under Π_t. This contradicts our choice of Π_t'' since this embedding has fewer unstable cycles than Π_t''.

If D_{i+2} is Π_t''-onesided, then we define G_t' to be the subgraph of G_t as defined above, together with a vertex $w \in V(Q) \setminus \{x, x', y, z\}$ and together with three paths in Q from w to $\{x, x', y, z\}$ that have pairwise only w in common. The paths can be chosen so that the obtained subgraph G'_t contains at most a segment from the Π''_t-onesided cycle D_{i+2}. Then we either have $\gamma(\Pi''_t|G'_t) \leq \gamma(\Pi''_t) - 1$ or the same inequality can be achieved by a simple re-embedding similar to the operation in Lemma 3.2. By applying Lemma 3.2 (if necessary), we can get an embedding of G'_t such that w is on common facial walks with any of x, y, z, x' (in the correct order), and the characteristic is increased at most by 1. Using Lemma 3.4, we can extend this embedding to an embedding of G_t of the same characteristic such that Q is embedded in the same way as under Π_t. This contradicts our choice of Π''_t.

A corollary to the above claims is that the union of induced nonseparating cycles D_1, D_5, D_9, D_{13}, ... from the daisy is an induced and nonseparating subgraph of G. The number of cycles is at least

$$\left\lfloor \frac{r}{4} \right\rfloor \geq \left\lfloor \frac{1}{4}\lceil(\beta(t-1)/2\rceil - \frac{1}{4} \right\rfloor \geq \frac{\beta(t-1)}{8} - 1. \tag{14}$$

Since the daisy $D_1, \ldots, D_r$ is contained in $C_{t+1} \cup \mathrm{int}(\Pi, C_{t+1})$, we get

$$\beta(t+1) \geq \left\lfloor \frac{r}{4} \right\rfloor + \beta(t-2) \geq \frac{\beta(t-1)}{8} + \beta(t-2) - 1. \tag{15}$$

Of course, $\beta(0) = 1$, $\beta(t)$ is nondecreasing, and by the same method as above we trivially get $\beta(t+3) \geq \beta(t) + 1$ for every $t \geq 0$. This implies that $\beta(23) \geq 8$ and $\beta(24) \geq 9$. Using this fact and (15), we see that for $t \geq 24$:

$$\beta(t) \geq \left(\frac{9}{8}\right)^{(t-23)/3} + 8 > 1.04^{t-23} + 8. \tag{16}$$

This estimate can be used up to $t = \tau$. From the definition of $\beta(\tau)$ and Lemma 2.6 we see that $\beta(\tau)$ is a lower bound for $\gamma(\Pi')$. A routine calculation shows that this implies (9).

The recursion (15) with constant coefficients can be solved exactly by standard methods. It follows that

$$\beta(t) \geq c\alpha^t + O(1) \tag{17}$$

where $\alpha > 1.04$ is the real root of the polynomial $x^3 - x/8 - 1$ and c is a constant. This implies (10).

This completes the proof of Theorem 6.1. ∎

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