Embedding graphs in the torus in linear time *

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Abstract. A linear time algorithm is presented that, for a given graph G, finds an embedding of G in the torus whenever such an embedding exists, or exhibits a subgraph Ω of G of small branch size that cannot be embedded in the torus.

1 Introduction

Let K be a subgraph of G, and suppose that we are given a (2-cell) embedding of K into a surface Σ. The *embedding extension problem* asks whether it is possible to extend the given embedding of K to an embedding of G in Σ. Every such embedding is called an *embedding extension* of K to G. An *obstruction* for embedding extensions is a subgraph Ω of $G - E(K)$ such that the embedding of K cannot be extended to $K \cup \Omega$. The obstruction is *small* if $K \cup \Omega$ is homeomorphic to a graph with a small number of edges. If Ω is small, then it is easy to verify (for example, by checking all the possibilities for the rotation systems of $K \cup \Omega$) that no embedding extension to $K \cup \Omega$ exists, and hence Ω is a good verifier that there are no embedding extensions of K to G as well. Though obstructions can be arbitrarily large, one can produce a small obstruction by changing some branches of K. Such changes are often applied in our algorithms. To indicate that the graph K might have been changed, we call a small obstruction obtained in this way a *nice obstruction.*

It is known [15] that the general problem of determining the genus, or the non-orientable genus of graphs is NP-hard. However, for every fixed surface there is a polynomial time algorithm which checks if a given graph can be embedded in the surface. Such algorithms were found first by Filotti *et al.* [3]. Unfortunately, even for the torus their algorithm has time complexity estimated only by $\mathcal{O}(n^{188})$. A special polynomial time algorithm for embedding cubic graphs in the torus has been published by Filotti [2]. Robertson and Seymour developed an $\mathcal{O}(n^3)$ algorithm using graph minors (with recent improvement by B. Reed to $\mathcal{O}(n^2 \log n)$) [12, 13, 14].

The main theorem of the present paper is:

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* The complete version of this work will appear elsewhere.

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Theorem 1. *There is a linear time algorithm that, for a given graph G, finds an embedding of G in the torus whenever such an embedding exists, or exhibits a subgraph Ω of G of small branch size that cannot be embedded in the torus.*

Moreover, an obstruction Ω obtained in Theorem 1 is homeomorphic to a minimal forbidden subgraph for embeddings in the torus, i.e., for each edge $e \in E(\Omega)$, $\Omega - e$ admits an embedding in the torus. Theorem 1 in particular implies that every such obstruction is homeomorphic to a graph with a small number of edges.

A proof of Theorem 1 is given by means of an algorithm consisting of five steps sketched below. Detailed proof of this result appears in [7, 8].

It is easy to see that we can restrict our attention to 2-connected nonplanar graphs [1]. In the paper it is shown that we can further restrict the problem to 3-connected graphs containing a Kuratowski subgraph K_0. This step uses the algorithm of Hopcroft and Tarjan [4] for determining 3-connected components of a graph and an extension of linear time planarity testing algorithms [1, 5] due to Williamson [16, 17] that finds a Kuratowski subgraph in G if G is nonplanar. Our reduction is not entirely obvious but it is much simpler than the 3-connected case considered in the sequel.

Next we try to replace the subgraph $K_0 \subseteq G$ by a graph homeomorphic to K_0, having the same main vertices and such that there are no local bridges with respect to the new subgraph. This can be achieved in linear time by using an algorithm from [6]. It is shown in the paper that if the algorithm does not yield a replacement, the 2-restricted embedding extension algorithm of [10] can be applied to find an embedding (or to discover an obstruction).

We reduce the torus embeddability problem of G (having the subgraph K_0 constructed in previous steps) to a small number of embedding extension problems. We try to extend every possible embedding of K_0 into the torus to an embedding of G. Having found an embedding in any of the cases, we get an embedding of the original graph G. On the other hand, if we are unsuccessful in each case, we combine the obtained nice obstructions $\Omega_1, \ldots, \Omega_N$ (one for each embedding extension problem) into a small obstruction for the torus embeddability of G.

Let us now fix an embedding of K_0 in the torus. Note that the embedding is 2-cell since K_0 is a Kuratowski subgraph. Then we systematically browse through all possible ways to extend this embedding. We order the possible embeddings to be tested with a partial order, which is based on appearances of basic pieces of K_0 actually used by an embedding. Furthermore, for every subset of appearances of non-singular basic pieces, we are able to select a small set of bridges attached to these appearances with the property that every their embedding that extends the considered embedding of K_0 use all the freedom that is offered by the current possible way of extending embedding of K_0. These bridges are called *representatives*. Then we embed the representatives in every possible way. Since the number of representatives is bounded, we reduce the problem of extending the (fixed) embedding of K_0 to G to a small number of subproblems. If no embedding is found, we combine obstructions from every subcase into a single nice obstruction.

The embedding of K_0 and the representatives may have no 2-singular faces. In this case, a variant of 2-restricted embedding extension algorithm [10] can be used to produce an embedding of G (or to find a nice obstruction). In the other case, it is shown that the remaining K_0-bridges can be split into two disjoint classes: those that have to be embedded into 2-singular face, and those that don't. Thus, we solve two independent problems. For bridges in the second class we simply use the 2-restricted embedding extension algorithm [10], while for the bridges from the first class we use the so called corner algorithm [7]. (Although the corner algorithm is not too complicated, its verification represents one of the major difficulties in our work.) When combining the obtained obstructions, we have to consider the fact that some of the embeddings of bridges of the second class might have been obstructed by the presence of the bridges in the first class (independently of their embedding). In this case, the combined obstruction must also include at most two paths from bridges in the first class.

This concludes the sketch of the algorithm.

As a consequence, our algorithm proves finiteness of the number of forbidden subgraphs for embeddability in the torus, a special case of Robertson and Seymour's generalized Kuratowski theorem [11].

These and other auxiliary results of this paper are used as the corner stones in the design of linear time algorithms for checking embeddability of graphs in general surfaces [9].

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