On Acyclic Colorings of Graphs on Surfaces

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Abstract

A proper k-coloring of a graph is acyclic if every 2-chromatic subgraph is acyclic. Borodin showed that every planar graph has an acyclic 5-coloring. This paper shows that the acyclic chromatic number of the projective plane is at most 7. The acyclic chromatic number of an arbitrary surface with Euler characteristic $\chi = -\gamma$ is at most $O(\gamma^{4/7})$. This is nearly tight; for every $\gamma > 0$ there are graphs embeddable on surfaces of Euler characteristic $-\gamma$ whose acyclic chromatic number is at least $\Omega(\gamma^{4/7}/(\log \gamma)^{1/7})$. Therefore, the conjecture of Borodin that the acyclic chromatic number of any surface but the plane is the same as its chromatic number is false for all surfaces with large γ (and may very well be false for all surfaces).

1. Introduction

A **coloring** of a (simple) graph G is a function from the vertices of the graph to the natural numbers (the colors). A k**-chromatic subgraph** of a graph with a coloring is a subgraph whose vertices receive at most k distinct colors. A coloring of G is **proper** if each 1-chromatic subgraph of G is edgeless. A coloring of G is **acyclic** if it is proper, and if each 2-chromatic subgraph of G is acyclic. A cycle C of a graph G is **induced** if there is no edge in G between two vertices of C which is not also an edge of C. The following alternate definition is useful: a coloring of G is acyclic if it is proper, and if it contains no 2-chromatic induced cycle.

Grünbaum [9] conjectured that planar graphs have acyclic 5-colorings, mentioning that this would imply several known results in point-arboricity. In the same paper, he showed that every planar graph has an acyclic 9-coloring. This result was steadily improved. Mitchem [12] proved planar graphs have acyclic 8-colorings, Albertson and Berman [2] acyclic 7-colorings, Kostochka [10] acyclic 6-colorings. Borodin [6] finally proved the conjecture, stated as Lemma 1 below.

Lemma 1 (Borodin). *Every planar graph has an acyclic 5-coloring.*

This bound is best possible. Grünbaum [9] demonstrated an infinite family of planar graphs which have no acyclic 4-coloring. These are the double wheels with at least six vertices, the simplest of which is the octahedron. Wegner [15] even showed a planar graph in which in every proper 4-coloring, every maximal 2-chromatic subgraph has a cycle. Grötzsch's theorem [8] says that a planar graph with no triangles has a 3-coloring. This condition does not help with acyclic colorings; Kostochka and Melnikov [11] showed a bipartite planar graph with no acyclic 4-coloring.

The **acyclic chromatic number** of G, denoted by $A(G)$, is the minimum number of colors in an acyclic coloring of G. Given a surface, the **chromatic number** (**acyclic chromatic number**) of that surface is the smallest number k such that every graph that can be embedded on that surface has a proper (acyclic) k-coloring. The Map Color Theorem (see [13]) shows that the chromatic number of an arbitrary surface with Euler characteristic $\chi = -\gamma$ is $\Theta(\gamma^{1/2})$. For surfaces other than the plane, Borodin [14] conjectured that the acyclic chromatic number equals the chromatic number. Albertson and Berman [3] proved that the acyclic chromatic number of the orientable surface of genus g is at most $4g + 4$. In this paper their bound is improved by showing that the acyclic chromatic number of an arbitrary surface with Euler characteristic $\chi = -\gamma$ is at most $O(\gamma^{4/7})$ (Theorem 3). This is nearly tight; for every $\gamma > 0$ there are graphs embeddable on surfaces of Euler characteristic $-\gamma$ whose acyclic chromatic number is at least $\Omega(\gamma^{4/7}/(\log\gamma)^{1/7})$ (Theorem 4). Therefore, the conjecture of Borodin is false for all surfaces with large γ (and may very well be false for all surfaces).

2. Connectivity

This section will show that graphs with certain separations do not need to be considered when dealing with acyclic colorings of graphs on surfaces. Let a cycle C of a graph G embedded in a surface S be **contractible** if $S \setminus C$ has one component whose complement is a closed disk D which is bounded by C; in this case let G_C be the planar graph consisting of all vertices and edges of G which are embedded in D, and let I_C, the set of interior vertices, be defined by $I_C := V(G_C) \setminus V(C)$. Otherwise, the cycle is **essential**.

Lemma 2. *Given an integer $k \geq 7$ and a minimal triangulation G of a surface S which has no acyclic k-coloring, if C is a contractible cycle of G with $|V(C)| \leq 4$ and $|I_C| > 0$, then $|V(C)| = 4$ and $|I_C| = 1$.*

Proof:

Let k, G, S, C be as in the statement of the theorem. Let the vertices of C, in its cyclic order, be $v_1, \ldots, v_j$. Clearly $j \geq 3$.

If $j = 3$, then $G - I_C$ is also a triangulation of S. Since G was minimal, $G - I_C$ has an acyclic k-coloring. Also, G_C has an acyclic 5-coloring by Lemma 1. These colorings can be permuted so that v_i is colored i for $i = 1, 2, 3$. The union of the colorings is then an acyclic k-coloring of G, since every induced cycle in G is either a cycle in $G - I_C$ or G_C. Similarly, G has no separating essential triangles. For the remainder, assume that G has no separating triangles.

If $j = 4$, then the lemma is true unless $|I_C| > 1$, thus assume so. In this case, without loss of generality through symmetry of rotation, there is no vertex of I_C adjacent to both v_1 and v_3. Note that $G - I_C + v_2v_4$ is a simple triangulation of S (since G has no separating triangles) with fewer vertices than the minimal G. Thus it has an acyclic k-coloring. Permute these colors so that v_1, v_2, v_4 are colored $6, 7, 1$ respectively, while v_3 is either colored 2 or 6. Give G_C an acyclic 5-coloring by Lemma 1, permuting the colors so that v_3 is colored 2 and v_4 is colored 1. Color G with k colors as follows: First color the vertices of $G - I_C + v_2v_4$ the colors they received previously. Then color the vertices of I_C the colors they received in the coloring of G_C. Since colors 6 and 7 do not appear in I_C, this coloring is a proper coloring. Further, this coloring is acyclic, as follows: If D is an induced cycle of G which is not a cycle of $G - I_C + v_2v_4$ or G_C, then D contains vertices of I_C, vertices of $G - I_C - C$, and either contains both v_1 and v_3 or contains both v_2 and v_4. If it contains both v_1 and v_3, then it contains a vertex colored 6, as well as two adjacent vertices of I_C, neither of which is colored 6. If it contains both v_2 and v_4, then it contains vertices colored 1 and 7, as well as a vertex of I_C adjacent to v_4, which cannot be colored either 1 or 7. Thus D has vertices of at least three different colors. ∎

For an alternate proof of Lemma 2, see [2].

3. Projective Plane Graphs

This section presents a 7-color theorem for projective planar graphs. This result may not be best possible.

Let the **representativity** of a graph G (in symbols $\text{rep}(G)$) embedded in a surface other than the sphere be the minimum number of times an essential closed curve in the surface intersects G. For a triangulation G, the representativity is equal to the number of vertices of the shortest essential cycle in G. Also note that any graph G embedded with $\text{rep}(G) \geq 3$ can be triangulated without introducing multiple edges.

Theorem 1. *Every projective planar graph has an acyclic 7-coloring.*

Proof:

Let G be a minimal (with respect to number of vertices) projective plane graph with no acyclic 7-coloring. If G is embedded with $\text{rep}(G) \leq 3$, then let C be a shortest essential closed curve of the projective plane which intersects G in at most the vertices x, y, and z. The graph $G - x - y$ is a planar graph, and thus has an acyclic 5-coloring by Lemma 1. If x and y are colored with colors 6 and 7, respectively, this clearly gives an acyclic 7-coloring of G. Thus G is embedded with $\text{rep}(G) \geq 4$.

Without loss of generality, then, G is a triangulation. Let C be a shortest essential cycle of G. Note from Lemma 2 that G has no vertices of degree at most 3, and no two adjacent vertices of degree 4. For each vertex x of C with $\deg(x) = 4$, delete x from C and replace it by one of its neighbors not previously in C. This new graph, D, will also be a shortest essential cycle of G. Note that no vertex of D has degree 4.

Note that $H = G \setminus D$ is a plane graph with one face (the exterior face) containing in its boundary all the neighbors of the vertices of D. Let z be a vertex not in H, and let K be the plane graph formed from H by adding z to the exterior face, and adding edges from z to each vertex in the boundary of the exterior face. Give K an acyclic 5-coloring by Lemma 2, such that z is colored 1.

First consider the case where $\text{rep}(G) > 4$. Now, give G an acyclic 7-coloring as follows: Color $V(H)$ with the colors they received in the coloring of K. Then choose a starting point, and color the first four vertices $1, 6, 7, 1$, and then alternately color the remaining vertices of D with 6 and 7. Clearly this is a proper coloring of G; the claim is that this is an acyclic coloring of G. Assume there is a 2-chromatic subgraph J of G which is an induced cycle. Since K had an acyclic 5-coloring, J must contain a vertex of D. Also note that, since D is a shortest essential cycle, if J has a vertex x adjacent to two vertices of D, then these two vertices are exactly distance two apart in D; also, x is not colored 1 since K had a proper coloring. If J contained two consecutive vertices of D, then, without loss of generality, it would be colored with 1 and 6. ¿From the previous statement, it would be of length four and contain three consecutive vertices of D, colored 6,1,6. But the previous statement says that the vertex of J off D cannot be colored 1, a contradiction. Since K received an acyclic coloring, exactly half of its vertices are on D. Together with the choice of the coloring of D, this shows that K cannot be essential, because it is shorter than D. If J were a contractible cycle, it must then be of length four. ¿From the assumption that D contains no vertices of degree four, this contradicts Lemma 2.

The final case is when $\text{rep}(G) = 4$. Label D with $v_1, \ldots, v_4$ so that v_1, v_3 are not adjacent. Note that, since D is a shortest essential cycle, no vertex of G is adjacent to each of $v_1, \ldots, v_4$. It follows that if there is an essential cycle av_1bv_3 such that $\{a, b\} \cap \{v_1, \ldots, v_4\} = \emptyset$, then there is no essential cycle cv_2dv_4 such that $\{c, d\} \cap \{v_1, \ldots, v_4\} = \emptyset$. Thus without loss of generality, by Lemma 2, there are no x, y such that both x and y are adjacent to both v_1 and v_3, but not to each other. Color v_1 and v_3 with 1, v_2 with 6, and v_4 with 7. This is an acyclic 7-coloring of G. ∎

4. Klein Bottle Graphs

In the introduction this conjecture of Borodin was mentioned, that for every surface other than the plane, the acyclic chromatic number equals the chromatic number. This section presents a counterexample in the case of the Klein bottle. Franklin [7] showed that the chromatic number of the Klein bottle is six. This section shows, by means of a Klein bottle graph which has no acyclic 6-coloring, that the acyclic chromatic number of the Klein bottle is not six.

Theorem 2. *The acyclic chromatic number of the Klein bottle is at least seven.*

Proof:

Consider the graph $G := K_8 \setminus (2K_{1,2} + K_2)$, which is embeddable on the Klein bottle (for an embedding, see Figure 1). Let G be given an acyclic 6-coloring. First note that for each triple of vertices, one pair of them is joined by an edge. Thus G has at most two vertices using each color. There must be two colors, each having two vertices of that color. But note that in G, if x is not adjacent to y, then either x and y are adjacent to every vertex of $V(G) \setminus \{x, y\}$ or there is a vertex z such that x, y, and z are each adjacent to every vertex of $V(G) \setminus \{x, y, z\}$. This is a contradiction. ∎

5. Graphs on General Surfaces

This section shows that Borodin's conjecture [14] is false for all surfaces with sufficiently large genus. Moreover, it determines the correct magnitude of the acyclic chromatic number of an arbitrary surface.

It is difficult to obtain a good upper bound on the acyclic chromatic number for a general surface. Albertson and Berman [3] showed that the sphere with g handles has acyclic chromatic number at most $4g + 4$. This section gives an improvement to this, as well as an extension to non-orientable surfaces. The proof uses the following result from [4].

Lemma 3 (Alon, McDiarmid, Reed). *For every graph G with maximum degree d, $A(G) \le \lceil 50d^{4/3} \rceil$.*

The following theorem is proved without any attempt to improve the constants:

Theorem 3. *If G is a (simple) graph embeddable on a surface of Euler characteristic $-\chi \le 0$, then $A(G) \le 100\gamma^{4/7} + 10000$.*

Proof:

Assume the theorem is false for a surface S with Euler characteristic $-\gamma \le 0$, and let G be a graph embeddable on it, with a minimum number of vertices, that violates the assertion of the theorem. Let H be G with (possibly multiple) edges added to triangulate S. Clearly $\deg_G(v) \le \deg_H(v)$ for every vertex v of G. Suppose $V(G) = V(H) = \{v_1, \dots, v_n\}$, where $\deg_H(v_1) \le \deg_H(v_2) \le \cdots \le \deg_H(v_n)$. If $\gamma = 0$, define $h_1 = 0$ and $h_2 = 0$. Otherwise, define $h_1 := \lceil c\gamma^{4/7} \rceil$ and $h_2 = \lfloor 6\gamma/h_1 \rfloor$ $(\le 6\gamma^{3/7}/c)$, where c is an absolute constant, to be chosen later. Let $d := \deg_H(v_{n-h_1})$. The proof will split on the size of d.

Case 1: $d \le (4/3)h_2 + 9$.

In this case, the induced subgraph of G on $\{v_1, \dots, v_{n-h_1}\}$ has maximum degree at most d, and thus has an acyclic coloring with at most $\lceil 50d^{4/3} \rceil$ colors, by Lemma 3. Coloring the remaining vertices of G with h_1 new colors that have not been used before gives an acyclic coloring of G with at most

$$\lceil 50((4/3)h_2 + 9)^{4/3} \rceil + h_1 \le 50(8\gamma^{3/7}/c + 9)^{4/3} + 1 + c\gamma^{4/7} + 1$$

colors. An appropriate choice of c shows that this is smaller than $100\gamma^{4/7}+10000$, implying that in this case G cannot be a counterexample.

Case 2: $d \ge (4/3)h_2 + 28/3$.

Define $\text{charge}(v_i) = 6 - \deg_H(v_i)$, and observe that $\Sigma_{i=1}^{n}\text{charge}(v_i) = -6\chi$ is true by Euler's formula. Since $\deg_H(v_i) \ge d$ for all $i \ge n - h_1$, it follows that $\Sigma_{i=n-h_1+1}^{n}(\mathrm{charge}(v_i) + \deg_H(v_i)/4) \leq h_1(6 - (3/4)((4/3)h_2 + 28/3)) = -h_1(h_2+1) < -6\chi$. Therefore,

$$\sum_{i=1}^{n-h_1} \mathrm{charge}(v_i) + \sum_{i=n-h_1+1}^{n} -\deg_H(v_i)/4 > 0.$$

Define $\mathrm{charge}'(v_i)$ as follows: for $1 \leq i \leq n-h_1$, $\mathrm{charge}'(v_i) := \mathrm{charge}(v_i)$. For $n-h_1+1 \leq i \leq n$, $\mathrm{charge}'(v_i) := -\deg_H(v_i)/4$. Finally, define new charges, $\mathrm{charge}''(v_i)$ to be the charges obtained from the charges $\mathrm{charge}'(v_i)$ by the following discharge rules. Send a charge of $1/2$ from each vertex of degree four to each of its neighbors of degree at least eight. Send a charge of $1/4$ from each vertex of degree five to each of its neighbors of degree at least seven. (All degrees are the degrees in H.) By the above discussion, $\Sigma_{i=1}^{n}\mathrm{charge}''(v_i) > 0$. Thus there is a j such that $\mathrm{charge}''(v_j) > 0$.

If $\deg_H(v_j) \leq 3$, so is $\deg_G(v_j)$, and one can delete v_j from G and add edges to make all its neighbors pairwise connected. By induction, there is an acyclic coloring of the resulting graph with the allowed number of colors, and this coloring gives a coloring of G by coloring v_j with a color that differs from that of all its neighbors.

If $\deg_H(v_j) = 4$, then $\mathrm{charge}'(v_j) = 2$, and thus v_j must have a neighbor v_k in H with $\deg_H(v_k) \leq 7$. Let K be G with v_j deleted, and edges added so that the neighbors of v_j in G except (possibly) v_k are pairwise adjacent. Give K an acyclic coloring by induction. If v_j is colored a different color from each of its neighbors as well as the neighbors of v_k, G will have an acyclic coloring with the right amount of colors.

If $\deg(v_j) = 5$, $\mathrm{charge}'(v_j) = 1$. Thus v_j must have two neighbors v_k, v_m in H such that $\deg_H(v_k) \leq 6$ and $\deg_H(v_m) \leq 6$. Let K be G with v_j deleted, and edges added so that the neighbors of v_j in G except (possibly) v_k, v_m are pairwise adjacent. Give K an acyclic coloring by induction. If v_j is colored a different color from each of its neighbors, as well as the neighbors of v_k and v_m, then G will have an acyclic coloring with the right amount of colors.

Clearly $\deg(v_j) \neq 6$, since in this case $\mathrm{charge}'(v_j) = \mathrm{charge}''(v_j) = 0$.

If $\deg(v_j) = 7$, $\mathrm{charge}'(v_j) = -1$, and thus v_j either has at least one neighbor in H of degree four, or at least five neighbors in H of degree five. Each of these yields a previous case.

If $\deg(v_j) \geq 8$, then $\mathrm{charge}'(v_j) \leq -\deg(v_j)/4$. Thus v_j must have neighbors

v_k, v_m in H such that $\deg_H(v_k) = 4$, $\deg_H(v_m) \leq 5$, and v_k is adjacent to v_m in H. This is also handled in a previous case, and thus completes the proof. ∎

The following result is used in the proof of Theorem 4 below, which shows that the bound of Theorem 3 is almost tight.

Lemma 4. *Let $G = G_n$ be the random graph on a set $V = \{1, 2, \ldots, n\}$ of n labelled vertices in which each pair of distinct vertices is chosen to be an edge, randomly and independently, with probability $p = 3((\log n)/n)^{1/4}$. Then, almost surely (that is, with probability that tends to 1 as n tends to infinity), G has at most $2n^{7/4}(\log n)^{1/4}$ edges and its acyclic chromatic number exceeds $n/2$.*

Lemma 4 can be derived from the proof in [4, pages 282–283]. Since the proof is not difficult, it is sketched here for the sake of completeness. For convenience, let n be divisible by 4. The fact that G has at most $2n^{7/4}(\log n)^{1/4}$ edges almost surely, follows trivially from the standard estimates for binomial distributions (cf., e.g., [5, Appendix A]).

The following suffices to prove that $A(G) > n/2$ almost surely. Observe, first, that for every fixed partition of V into $r \leq n/2$ disjoint color classes $V_1, \ldots, V_r$, on can omit a vertex from each V_i of odd cardinality and partition the remainder of each such V_i into pairs, thus obtaining at least $k = n/4$ pairwise disjoint subsets $U_1, \ldots, U_k$ of V, each of cardinality 2, and each a subset of some color class in the original given partition. Therefore, if, for some $1 \leq i < j \leq k$, the four edges between U_i and U_j are all in G, then the coloring is not acyclic. The probability that these four edges are not in G is $1-p^4$, and, crucially, all the $\binom{n/4}{2}$ events of this type corresponding to all possible choices of i and j are mutually independent. It follows that the probability that the given, fixed, partition is an acyclic coloring of G is at most

$$(1-p^4)^{\binom{n/4}{2}} \leq \exp(-81\binom{n/4}{2}(\log n)/n) < n^{-2n}$$

for all sufficiently large n. Since there are less than n^n partitions of V into at most $n/2$ color classes, it follows that the probability that there is an acyclic coloring with at most $n/2$ colors is at most n^{-n}, which tends to 0 as n tends to infinity. ∎

Theorem 4. *For every large $\gamma \geq 0$, there is a graph G embeddable on an (orientable) surface with Euler characteristic $\chi = -\gamma$ that satisfies $A(G) \geq (1/3)\gamma^{4/7}/(\log\gamma)^{1/7}$.*

Proof:

Let G satisfy the assertion of Lemma 4. Let G be embedded in a surface Σ' of the maximum Euler characteristic χ' such that G embeds in Σ'. Euler's formula gives $n - e + f = \chi'$, where $e := |E(G)|$ and $f := |F(G)|$. By Lemma 4, $e \leq 2n^{7/4}(\log n)^{1/4}$. Trivially $n \geq 0$ and $f \geq 0$. Thus $\chi' \geq -2n^{7/4}(\log n)^{1/4}$. Clearly G embeds in the orientable surface Σ of Euler characteristic χ equal to the greatest even number less than $-2n^{7/4}(\log n)^{1/4}$. If $\gamma := -\chi$, then $\gamma > n^{7/4}$, and $\log\gamma > (7/4)\log n$. Substituting, $\gamma < 2n^{7/4}((4/7)\log\gamma)^{1/4}$, or $n > (7/64)^{1/7}\gamma^{4/7}/(\log\gamma)^{1/7}$. By Lemma 4, $A(G) > n/2$, and Theorem 4 follows. ∎

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