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FACE-WIDTH OF EMBEDDED GRAPHS

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ABSTRACT. In their work on graph minors, Robertson and Seymour introduced the face-width (or representativity) as a measure of how densely a graph is embedded on a surface. The face-width of a graph embedded in S is the smallest number k such that S contains a noncontractible closed curve that intersects the graph in k points. We survey recent developments in the theory of embeddings of graphs in surfaces that concern this important invariant of embedded graphs.

1. Introduction

In their work on graph minors, Robertson and Seymour [RS88] introduced the concept of representativity as a measure of how densely a graph is embedded on a surface. The representativity is also widely known as the face-width. Since this other name has been used in most recent works on this subject, we have decided to use it also in this survey work. The *face-width* of a graph embedded in S is the smallest number k such that S contains a noncontractible closed curve that intersects the graph in k points. Alternatively, if G is 2-cell embedded, then the face-width is equal to the smallest integer k such that there exist facial walks $W_1, \dots, W_k$ whose union contains a noncontractible cycle. Robertson and Seymour proved that any infinite sequence of graphs embedded in a fixed surface S with increasing face-width can serve as a generic class of graphs on S in the sense that every embedding in S is a minor of one of these embeddings. We discuss this aspect of face-width in Section 8. Robertson and Vitray [RV90] presented an expository article on face-width, where they also developed the basic theory of the subject. They showed that embeddings with large face-width are minimum genus embeddings,

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and that they share many important properties with planar embeddings. These results stimulated further research on the face-width of embeddings that led to discoveries of many interesting properties of embeddings. It is the purpose of this paper to survey all these achievements in order to demonstrate the importance of this notion.

The reader should be aware that the research on graphs on surfaces involving the face-width is rather extensive. Therefore several of the presented results may not be entirely up to date at the time when the paper appears. The reader is also invited to consult [RV90], where several additional results can be found, and many results presented here are treated in more depth.

A related invariant of embedded graphs is the *edge-width*. It is defined as the length of a shortest noncontractible cycle of the embedded graph. The edge-width was introduced and studied by Thomassen [Th90]. It coincides with the face-width when G is a triangulation. Consequently, some of the results of this survey can also be viewed as results on the edge-width of triangulations (and conversely).

The reader can find more on the face-width and the edge-width in the forthcoming book by Thomassen and the author [MT97p].

2. Basic definitions

Graphs in this paper are undirected, finite and simple. We follow standard terminology as used, for example, in [BM81]. If we use multiple edges and loops, we speak about multigraphs. A subgraph C of a graph G is *induced* if every pair of non-adjacent vertices in C is also non-adjacent in G. It is *non-separating* if $G - V(C)$ is connected.

We will consider only 2-cell embeddings in closed surfaces. They can be described in a purely combinatorial way by specifying:

(1) A *rotation system* $\pi = \big(\pi_v\,;\ v \in V(G)\big)$; for each vertex v of the given graph G we have a cyclic permutation π_v of edges incident with v, representing their circular order around v on the surface.

(2) A *signature* $\lambda\colon E(G) \to \{-1, 1\}$. Suppose that $e = uv$. Following the edge e on the surface, we see if the local rotations π_v and π_u are chosen consistently or not. If yes, then we have $\lambda(e) = 1$, otherwise we have $\lambda(e) = -1$.

The reader is referred to [GT87] for more details. We will use this description as a definition: An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$, where π is a rotation system and λ is a signature. Having an embedding Π of G, we say that G is Π-*embedded*. A cycle with an odd number of edges e having $\lambda(e) = -1$ is Π-*onesided*. Other cycles are Π-*twosided*.

Given an embedding $\Pi = (\pi, \lambda)$, an *angle* of Π is any pair of edges $\{e, \pi_v(e)\}$, where $v \in V(G)$ is an endvertex of e. The cyclic sequence $e, \pi_v(e), \pi_v^2(e), \pi_v^3(e), \ldots$ is called Π-*clockwise ordering* around v. We define Π-*facial walks* (or simply Π-*faces*) as closed walks in the graph which correspond to traversals of face boundaries (in one or the other direction) of the topological embedding related to Π. Two embeddings are *equivalent* if they have the same facial walks.

Let $\mathcal{F}(\Pi, G)$ be the set of Π-facial walks. The number

$$\chi(\Pi) = |V(G)| - |E(G)| + |\mathcal{F}(\Pi, G)|$$

is the *Euler characteristic* of the embedding Π. If $\chi(\Pi) \neq 2$, then the embedding is *nonplanar.* We shall assume that the reader is familiar with the definition of the Π-*dual graph* (or multigraph) G^* of G. The corresponding Π-*dual embedding* of G^* will be denoted by Π^*. The graph and its dual can be simultaneously represented by the *vertex-face multigraph* $\Gamma = \Gamma(G, \Pi)$. The multigraph Γ is an embedded bipartite multigraph with vertex set $V(\Gamma) = V(G) \cup V(G^*)$ such that the vertex $w \in V(G^*)$ corresponding to a Π-facial walk W is joined to the vertices of W (where a repetition of vertices of W corresponds to multiple edges between w and the repeated vertex). Γ has a natural embedding in the surface of Π such that all facial walks of Γ have length 4. We denote this embedding by Π_Γ. Note that the edges of Γ are in bijective correspondence with the angles of Π. The vertex-face multigraph appeared in the literature under various names, e.g., the *radial graph* ([AR92]), the *angle graph* ([MR95]), the *primal-dual graph* ([BS93]), etc.

If Π is an embedding of a graph G, and H is a subgraph of G, then the *induced embedding* of H, which we denote by Π as well, is obtained from that of G by ignoring all edges in $E(G) \backslash E(H)$ and by restricting λ to $E(H)$. More precisely, if $e = uv \in E(H)$, then the successor of e in the clockwise ordering around v is the first edge of H in the sequence $\pi_v(e), \pi_v^2(e), \ldots$.

Let e be an edge of a Π-embedded multigraph G that is not a loop. Suppose that the signature $\lambda(e)$ of e is equal to 1. Then Π naturally defines the *induced embedding* of the edge-contracted graph G/e. Hence, starting with the embedded graph G and performing edge-deletions (excluding cutedges and edges whose removal changes the surface of the embedding) and edge-contractions (excluding loops), we get a minor H of G together with the corresponding induced embedding in the same surface. The embedded graph H is said to be a (*surface*) *minor* of the embedded graph G.

Let C be a Π-twosided cycle of a Π-embedded graph G. If it separates the surface of the embedding, then C is Π-*bounding* (also Π-*separating*). In the special case when C bounds a disk D on the surface, C is Π-*contractible.* In that case, the subgraph of G embedded inside D is the Π-*interior* of C, denoted by $\mathrm{int}(C, \Pi)$. Π-onesided cycles are always Π-nonbounding, and hence

Π-noncontractible. The *edge-width* $\mathrm{ew}(\Pi)$ of the embedding Π is the length of a shortest Π-noncontractible cycle of G.

3. Face-width

Let Π be a nonplanar embedding of G. Then we define the *face-width* $\mathrm{fw}(\Pi)$ (or *representativity*) of Π as the smallest number of Π-facial walks whose union (viewed as a subgraph of G) contains a Π-noncontractible cycle. (If we consider Π as a topological embedding in a surface S, then the face-width of Π equals the maximum w such that every noncontractible closed curve in S intersects the graph at least w times. Alternatively, this is equal to the smallest number k such that S contains a noncontractible closed curve that intersects the graph in k points.) If Π is a planar embedding, then the face-width is not defined. Whenever speaking of the face-width, we thus implicitly assume that the considered embedding is nonplanar.

The face-width can also be expressed by using the vertex-face multigraph $\Gamma(G,\Pi)$.

PROPOSITION 3.1. *Let Π_Γ be the embedding of the vertex-face graph of a Π-embedded graph G. Then*

$$\mathrm{fw}(\Pi) = \frac{1}{2}\,\mathrm{ew}(\Pi_\Gamma)\,.$$

Let G^* be the Π-dual of G, and let Π^* be the Π-dual embedding. Then $\Gamma(G,\Pi) = \Gamma(G^*,\Pi^*)$. It is easy to see that the vertex-face embedding Π^*_Γ is equal to Π_Γ. This implies:

PROPOSITION 3.2. *The face-width of a Π-embedded graph G and its Π-dual multigraph G^* are the same,* $\mathrm{fw}(\Pi) = \mathrm{fw}(\Pi^*)$.

Another consequence of Proposition 3.1 and a result of Thomassen [Th90] that the edge-width can be computed in polynomial time is:

Proposition 3.3. *The face-width of an embedded graph can be determined in polynomial time.*

It is helpful to consider a local version of face-width. Let ν be a vertex of $\Gamma(G,\Pi)$, i.e., ν is either a vertex of G or a Π-face of G.

Then we define the *face-width* from ν, $\mathrm{fw}(\Pi,\nu)$, as one half of the minimum length of a closed walk W in $\Gamma(G,\Pi)$ containing ν and a Π_Γ-noncontractible cycle. Note that the length of a shortest Π_Γ-noncontractible cycle through ν can be larger than $2\,\mathrm{fw}(\Pi,\nu)$. Such an example is shown in Figure 1.

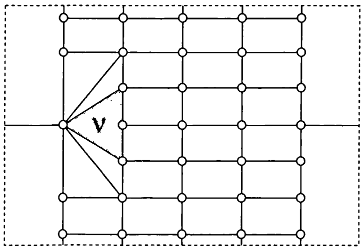

FIGURE 1. Shortest noncontractible walk through ν is shorter than the shortest cycle.

Viewing embeddings of graphs in surfaces from topological point of view, it makes sense to consider also embeddings that are not 2-cell. Such embeddings contain noncontractible curves that are disjoint from the graph and can be considered as embeddings of face-width 0. Similarly, an embedding of face-width 1 is in some sense degenerate. If the face-width is 2, then we have a slightly more regular behaviour as shown by the results presented below. The following lemma is easy to prove.

LEMMA 3.4. *Suppose that W is a Π-facial walk, and that $w \in V(G^*)$ is the vertex of $\Gamma(G,\Pi)$ corresponding to W. If $\mathrm{fw}(\Pi, w) \geq 2$, then W contains a Π-contractible cycle C such that $W \subseteq \mathrm{int}(C,\Pi) \cup C$.*

Let W and w be as in Lemma 3.4. In the sequel, we shall consider W and w as the same object, i.e., we shall assume that the vertices of the dual G^* are Π-facial walks. Suppose that $\mathrm{fw}(\Pi, w) \geq 2$, and let $C \subseteq w$ be the Π-contractible cycle such that $w \subseteq \mathrm{int}(C,\Pi) \cup C$. Let us delete $\mathrm{int}(C,\Pi)$ and contract all edges of C but one, and finally delete the remaining edge of C. If w is a cycle, this operation geometrically means shrinking of a face to a point. Thus we call it *face-shrinking.* The resulting graph is denoted by G/w, and the corresponding embedding of G/w is Π/w.

PROPOSITION 3.5. *Let w be a Π-facial cycle such that $\mathrm{fw}(\Pi, w) \geq 2$. Let v be the vertex of G/w resulting from face-shrinking w. Then the embedding Π/w of G/w is an embedding in the same surface as Π. If ν is either a vertex of G/w distinct from v or a (Π/w)-face, then we have*

$$\mathrm{fw}(\Pi,\nu) - 2 \leq \mathrm{fw}(\Pi/w,\nu) \leq \mathrm{fw}(\Pi,\nu)\,, \tag{1}$$

where the lower bound is attained if and only if there is a closed walk W_ν in $\Gamma(G,\Pi)$ determining $\mathrm{fw}(\Pi,\nu)$ which runs through ν twice. Similarly,

$$\mathrm{fw}(\Pi, w) - 1 = \mathrm{fw}(\Pi/w, v)\,. \tag{2}$$

Consequently,

$$\mathrm{fw}(\Pi) - 1 \le \mathrm{fw}(\Pi/w) \le \mathrm{fw}(\Pi). \tag{3}$$

P r o o f. The vertex-face graph $\Gamma/w = \Gamma(G/w, \Pi/w)$ is obtained from $\Gamma = \Gamma(G, \Pi)$ by contracting all edges incident to w in Γ and deleting one edge from each of the resulting digons. The rest is an obvious comparison of noncontractible closed walks in Γ and Γ/w. Let us remark that the lower bound in (1) may be attained, but the characterization when this happens shows that the same does not occur in (2) or (3). □

The dual operation to face-shrinking is the removal of a vertex. This gives us the dual form of Proposition 3.5.

PROPOSITION 3.6. *Let v be a vertex of G such that $\mathrm{fw}(\Pi, v) \ge 2$. Then the induced embedding of precisely one of the components of $G - v$, say G', is nonplanar. Let Π' be the restriction of Π to G'. Then Π' is an embedding in the same surface as Π, and there is precisely one Π'-facial walk w that is not Π-facial. If ν is either a vertex of G' or a Π'-face distinct from w, we have*

$$\mathrm{fw}(\Pi, \nu) - 2 \le \mathrm{fw}(\Pi', \nu) \le \mathrm{fw}(\Pi, \nu), \tag{4}$$

where the lower bound is attained if and only if there is a closed walk W_ν in $\Gamma(G, \Pi)$ determining $\mathrm{fw}(\Pi, \nu)$ which runs through ν twice. Similarly,

$$\mathrm{fw}(\Pi, v) - 1 = \mathrm{fw}(\Pi', w). \tag{5}$$

Consequently,

$$\mathrm{fw}(\Pi) - 1 \le \mathrm{fw}(\Pi') \le \mathrm{fw}(\Pi). \tag{6}$$

The face-width of an embedding is a measure of how densely the graph is embedded in the surface. The following basic result that indeed motivated introduction of face-width in [RS88] shows that large face-width implies that the embedding is highly locally planar.

If ν is a vertex of G or a Π-face, we define for $k = 0, 1, 2, \ldots$ subgraphs $B_k(\nu)$ of G recursively as follows: $B_0(\nu) = \nu$, and for $k > 0$, $B_k(\nu)$ is the union of $B_{k-1}(\nu)$ and all Π-facial walks that have a vertex in $B_{k-1}(\nu)$. Let $\partial B_k(\nu)$ be the set of edges of $B_k(\nu)$ that are contained in a Π-face that has not been used in defining $B_k(\nu)$.

PROPOSITION 3.7.

(a) *Let ν be a vertex of G, and let $k = \lfloor (\mathrm{fw}(\Pi, \nu) - 1)/2 \rfloor$. Then there exist disjoint Π-contractible cycles $C_1, \ldots, C_k$ such that for $i = 1, \ldots, k$, $C_i \subseteq \partial B_i(\nu)$ and $B_i(\nu) \subseteq \mathrm{int}(C_i, \Pi) \cup C_i$.*

(b) *Let ν be a Π-face, and let $k = \lfloor \mathrm{fw}(\Pi, \nu)/2 \rfloor - 1$. Then there exist disjoint Π-contractible cycles $C_0, \ldots, C_k$ such that for $i = 0, 1, \ldots, k$, $C_i \subseteq \partial B_i(\nu)$ and $B_i(\nu) \subseteq \mathrm{int}(C_i, \Pi) \cup C_i$.*

P r o o f . By induction on $\mathrm{fw}(\Pi, \nu)$. The cases when $k = 0$ are trivial. (If $\mathrm{fw}(\Pi, \nu) = 2$ and ν is a Π-face, the result follows by Lemma 3.4.) The inductive step is easy by applying relations (2) and (5) of Propositions 3.5 and 3.6. □

Embeddings of face-width more than one are easily recognized. If the graph is 2-connected, they have properties similar to embeddings of 2-connected graphs in the plane.

PROPOSITION 3.8. *Let G be a Π-embedded multigraph, and let G^* and Π^* be its Π-dual multigraph and the Π-dual embedding, respectively. If Π is a nonplanar embedding, then the following conditions are equivalent:*

(a) *All Π-facial walks are cycles.*
(b) $\mathrm{fw}(\Pi) \geq 2$ *and G is 2-connected.*
(c) $\Gamma(G, \Pi)$ *is a graph.*
(d) $\mathrm{fw}(\Pi^*) \geq 2$ *and G^* is 2-connected.*

P r o o f . (a) implies that $\Gamma(G, \Pi)$ has no multiple edges, so it is a graph. (c) clearly implies that $\mathrm{fw}(\Pi) = \mathrm{ew}(\Pi_\Gamma)/2 \geq 2$. If v is a cutvertex of G, there is a Π-facial walk that visits v at least twice. Thus (c) implies (b). If a Π-facial walk is not a cycle, we have a 2-cycle C in $\Gamma(G, \Pi)$. If C is contractible, then G is not 2-connected, otherwise $\mathrm{ew}(\Pi_\Gamma) = 2$. Hence (b) implies (a).

Since $\Gamma(G, \Pi) = \Gamma(G^*, \Pi^*)$, (d) follows by duality. □

Embeddings characterized in Proposition 3.8 are sometimes called *closed-2-cell embeddings* or *circular embeddings.*

In the same way, we prove the next result which shows that embeddings of 3-connected graphs having face-width more than two share various properties with embeddings of 3-connected graphs in the plane.

PROPOSITION 3.9. *Let G be a graph. If Π is a nonplanar embedding of G, then the following conditions are equivalent:*

(a) *For each vertex $v \in V(G)$, $\partial B_1(v)$ is a Π-contractible cycle.*
(b) $\mathrm{fw}(\Pi) \geq 3$ *and G is 3-connected.*
(c) $\Gamma(G, \Pi)$ *is a graph and all its 4-cycles are Π_Γ-facial.*
(d) *All Π-facial walks are cycles, and any two of them are either disjoint or their intersection is just a vertex or an edge.*
(e) $\mathrm{fw}(\Pi^*) \geq 3$ *and G^* is 3-connected.*

P r o o f . By using Proposition 3.8, we get parts of equivalences for free. The rest is similar to the proof of Proposition 3.8. Equivalence of (c) and (e) follows by duality. □

Embeddings with properties of Proposition 3.9 are also known as *polyhedral embeddings* as they appear in geometry of maps as a natural generalization of convex 3-polytopes.

Property (a) of Proposition 3.9 is called the "wheel neighborhood property" ([RV90]) since the neighborhood $B_1(v)$ of each vertex v is a wheel graph (with possibly subdivided edges). Condition (c) of Proposition 3.9 implies:

PROPOSITION 3.10. *Let G be a 3-connected graph embedded with face-width at least 3. Then all Π-facial walks are induced nonseparating cycles.*

One can easily construct embeddings with all faces induced and nonseparating cycles but with face-width only 2.

PROPOSITION 3.11. (Robertson and Vitray [RV90]) *Let G be a Π-embedded graph. If $2 \le \mathrm{fw}(\Pi) < \infty$, then there is precisely one block Q of G that contains a Π-noncontractible cycle. Moreover, the embedding restricted to Q is an embedding in the same surface and its face-width is equal to* $\mathrm{fw}(\Pi)$.

A similar result holds for embeddings with face-width three or more.

By Proposition 3.11, we can usually restrict our attention to 2-connected graphs. Similarly, in case of embeddings of face-width at least three, one can concentrate on 3-connected graphs as shown below.

Suppose that G is a 2-connected graph and that $G = G_1 \cup G_2$, where G_1 and G_2 are connected, edge-disjoint subgraphs that intersect only in two vertices x, y of G, and none of G_1, G_2 is just a path. For $i = 1, 2$, let G_i' be the subgraph of G obtained from G_i by adding a path $P_{xy}^{(i)}$ from x to y in G_{3-i}. If G_1' and G_2' can be further decomposed in a similar way, we split them into smaller graphs and repeat the process on all resulting graphs as long as possible. The graphs that cannot be further decomposed are called *3-connected blocks* of G. Each of them is a subgraph of G and is either a subdivision of a 3-connected graph, or consists of two vertices joined by two or three internally disjoint paths. The latter kind of 3-connected blocks are called *trivial.* The nontrivial 3-connected blocks of G are uniquely determined up to the choice of the paths $P_{xy}^{(i)}$, while the obtained trivial 3-connected blocks may depend on the splitting process.

PROPOSITION 3.12. (Robertson and Vitray [RV90]) *Let G be a 2-connected Π-embedded graph. If $3 \le \mathrm{fw}(\Pi) < \infty$, then there is precisely one nontrivial 3-connected block Q of G such that the restriction Π' of Π to Q is an embedding in the same surface as Π and* $\mathrm{fw}(\Pi') = \mathrm{fw}(\Pi)$.

Proposition 3.12 follows from Theorem 3.13 below, which shows that in a graph embedded with large face-width, no small vertex set separates the graph into two pieces of positive genus.

THEOREM 3.13. (Mohar [Mo95]) *Let G be a 2-connected Π-embedded graph. Suppose that X is a set of vertices of G with $|X| < \mathrm{fw}(\Pi)$ such that $G - X$*

is disconnected but no proper subset of X *is separating. Then* $G = G_1 \cup G_2$, *where* $G_1 \cap G_2 = X$ *such that:*

(i) X *is an induced and nonseparating set in* G_1.

(ii) *Let* Π' *be the restriction of* Π *to* G_1. *Then* Π' *is an embedding in the same surface as* Π.

(iii) *It is possible to add to* G *a cycle* C *with vertex set* $V(C) = X$ *such that* Π *can be extended to* $G \cup C$. *The cycle* C *is* Π*-contractible and* $G_2 = \mathrm{int}(C, \Pi)$. *In particular,* G_2 *contains no* Π*-noncontractible cycles.*

Similar result can be proved if G is not 2-connected and $|X| = 1$, but the property (iii) no longer holds.

The proof of Theorem 3.13 also works if the requirement that X is a minimal separating set is replaced by the following weaker condition:

(a) An X-bridge B containing Π-noncontractible cycles is attached to all vertices of X.

(b) For every proper separating subset X' of X there is an X-bridge in G that is attached to two or more vertices of X' and has a vertex of attachment in $X \backslash X'$.

4. Face-width minimal embeddings

Let $k \geq 2$ be an integer. An embedding Π of G is (*minor*) *minimal* of face-width k (or *face-width* k*-minimal*) if $\mathrm{fw}(\Pi) = k$, but for any edge e, the face-width of G/e is less than k, and if e is not a cutedge, the face-width of $G-e$ is also less than k. One can also speak of minimal embeddings of face-width 1.

THEOREM 4.1. *For every surface* S *and every integer* $k \geq 2$ *there are only finitely many face-width* k*-minimal embeddings in* S.

Theorem 4.1 follows from Robertson and Seymour's proof of Wagner's Conjecture ([RS95p] and the graph minors papers preceding that one) if all contractions in that proof are replaced with surface contractions (cf. [GRS96]). The first elementary proof of Theorem 4.1 was given by Malnič and Mohar [MM92] for the case $k = 2$, and their proof was extended to the general case by Malnič and Nedela [MN95]. A similar but shorter proof was obtained by Gao, Richter, and Seymour [GRS96]. Recently, Juvan, Malnič, and Mohar [JMM96] presented an even shorter proof of the same result.

Hutchinson [Hu88], [Hu89] showed that there is a constant c_1 such that every triangulation of S_g with face-width w has at least $c_1 w^2 g / \log^2 g$ vertices, and she raised the question of the existence of triangulations of large face-width and not too many vertices. Przytycka and Przytycki [PP93p] constructed triangulations having face-width w and with at most $c_2 w^2 g/\log g$ vertices, where c_2 is a constant. Similar results hold also for nonorientable surfaces. G a o et al. [GRS96] showed that triangulations in which every edge lies in a noncontractible cycle of length w contains at most $c_3 w^2 w!(4ww!)^w g^2$ vertices. (For the case $w = 3$, an elementary and very short proof is given in [GRT91p] with an explicit upper bound $O(g^4)$ and in [NO95] with a linear bound.) The barycentric subdivision of a minor minimal embedding of face-width k is a triangulation in which each edge is in a noncontractible cycle of length $2k$ (and not less). The results above thus imply that a minor minimal embedding of face-width w contains at least $c_4 w^2 g/\log^2 g$ and at most $c_5 w^2 (2w)!\big(8w(2w)!\big)^{2w} g^2$ edges.

Let G be a Π-embedded graph. A $Y\Delta$*-exchange* is an operation that replaces a triangular face F by a vertex of degree three joined to the vertices of F, or conversely.

PROPOSITION 4.2. *Suppose that* Π *is a face-width* k*-minimal embedding. Then*

(a) *Its dual embedding* Π^* *is also face-width* k*-minimal.*
(b) *If* Π' *is obtained from* Π *by a sequence of* $Y\Delta$*-exchanges, then* Π' *is also face-width* k*-minimal.*

Proposition 4.2 (a) follows from Proposition 3.2 and the fact that edge deletion and edge contraction are dual operations, while the claim (b) is easy to verify.

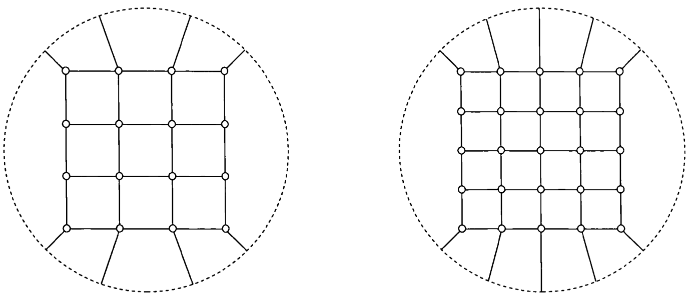

FIGURE 2. The projective 4×4 and 5×5 grids.

Face-width k-minimal embeddings in the projective plane for $k \in \{1, 2, 3\}$ have been determined by V i t r a y [Vi92]. The same result was obtained independently by B a r n e t t e (cases $k = 1$ and $k = 2$ in [Ba87], and $k = 3$ in [Ba91a]). B a r n e t t e also determined the 16 subgraph minimal embeddings of face-width three in the projective plane ([Ba91b]). For other values of k, one can easily check that the *projective* $k \times k$ *grid* (see Figure 2 for 4×4 and 5×5 grids) is a minimal embedding of face-width k. Due to the following result of Randby [Ra95p], all minimal embeddings of face-width k in the projective plane can be generated from the projective $k \times k$ grid by applying $\mathrm{Y}\Delta$-exchanges.

THEOREM 4.3. (Randby [Ra95p]) *Every minor minimal embedding of face-width k in the projective plane can be obtained from the projective $k \times k$ grid by a sequence of $\mathrm{Y}\Delta$-exchanges. In particular, the graph of such an embedding has precisely $2k^2 - k$ edges.*

This result does not hold for other surfaces since already on the torus and the Klein bottle there are minimal embeddings without triangles and without vertices of degree 3.

Call two embeddings *similar* if one can be obtained from the other by a finite sequence of operations each of which is a $\mathrm{Y}\Delta$-exchange or the replacement of a graph by its dual. Schrijver [Sch94] classified similarity classes of face-width k-minimal embeddings on the torus.

THEOREM 4.4. (Schrijver [Sch94]) *There are precisely $\frac{1}{6}(k^3 + 5k)$ (if k is odd) and precisely $\frac{1}{6}(k^3 + 8k)$ (if k is even) similarity classes of face-width k-minimal embeddings on the torus.*

The similarity classes in Theorem 4.4 are classified by means of certain symmetric integer polygons in the plane $\mathbb{R}^2$. (An *integer polygon* P in $\mathbb{R}^2$ is the convex hull of a finite number of points in $\mathbb{R}^2$ with integer coordinates. The polygon P is *symmetric* if $P = -P$.) The simplest case $k = 2$ is also elaborated in [Ba87].

Several other results about face-width of graphs on the torus are contained in [Sch92a], [Sch93], [GS94].

Let G be a Π-embedded graph, and let D be a closed walk in G. If there is an edge $e \in E(G)$ such that $D = D_1 e e^- D_2$ (where e^- denotes the traversal of e in the opposite direction), let $D' = D_1 D_2$. Similarly, if $D = D_1 e D_2$ and $W = eR$ is a Π-facial walk containing e, we can change D into a closed walk $D' = D_1 R^- D_2$. These two operations of replacing D by D' and their inverses are called *elementary homotopic shifts.* Two closed walks in G are (*freely*) Π-*homotopic* if one can be obtained from the other by a sequence of elementary homotopic shifts. It is easy to see that any two Π-contractible cycles are homotopic. Conversely, if C is Π-homotopic to a Π-contractible cycle, then C is also Π-contractible.

Let D be a closed walk in the vertex-face graph $\Gamma = \Gamma(G, \Pi)$. Then we define

$$\mu(G, \Pi, D) = \frac{1}{2} \min |D'|, \tag{7}$$

where the minimum ranges over all closed walks D' in Γ that are Π_Γ-homotopic to D, and $|D'|$ is the length of D'. This determines a function $\mu(G,\Pi)$ defined on Π_Γ-homotopy classes of closed walks in Γ. Since $\Gamma(G,\Pi) = \Gamma(G^*,\Pi^*)$, we have $\mu(G,\Pi) = \mu(G^*,\Pi^*)$. If G', Π' is obtained from G, Π by a $\mathrm{Y}\Delta$-exchange, then the homotopy classes of closed walks in $\Gamma(G,\Pi)$ and $\Gamma(G',\Pi')$ are in a natural bijective correspondence. An obvious generalization of Proposition 4.2 (b) is the fact that

$$\mu(G,\Pi) = \mu(G',\Pi'). \tag{8}$$

Suppose that G', Π' is a surface minor of G, Π. Then the homotopy classes of closed walks in $\Gamma(G',\Pi')$ are in a bijective correspondence with homotopy classes in $\Gamma(G,\Pi)$ since each edge-deletion or edge-contraction in G corresponds to the elimination of a face in Γ by identifying opposite vertices of a Π_Γ-facial quadrangle. Consequently,

$$\mu(G',\Pi') \le \mu(G,\Pi). \tag{9}$$

The Π-embedded graph G is a *kernel* if $\mu(G',\Pi') \ne \mu(G,\Pi)$ for each proper surface minor G', Π' of G, Π. Since $\mathrm{fw}(\Pi) \ge k$ if and only if for each Π_Γ-noncontractible cycle D of Γ, $\mu(G,\Pi,D) \ge 2k$, kernels generalize the concept of minimal embeddings of given face-width. Schrijver [Sch92b] proved that kernels with the same function μ are closely related to each other.

THEOREM 4.5. (Schrijver [Sch92b]) *If G, Π and G', Π' are kernels in the same orientable surface, then $\mu(G,\Pi) = \mu(G',\Pi')$ if and only if G', Π' can be obtained from G, Π by a sequence of $\mathrm{Y}\Delta$-exchanges and taking surface duals.*

On the projective plane, there is only one homotopy class of noncontractible closed walks. Theorem 4.5 therefore implies a weak version of Theorem 4.3. (It is weaker since Theorem 4.3 does not involve surface duality.)

5. Embedding flexibility

Whitney's 2-switching theorem ([Wh33]) determines the variety of embeddings of a 2-connected graph in the sphere as the set of embeddings obtained from any embedding in the sphere by a sequence of *flippings* (sometimes also called *Whitney's 2-switchings*). This operation can be defined for general embeddings as follows. Suppose that G is a 2-connected Π-embedded graph, and C is a Π-contractible cycle such that only two vertices of C, say u and v, have incident edges that are not in $C \cup \mathrm{int}(C,\Pi)$. We may assume that all edges in $C \cup \mathrm{int}(C,\Pi)$ have positive signature. Let $\pi_u = (e_1 Q_1 e_2 Q_2)$ and $\pi_v = (e'_1 Q'_1 e'_2 Q'_2)$ be the Π-clockwise ordering around u and v, respectively, where e_1, e_2, $e'_1 e'_2$ are edges of C, and Q_1, Q'_1 are edges in $\mathrm{int}(C,\Pi)$. A *flipping* with respect to C is the change of the embedding Π such that the Π-clockwise ordering around vertices in $\big(C \cup \mathrm{int}(C,\Pi)\big)\backslash\{u,v\}$ is changed to anticlockwise, π_u and π_v are changed to $\pi'_u = (e_2 Q_1^- e_1 Q_2)$ and $\pi'_v = (e'_2 {Q'_1}^- e'_1 Q'_2)$, respectively, and other vertices retain the original rotations. Note that the new embedding is in the same surface as the former one. Two embeddings of G are *Whitney equivalent* if one can be obtained from the other by a sequence of flippings.

PROPOSITION 5.1. (Mohar [Mo92]) *Let G be a 2-connected graph, and let Π and Π' be embeddings of G such that $3 \le \mathrm{fw}(\Pi) < \infty$. Let $H \subseteq G$ be a nontrivial 3-connected block of G containing some Π-noncontractible cycles. Then the following assertions are equivalent:*

(a) *Π and Π' are Whitney equivalent (and hence they are embeddings in the same surface and have equal face-width).*
(b) *A cycle C of G is Π-contractible if and only if it is Π'-contractible.*
(c) *An induced nonseparating cycle C of H is Π-contractible (in G) if and only if it is Π'-contractible.*
(d) *The induced embeddings Π_1 and Π'_1 of Π and Π' (respectively) to H are equivalent.*

Even if we restrict to 3-connected graphs, where the Whitney equivalence is trivial, there are graphs with arbitrarily many embeddings (for every surface except the sphere and the projective plane).

If we speak of different ways of embedding a graph in a fixed surface, we say that we speak of *embedding flexibility* of the graph in that surface.

For particular surfaces, embedding flexibility has been investigated by several authors. Negami [Ne88] solved the embedding flexibility of non-planar graphs in the projective plane by describing the structures (and their flexibility) of embeddings that are not unique. Independently, the same results have been obtained by Vitray [Vi93]. A very special case of Negami's and Vitray's results is a discovery of Barnette [Ba89] that a cubic graph in the projective plane having face-width three has more than one embedding in the projective plane if and only if its dual contains a subgraph isomorphic to K_6 or a subgraph isomorphic to K_4 that is embedded with face-width 2. Lawrenchenko [La92] considered flexibility of projective plane triangulations. Mohar, Robertson, and Vitray [MRV96] describe the structure and flexibility of embeddings of planar graphs in the projective plane.

Embedding flexibility in the torus of graphs that have embeddings of face-width four or more in the torus is solved in recent work of Robertson, Zha, and Zhao [RZZ95p]. Results about graphs that are embeddable in the torus and are not embeddable in the Klein bottle have been obtained by Riskin [Ri94p]. Lawrencenko and Negami [LN94p] characterized all graphs that triangulate the torus and the Klein bottle. In particular, all other triangulations of one of these surfaces do not embed in the other one.

The above results show that embeddings of large face-width have little flexibility, and that such embeddings are close to minimum genus embeddings. Another extreme is to consider flexibility of non-genus embeddings of graphs. Nowadays, this problem is totally understood in the case of planar graphs. Mohar, Robertson and Vitray [MRV96] describe flexibility of embeddings of planar graphs in the projective plane. Embeddings of planar graphs in general surfaces are treated in [MR96b]. The basic tool used in the latter work is a generalization of Robertson's observation [RV90] that the face-width of a planar graph in a nonplanar surface cannot be more than two. (Robertson's result was also proved by Thomassen [Th90] and Barnette and Riskin [BR91p].)

THEOREM 5.2. (Mohar and Robertson [MR96b]) *Let G be a 2-connected planar graph that is Π-embedded in a nonplanar surface. Suppose that α is a Π-angle at a vertex ν that is not an angle in planar embeddings of G, and that* $\mathrm{fw}(\Pi, \nu) \geq 2$. *Then there is a Π_Γ-noncontractible 4-cycle of $\Gamma(G, \Pi)$ through α. In particular,* $\mathrm{fw}(\Pi, \nu) = 2$.

One can generalize flippings as follows. Suppose that $\Gamma = \Gamma(G, \Pi)$ contains a Π_Γ-contractible 6-cycle $R = v_1 w_1 v_2 w_2 v_3 w_3$ such that the Π-face w_2 contains also the vertex v_1 of G. Then we can re-embed the part of G inside R into the face w_2 (by changing clockwise to anticlockwise in the interior and by a partial reversal of the rotation at v_1, v_2, v_3 as we have in the case of flippings). The new embedding of G, if it is in the same surface, is said to be obtained from Π by a 3-*flipping*. Similarly, we define a 4-*flipping*, where we need a Π_Γ-contractible walk $R = v_1 w_1 v_2 w_2 v_3 w_3 v_4 w_4$ such that $w_2 = w_4$, and then we re-embed the part of G inside R into the face w_2. Mohar and Robertson [MR96b] proved that adding 3-flippings and 4-flippings yields bounded flexibility (up to Whitney's flippings and these two operations) in the case of planar graphs embedded in arbitrary surfaces.

Theorem 5.3. (Mohar and Robertson [MR96b]) *Let S be a surface. There is a constant $q = q(S)$ such that up to equivalence generated by Whitney's flippings, 3-flippings, and 4-flippings, every 2-connected planar graph has at most q embeddings in S.*

We conjecture that the same result holds also for nonplanar graphs.

A graph is an *apex graph* if it contains a vertex whose removal leaves a planar graph. Robertson and Vitray [RV90] asked if apex graphs can have embeddings with face-width three. Their question was partially solved by Robertson, Seymour, and Thomas [RST93] in their work about linkless embeddings of graphs in 3-space. A simpler proof of the same result was independently obtained by Mohar in [Mo94q].

THEOREM 5.4. *If G is an apex graph embedded in a nonorientable surface, then the embedding has face-width at most two.*

Theorem 5.4 generalizes Theorem 5.2. On the other hand, Mohar [Mo94q] characterized the structure of apex graphs and their embeddings of face-width three. Surprisingly, there are such embeddings of arbitrarily large genus.

6. Uniqueness of embeddings

A well-known result of Whitney [Wh33] says that 3-connected planar graphs admit unique embedding in the sphere, in the sense that local rotations are uniquely determined. This result does not extend to other surfaces even under stronger requirements on connectivity. For example, it is easy to describe 4-connected triangulations of the projective plane or the torus that are not uniquely embeddable in that surface. Lavrenchenko [La87] described an infinite family of 5-connected triangulations of the torus that are not uniquely embeddable in the torus. On the positive side is a result of Negami who proved that 6-connected toroidal graphs are "uniquely" embeddable in the torus up to automorphisms of the graph [Ne83] (but the family of such graphs is very restricted [Al72]). Similarly, if G is a 5-connected projective planar graph distinct from K_6 that contains a subgraph homeomorphic to K_6, then G admits a unique embedding in the projective plane ([Ne85]).

A breakthrough in research about uniqueness of embeddings of graphs in surfaces was reached when Robertson (cf. [RV90]) proved that a 3-connected graph embedded in a surface of genus g with face-width at least $2g + 3$ is uniquely embeddable in that surface, and that this embedding is a minimal genus embedding. Here we say that G has a *unique embedding* in S if G has an embedding Π in S and any other embedding in S is equivalent with Π. Roughly at the same time Thomassen [Th90] proved uniqueness and genus minimality of embeddings under a similar condition on large edge-width. Robertson's result was slightly improved by Mohar [Mo92] who obtained a local version of the theorem and replaced g in the bound on the face-width by the genus of the graph. This result restricted to the projective plane, the torus, and the Klein bottle was also observed by Barnette and Riskin [BR91p], [Ba91s].

Robertson then asked if the face-width larger than some constant forces uniqueness of embeddings. Thomassen [Th90] and independently Barnette and Riskin [BR91p] proved that face-width four does not suffice by exhibiting toroidal embeddings of arbitrarily large face-width whose graphs admit embeddings of face-width four in some other surface. Later, Archdeacon proved by a more sophisticated construction:

THEOREM 6.1. (Archdeacon [Ar92]) *For each integer k there exists a 3-connected graph R_k that has nonequivalent embeddings Π and Π', both of face-width more than k.*

The exact values of the genera of embeddings Π and Π' of R_k (expressed in terms of k) have not yet been calculated. However, the following result of Seymour and Thomas [ST96], and Mohar [Mo95] shows that the genera of Π and Π' are more than exponential in terms of the face-width. Let us recall that the *Euler genus* $\varepsilon(\Pi)$ of an embedding Π is equal to

$$\varepsilon(\Pi) = 2 - \chi(\Pi).$$

This notion "unifies" embeddings in orientable and nonorientable surfaces with the same Euler characteristic.

THEOREM 6.2. (Seymour, Thomas, Mohar) *Let G be a connected graph and Π, Π' its embeddings having face-widths k and k', respectively.*

(a) ([Mo95]) *Suppose that $k \geq 7$ and*

$$\left\lfloor \frac{k-3}{2} \right\rfloor^{\lfloor k/2 \rfloor - 2} > \varepsilon(\Pi), \qquad \left\lfloor \frac{k'-3}{2} \right\rfloor^{\lfloor k'/2 \rfloor - 2} > \varepsilon(\Pi'). \tag{10}$$

Then Π and Π' are embeddings in the same surface, and Π' is Whitney equivalent with Π.

(b) ([ST96]) *If $k' \geq 3$ and $k \geq \min\{400, \log \varepsilon(\Pi)\}$, then either Π' is Whitney equivalent with Π (and hence it is an embedding in the same surface), or $\varepsilon(\Pi') \geq \varepsilon(\Pi) + 10^{-6}k^2$.*

(c) ([ST96]) *If $k \geq 100 \log \varepsilon(\Pi)/\log\log \varepsilon(\Pi)$ (or $k \geq 100$ if $\varepsilon(\Pi) \leq 2$), then $\varepsilon(\Pi') \geq \varepsilon(\Pi)$, and if $\varepsilon(\Pi) = \varepsilon(\Pi')$, then Π and Π' are Whitney equivalent.*

In particular, if the graph G in Theorem 6.2 is 3-connected, then Π' is equivalent to Π, i.e., such embeddings are unique. In order to compare (a) with (b) and (c) of Theorem 6.2, we note that (10) is equivalent to $k \geq c \log \beta / \log\log \beta$ (and the same for k'), where $c = 2 + \delta$ and $\delta > 0$ is arbitrarily small.

Riskin and Barnette [RB94] proved that there are embeddings Π, Π' of the same graph G, both of face-width at least three and of genera g and $g' > g+1$, respectively, but no such embeddings of genus $g+1$ exist. Theorem 6.2 (b) does not only prove this result, but also shows that this is always true if $\mathrm{fw}(\Pi)$ is large enough.

Recall that a graph is *orientably simple* if its nonorientable genus is more than twice the orientable genus, i.e., the minimal Euler genus embeddings of the graph are all in an orientable surface. The following corollary of Euler genus minimality results was observed in [Mo92].

COROLLARY 6.3. *If G is Π-embedded in an orientable surface of Euler genus ε and* $\mathrm{fw}(\Pi) \geq 100 \max\{1, \log \beta / \log\log \beta\}$, *then G is orientably simple.*

Bender, Gao, and Richmond [BGR94] proved that there is a constant $c > 0$ such that almost all (rooted) embeddings of graphs with q edges in a fixed surface have face-width at least $c \log q$. They proved that the same conclusion holds for some other families of (rooted) embedded graphs, e.g., triangulations. Combined with the results of this section, this shows that almost all graphs on a fixed surface are minimum genus embeddings sharing many properties with planar graphs.

Thomassen [Th93a] proved that finding a minimum genus (or minimum nonorientable genus, or minimum Euler genus) embedding of face-width at least three is still an **NP**-hard problem. However, the following problem is still unsolved.

PROBLEM 6.4. *Let $k \geq 3$ be an integer. How difficult is to decide if a given graph G admits an embedding of face-width k or more? If it does, can we find such an embedding in polynomial time? If $k \geq 4$, can we find such an embedding of G with minimal possible genus (or Euler genus) in polynomial time?*

7. Orientable genus of graphs with given nonorientable embeddings

It is easy to see that the nonorientable genus $\tilde{\gamma}(G)$ of every graph G is bounded by a linear function of the genus $\gamma(G)$, $\tilde{\gamma}(G) \leq 2\gamma(G)+1$. On the other hand, Auslander, Brown, and Youngs [ABY63] proved that there are graphs embeddable in the projective plane whose orientable genus is arbitrarily large. This phenomenon is now appropriately understood since Fiedler et al. [FHRR95] proved the following result.

THEOREM 7.1. (Fiedler, Huneke, Richter, Robertson) *Let G be a graph that is Π-embedded in the projective plane. If* $\mathrm{fw}(\Pi) \neq 2$, *then the orientable genus $\gamma(G)$ of G is*

$$\gamma(G) = \left\lfloor \frac{1}{2} \mathrm{fw}(\Pi) \right\rfloor . \tag{11}$$

If $\mathrm{fw}(\Pi) = 2$, *then $\gamma(G)$ is either 0 or 1.*

Theorem 7.1 has been generalized to the next nonorientable surface by Robertson and Thomas [RT91] as follows. Let Π be an embedding of G in the Klein bottle. Then we denote by $\mathrm{ord}_2(G, \Pi)$ the minimum of $\lceil |W|/4 \rceil$ taken over all Π_Γ-nonbounding Π_Γ-twosided closed walks W in the vertex-face graph $\Gamma(G,\Pi)$, where $|W|$ denotes the length of the closed walk W. Similarly, we denote by $\operatorname{ord}_1(G,\Pi)$ the minimum of $\lfloor |W_1|/4\rfloor + \lfloor |W_2|/4\rfloor$ taken over all pairs of Π_Γ-nonhomotopic Π_Γ-onesided closed walks W_1, W_2 in $\Gamma(G,\Pi)$. The latter minimum restricted to all noncrossing pairs W_1, W_2 is denoted by $\operatorname{ord}_1'(G,\Pi)$.

THEOREM 7.2. (Robertson and Thomas [RT91]) *Let G be a graph that is Π-embedded in the Klein bottle. Let*

$$g = \min\{\operatorname{ord}_1(G,\Pi), \operatorname{ord}_2(G,\Pi)\}. \tag{12}$$

If $g \geq 4$, then g is equal to the orientable genus of the graph G. Moreover, g can be determined in polynomial time.

Robertson and Thomas also proved that

$$g = \min\{\operatorname{ord}_1'(G,\Pi), \operatorname{ord}_2(G,\Pi)\}.$$

By Theorems 7.1 and 7.2 we have the following corollary.

COROLLARY 7.3. *The genus of graphs that can be embedded in the projective plane, or the Klein bottle can be computed in polynomial time.*

By [Th89], genus testing is **NP**-complete for general graphs. Therefore it is interesting that the classes of projective graphs and graphs embeddable in the Klein bottle admit a polynomial time genus testing algorithm. Very likely the genus problem for graphs with bounded nonorientable genus is solvable in polynomial time as suggested in [RT91].

Robertson and Thomas [RT91] also proposed as a conjecture the following generalization of Theorems 7.1 and 7.2. Let Π be an embedding of G into the nonorientable surface N_k of genus k. Suppose that $\mathcal{F} = \{F_1, \ldots, F_p\}$ is a set of closed walks in the vertex-face graph $\Gamma(G,\Pi)$. Then $\mathcal{F}$ is *crossing-free* if

(a) no F_i crosses itself;
(b) for $1 \leq i < j \leq p$, F_i and F_j do not cross each other.

Suppose that every Π-onesided cycle of G crosses at least one F_i, and that there are disjoint simple closed curves $f_1, \ldots, f_p$ in the surface such that F_i is homotopic to f_i, $i = 1, \ldots, p$. Then we say that the family $\mathcal{F}$ is a *blockage* and that $\mathcal{F}$ *blocks* Π*-onesided cycles.* We define the *order* of a blockage $\mathcal{F}$ as

$$\operatorname{ord}(\mathcal{F}) = \frac{1}{2}(k-s) + \sum_{i=1}^{p} \operatorname{ord}(F_i), \tag{13}$$

where s is the number of Π_Γ-onesided cycles in $\mathcal{F}$, and

$$\operatorname{ord}(F_i) = \begin{cases} \lfloor |F_i|/4 \rfloor & \text{if } F_i \text{ is } \Pi_\Gamma\text{-onesided,} \\ \lfloor (|F_i|-1)/4 \rfloor & \text{if } F_i \text{ is } \Pi_\Gamma\text{-twosided}. \end{cases}$$

The conjecture of Robertson and Thomas [RT91] is that if G is embedded in N_k, the nonorientable surface of genus k, with sufficiently large face-width, the following are equivalent:

(a) The genus of G is at least g.
(b) Every crossing-free blockage has order at least g.
(c) Every blockage has order at least g.

However, it is not clear how one can find in polynomial time a (crossing-free) blockage of minimum order.

Mohar [Mo95q] disproved the above conjecture of Robertson and Thomas and proposed a slightly different conjecture that might be off by a constant (depending on the genus) from the minimal value of g in the above conjecture of Robertson and Thomas. Suppose that G is embedded in N_k. Consider a crossing-free blockage $\mathcal{F} = \{F_1, \dots, F_p\}$ and cut the surface N_k along $F_1, \dots, F_p$. This results in a graph G' embedded in an orientable surface. Now each vertex $v \in V(G) \cap \left(\bigcup_{i=1}^{p} V(F_i) \right)$ has two or more copies in G', and we add a new vertex v' and join it to all copies of v in G'. Call the resulting graph G'' and note that contraction of all new edges results in the original graph G. Now, the orientable embedding of G' defines local rotations of all vertices of G'' except for the new vertices v'. The minimum genus of an orientable embedding of G'' extending this partial embedding is called the *genus order* of the blockage $\mathcal{F}$. We note that in the case when no vertex of G is split into more than two vertices of G', the genus order coincides with (13), and that, in general, it is majorized by (13).

CONJECTURE 7.4. *If G is embedded in a nonorientable surface with sufficiently large face-width, then the orientable genus of G is equal to the minimal genus order of a crossing-free blockage.*

Mohar and Schrijver (private communication) proved that Conjecture 7.4 (and also the conjecture of Robertson and Thomas) holds up to a constant error term: There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for any graph G embedded in N_k the orientable genus of G differs from the minimum order of a crossing-free blockage at most by $f(k)$.

8. Minors and embeddings

It is well known that the grid graphs can serve as a generic class for planar graphs when working with problems involving graph minors.

THEOREM 8.1. *Let G_0 be a plane graph. Then there is a constant k such that G_0 is a (surface) minor of the $k \times k$ grid $P_k \times P_k$.*

Robertson and Seymour [RS88] found out that as an appropriate generalization of the grid graphs for graphs embedded in a fixed surface S one can take any sequence of graphs embedded in S whose face-width tends to infinity.

THEOREM 8.2. (Robertson and Seymour [RS88]) *Let G_0 be a graph that is Π_0-embedded in a surface S. Then there is a constant k such that for any graph G that is Π-embedded in S with the face-width at least k, G_0, Π_0 is a surface minor of G, Π.*

This result shows that graphs on nonplanar surfaces embedded with sufficiently large face-width contain as surface minors any fixed structure on the surface. In particular, it implies that embeddings of sufficiently large face-width are unique and minimum genus embeddings, the aspect that we have treated previously.

Theorem 8.2 does not give explicit bounds on the face-width that guarantees the presence of G_0, Π_0 as a minor. Such bounds are of great interest, and thus it is not surprising that several works treat special cases of Theorem 8.2 by presenting explicit bounds.

(1) **Existence of noncontractible bounding cycles:** Barnette conjectured that a triangulation of a surface of genus $g \geq 2$ contains a noncontractible bounding cycle. Zha independently proposed the conjecture that every embedding of face-width at least three and genus at least two contains such a cycle. Richter and Vitray [RV92p] proved that this holds provided the face-width is at least 11. Zha and Zhao [ZZ93] improved their result by showing that face-width 6 (even 5 for nonorientable surfaces) is sufficient. (Zha and Zhao's original estimate on the face-width was 7; it was improved to 6 upon adding ideas of Brunet, Mohar, and Richter from [BMR96], cf. [ZZ93].) Brunet, Mohar, and Richter [BMR96] also proved that a graph embedded with face-width w in a surface of genus at least 2 contains $\lfloor (w-9)/8 \rfloor$ disjoint noncontractible and pairwise homotopic bounding cycles. (In [BMR96], also an independent proof of the above mentioned result of Zha and Zhao was obtained.) Richter and Vitray [RV94p] proved that for any Π-embedded graph with $\mathrm{fw}(\Pi) \geq 4$ and any pair of Π-faces F_1, F_2 there is a Π-contractible cycle C such that F_1 and F_2 are in the Π-interior of C.

More generally, Thomassen conjectured that given a triangulation T of a surface of genus g and a number h, $1 \leq h < g$, T must contain a bounding cycle C such that the two surfaces separated by C have genus h and $g - h$, respectively. Mohar conjectured that the same is true for any embedding with face-width at least 3.

(2) **Disjoint noncontractible cycles:** Barnette [Ba88] proved that a graph embedded in the torus with face-width at least 3 contains two disjoint noncontractible cycles. If the graph is 3-connected, then the cycles can be taken so that one of the cylinders between them contains no vertices of the graph. Mohar and Robertson [MR93] characterized the structure of embeddings in the torus that do not have two disjoint noncontractible cycles. More generally, such a characterization is obtained for general surfaces by the same authors [MR96a]. Schrijver [Sch93] proved that a graph embedded in the torus with face-width w contains $\lfloor 3w/4 \rfloor$ disjoint noncontractible cycles. Such cycles on the torus are necessarily homotopic to each other. For general surfaces, Zha and Zhao [ZZ93] established existence of two disjoint homotopic noncontractible cycles. More generally, Brunet, Mohar, and Richter [BMR96] proved that a graph embedded with face-width w contains at least $\lfloor (w-1)/2 \rfloor$ disjoint noncontractible homotopic cycles. A special case of this result for $w = 5$ was obtained previously by Zha and Zhao [ZZ93].

(3) **Disjoint cycles of given homotopies:** Although it is not directly related to the face-width, it is worth mentioning the results of Schrijver about existence of disjoint cycles of prescribed homotopies in an embedded graph G. He obtained a characterization when a given family of homotopy classes of curves on a surface S can be represented by a set of disjoint cycles in the graph embedded in S ([Sch91a]). Schrijver's characterization yields a polynomial time algorithm for finding such cycles or proving their nonexistence. Also of great importance, but much harder, is the problem of when an embedded graph contains edge-disjoint cycles of prescribed homotopies. This problem is related to multicommodity flow problems and has also been successfully attacked by several authors, cf. [Li81], [Sch90], [Sch91b], [Gr94].

(4) **Planarizing cycles:** A set $\{Q_1, \ldots, Q_g\}$ of disjoint cycles of a Π-embedded graph is *planarizing* if, after cutting the surface along the cycles, we get a surface that is homeomorphic to a subset of the plane. Existence of planarizing cycles is useful since it enables the use of results about planar graphs to derive conclusions about general embedded graphs. Thomassen [Th93b] proved that if g and d are positive integers, and G is a triangulation of the orientable surface of genus g with face-width at least $8(d+1)(2^g-1)$, then G contains a planarizing set $\{Q_1, \ldots, Q_g\}$ of cycles such that any two distinct cycles Q_i, Q_j are at distance at least d. Moreover, any path starting on the Π-left side of some Q_i and ending on the Π-right side of it has length at least d. Thomassen's result was extended under the same premises to general embeddings by Yu [Yu93p]. Recently, the author of this survey improved the required bound on the face-width that implies existence of planarizing cycles (unpublished).

(5) **Grid minors:** Graaf and Schrijver [GS94] proved that a graph on the torus with face-width $w \geq 5$ contains a toroidal $\lfloor 2w/3 \rfloor \times \lfloor 2w/3 \rfloor$ grid as a surface minor, where the toroidal $k \times k$ grid means the graph $C_k \times C_k$ (the Cartesian product of two k-cycles) with the obvious embedding in the torus. Results of Brunet, Mohar, and Richter [BMR96] imply that every nonplanar orientable embedding of face-width w contains as a surface minor a $\lfloor w/2 \rfloor \times \lfloor w/4 \rfloor$ "twisted grid". Such a grid consists of $k = \lfloor w/2 \rfloor$ disjoint homotopic noncontractible cycles $C_1, \ldots, C_k$ of length $l = \lfloor w/4 \rfloor$ and internally disjoint homotopic paths $P_1, \ldots, P_l$ such that they start at distinct vertices $p_1, \ldots, p_l$ of C_1 (respectively), cross each of $C_2, \ldots, C_k$, and terminate at $p_i, p_{i+1}, \ldots, p_{i+l}$ (indices modulo l), respectively.

It is worth mentioning that proofs in most of the above cases yield polynomial time algorithms to find the desired structures whenever a given embedded graph fulfils the required conditions.

9. Highly locally planar embeddings

A graph G embedded with sufficiently large face-width can be considered as a *locally planar graph* since, by Proposition 3.7, a large neighborhood of any vertex constitutes a planar embedding. Existence of planarizing cycles (cf. Case (4) after Theorem 8.2) enables us to cut along these cycles and get a plane embedded graph H. Most of the faces of the resulting graph coincide with the faces of the original embedding. Only the small number of faces corresponding to the planarizing cycles is different. Using known results (or slight improvements of known results) about plane graphs, we can get an information about the original graph. For example, Thomassen [Th93b] used a strengthening of the planar 5-color theorem to prove:

THEOREM 9.1. (Thomassen [Th93b]) *Let G be a graph that is Π-embedded in an orientable surface of genus g such that the edge-width of Π is at least 2^{14g+6}. Then G is 5-colorable.*

Similarly, Hutchinson extended the fact that planar graphs with all faces of even size are 2-colorable to locally planar graphs.

THEOREM 9.2. (Hutchinson [Hu95]) *Let G be a graph that is Π-embedded in an orientable surface of genus g such that all faces are of even size and the edge-width of Π is at least 2^{3g+5}. Then G is 3-colorable.*

Another well-known result about planar graphs that has an extension to highly locally planar embeddings is Tutte's theorem that every 4-connected planar graph contains a hamiltonian cycle [Tu56]. Thomas and Yu proved that every edge of a 4-connected projective planar graph is contained in a hamiltonian cycle [TY94]. They proved the same result for 5-connected toroidal graphs [TY93p]. For general surfaces we need an additional requirement on the face-width:

THEOREM 9.3. (Yu [Yu93p]) *Let G be a 5-connected triangulation of a surface with Euler genus ε. If the face-width of G is at least $96(2^{\varepsilon}-1)$, then G has a Hamilton cycle.*

Barnette [Ba66] extended Tutte's hamiltonicity theorem to 3-connected graphs by showing that every 3-connected planar graph contains a spanning tree with maximal vertex degree 3 (having maximal vertex degree 2, we would have a hamiltonian path). In [Ba92], the same result is proved for the projective plane, the torus, and the Klein bottle provided the face-width is at least 3. Brunet et al. [BEGMR95] have proved the same result for 3-connected graphs on the torus or the Klein bottle without any assumption on the face-width. (They even prove existence of a 2-walk; see the definition below.) However, as $K_{3,7}$ shows, the same result without assumptions on the face-width cannot be proved for surfaces with negative Euler characteristic. In general we have:

THEOREM 9.4. (Thomassen [Th94]) *If G is a triangulation of an orientable surface of genus g with face-width at least $8(2^{2g}+2)(2^{g}-1)$, then G has a spanning tree of maximum degree at most four.*

Ellingham and Gao [EG94] improved Thomassen's result by showing that every 4-connected triangulation with sufficiently large face-width contains a spanning tree with maximum degree at most three. A strengthening of their result generalized to arbitrary embeddings was obtained by Yu [Yu93p] who established existence of even more restricted structures. A *k-walk* in a graph G is a walk that visits each vertex at least once and at most k times. It is easy to see that the existence of a k-walk implies the presence of a spanning tree with maximum degree at most $k+1$.

THEOREM 9.5. (Yu [Yu93p]) *Let G be a 4-connected graph embedded in a surface of Euler genus ε. If the face-width of the embedding is at least $48(2^{\varepsilon}-1)$, then G contains a 2-walk and hence also a spanning tree of maximum degree 3.*

By restricting to 3-connected graphs, Yu obtained a generalization of Theorem 9.4.

THEOREM 9.6. (Yu [Yu93p]) *Let G be a 3-connected graph embedded in a surface of Euler genus ε. If the face-width of the embedding is at least $48(2^{\varepsilon}-1)$, then G contains a 3-walk and hence also a spanning tree of maximum degree 4.*

Some other results on 2-walks in graphs on surfaces can be found in [GR94], [BEGMR94p], [GRY93p], and a result on 4-walks in [GW94].

Archdeacon, Hartsfield, and Little [AHL96] have examples of k-connected graphs embedded with face-width more than k and with no spanning tree of maximum degree less than k ($k = 1, 2, 3, \dots$). These examples show that conditions on the face-width in above results cannot be dropped or replaced by constant bounds.

Despite the above results, some problems remained unanswered. Does Theorem 9.3 generalize to arbitrary 5-connected embedded graphs with sufficiently large face-width? We also have a fresh conjecture:

CONJECTURE 9.7. *Let G be a 4-connected graph embedded in a surface of genus g with sufficiently large face-width. Then G contains a spanning tree with maximum degree 3 such that the number of vertices of degree 3 is $O(g)$.*

Barnette [Ba94] proved that a 3-connected planar graph contains a 2-connected spanning subgraph with maximal vertex degree at most 15, and Gao [Ga95] proved that 15 can be replaced by 6 (which is the best possible bound). This implies:

THEOREM 9.8. *Let G be a 3-connected graph embedded in a surface of Euler genus ε. If the face-width of the embedding is at least $16(2^{\varepsilon}-1)$, then G contains a 2-connected spanning subgraph of maximal degree at most 14.*

Proof. By [Yu93p], G contains a planarizing set of induced nonadjacent cycles $Q_1, \dots, Q_g$. Cut along these cycles to get a plane graph and add a vertex for each of the new faces joined to all vertices on that face. The resulting plane graph is 3-connected and has a 2-connected spanning subgraph of maximum degree at most 6. The restriction of this subgraph to G together with the cycles $Q_1, \dots, Q_g$ is a 2-connected spanning subgraph of G of maximum degree at most 14. □

By the result of the author (unpublished), much smaller bounds on the face-width imply most of the above results.

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