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The number of matchings of low order in hexagonal systems

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Abstract

A simple way to calculate the number of k-matchings, $k \leqslant 5$, in hexagonal systems is presented. Some relations between the coefficients of the characteristic polynomial of the adjacency matrix of a hexagonal system and the number of matchings are obtained. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

A *hexagonal system* is a 2-connected plane graph G such that every interior face of G is a regular hexagon. A *k-matching* (or a *matching* of order k) of a graph G is a set of k pairwise nonadjacent edges of G.

A hexagonal system has only vertices of degree 2 or 3. Note also that each hexagonal system H is a bipartite graph. It is also easy to see that H does not contain cycles of lengths 4, 8.

Let G be a hexagonal system. Throughout the paper, n will denote the number of vertices whereas m will stand for the number of edges of G. By $A = \{a_{ij}\}_{i,j=1}^{n}$ we will denote the adjacency matrix of G, that is

$$a_{ij} = \begin{cases} 0, & ij \notin E(G), \\ 1, & ij \in E(G). \end{cases}$$

Since every hexagonal system is bipartite, coefficients of the characteristic polynomial of A at $x^{n-1}, x^{n-3}, \ldots$ are zero.

The following result is also well known (cf. [2]) and easily follows from the permutation expansion of the determinant $\det(xI - A)$.

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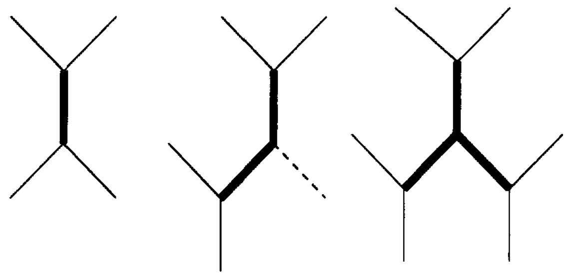


Fig. 1. Situations of types X_1, X_2, and Δ.

Lemma 1.1. *Let G be a hexagonal system. Then its characteristic polynomial is of the form*

$$\det(xI - A) = x^n - a_2x^{n-2} + a_4x^{n-4} - a_6x^{n-6} + \cdots,$$

where $a_2, a_4, a_6, \ldots$ are all nonnegative. If H is a subgraph of G, whose components are either cycles or edges, then we define $\alpha(H)$ as $2^{c(H)}$, where $c(H)$ is the number of cycles in H. Then for each k, $1 \leqslant k \leqslant \lfloor n/2 \rfloor$, a_{2k} is equal to the sum of $\alpha(H)$ over all subgraphs H of G with exactly 2k vertices such that each component of H is either a cycle or an edge.

Let m_k denote the number of k-matchings in a hexagonal system G. It is well known that in case when n is even

$$m_{n/2} = \sqrt{a_n}.$$

This is the connection between the number of perfect matchings and the determinant of the adjacency matrix in hexagonal systems (see, for example [2] or [6]).

In Sections 1, 2, and 3 some further relations will be obtained among the coefficients a_2, a_4, a_6 and m_1, m_2, m_3, respectively. In Proposition 2.2, and Theorems 3.1, 4.1, 5.1 we get simple formulas expressing m_2, m_3, m_4, m_5, respectively, in terms of simple parameters of the hexagonal system. Theorems 2.3, 3.2, 4.2 show how to compute a_4, a_6, a_8. Also, a linear time algorithm for computing m_k, $k = 2,3,4,5$ is presented. The reader is referred to [1,5–7] for some further results on matchings in hexagonal systems.

We will need some additional notation. Let G be a hexagonal system. Let Π_k be the number of paths in G that have exactly k edges. We will denote by X_1 the number of edges of G whose both endpoints have degree 3. Let X_2 be the number of paths in G that have exactly two edges and their endpoints both have degree 3. Further, let Δ be the number of vertices of degree 3 whose all neighbors are of degree 3, see Fig. 1. All these quantities, except Π_k, can be computed in linear time $\mathcal{O}(n)$ by a single search over all vertices of G. Π_k can also be computed in linear time for $k = 2,3,4,5,6$. This can be done by starting the breadth-first search at each vertex and counting how many different vertices we have reached after 2,3,4,5,6 steps. Summing over all the vertices and dividing the sum by 2, gives us Π_k for $k = 2,3,4,5,6$.

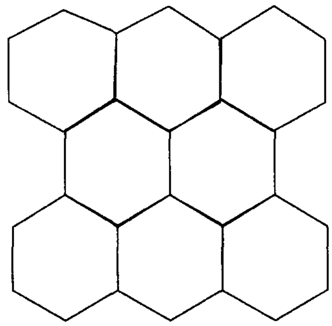
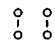
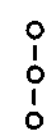


Fig. 2. A hexagonal system with 8 hexagons.

We will also use the symbol $\sharp$ followed by a figure of a graph to denote the number of subgraphs of G isomorphic to the graph shown. So, for example:

$\sharp$  $= \Pi_1$,

$\sharp$  $= m_2$,

$\sharp$  $= \Pi_2$, etc.

Example 1.1. Let G be the hexagonal system shown in Fig. 2. Then

$X_1 = 15, \quad X_2 = 24, \quad \Delta = 6,$
$\Pi_2 = 56, \quad \Pi_3 = 92, \quad \Pi_4 = 152, \quad \Pi_5 = 246.$

Note that G has $\Pi_1 = m = 35$ edges and $n = 28$ vertices.

2. Matchings of order two

In counting the matchings of order two we will derive some results valid for an arbitrary graph. Let G be a simple graph on n vertices and with m edges. Let G have n_i vertices of degree i, $i = 1, 2, \ldots, n$. Then

$$n = \sum_{i=1}^{n} n_i \quad \text{and} \quad 2m = \sum_{i=1}^{n} i n_i.$$

First of all we note that $a_2 = m_1 = m$ simply by applying Lemma 1.1.

Proposition 2.1. *In an arbitrary simple graph with m edges, n vertices and n_i vertices of degree i ($i \geqslant 1$) we have*

$$m_2 = \frac{m(m+1)}{2} - \frac{1}{2} \sum_{i=1}^{n} i^2 n_i.$$

Proof. $m_2 = \binom{m}{2} - \Pi_2$ since the first term is the number of all subsets of $E(G)$ of order 2 and we subtract number of subsets that do not represent a matching. (Two edges do not represent a matching if and only if they form a path.)

Suppose that a vertex v has degree i. Then v is the mid-point vertex of exactly $\binom{i}{2}$ paths of length 2. Summing over all the vertices we obtain $\Pi_2 = \sum_{i=1}^{n} \binom{i}{2} n_i$. Hence,

$$m_2 = \binom{m}{2} - \frac{1}{2}\sum_{i=1}^{n} i^2 n_i + \frac{1}{2}\sum_{i=1}^{n} i n_i. \tag{2.1}$$

The last sum is equal to m which yields the theorem. □

Corollary 2.1. *Suppose that G has only vertices of degrees i and j. Then*

$$m_2 = \tfrac{1}{2}(m^2 + m + ijn - 2im - 2jm).$$

Proof. From $n = n_i + n_j$ and $2m = in_i + jn_j$ we get

$$n_j = \frac{2m - in}{j - i} \quad \text{and} \quad n_i = \frac{nj - 2m}{j - i}.$$

Now we simply apply Proposition 2.1. □

From (2.1) we easily get the following corollary:

Corollary 2.2. *Let G be a k-regular graph on n vertices. Then*

$$m_2 = \tfrac{1}{8}(n^2k^2 + 2nk - 4nk^2).$$

From here on we will consider hexagonal systems only. Let G be an arbitrary hexagonal system with n vertices and m edges. The following theorem shows that n and m uniquely determine the number of 2-matchings.

Proposition 2.2. $m_2 = \frac{1}{2}(m^2 - 9m + 6n)$.

Proof. In Corollary 2.1 put $i = 3$ and $j = 2$. □

Clearly, $n_2 + n_3 = n, 2n_2 + 3n_3 = 2m, n_2 + 3n_3 = \Pi_2$. This implies

Lemma 2.1. $n_2 = 3n - 2m, n_3 = 2m - 2n, \Pi_2 = 4m - 3n$.

The number of 2-matchings is also related to the characteristic polynomial of a hexagonal system.

Observation 2.3. *In a hexagonal system we have $m_2 = a_4$.*

Proof. Since hexagonal systems do not have cycles of length 4, subgraphs H from Lemma 1.1 correspond precisely to 2-matchings in G, and for them we have $\alpha(H) = 1$. □

3. Matchings of order three

Counting the number of 3-matchings is slightly more complicated. For each vertex i let $\Pi_k(i)$ denote the number of paths which have k edges and one of whose endpoints is the vertex i. We will refer to a path which has k edges as a k-path. Clearly,

$$\Pi_k = \frac{1}{2}\sum_{i=1}^{n} \Pi_k(i) = \frac{1}{2}\left(\sum_{i,\deg(i)=3} \Pi_k(i) + \sum_{i,\deg(i)=2} \Pi_k(i)\right). \tag{3.1}$$

We also have

Lemma 3.1. *For* $2 \leqslant k \leqslant 5$,

$$\Pi_k = \frac{1}{2}\left(2 \cdot \sum_{i,\deg(i)=3} \Pi_{k-1}(i) + \sum_{i,\deg(i)=2} \Pi_{k-1}(i)\right).$$

Proof. k being smaller than the length of a shortest cycle in G, any $(k-1)$-path P that has an end vertex v of degree i can be extended through v to exactly $i-1$ k-paths. Using this observation and the fact that in such a way we have counted each k-path twice, the result of the lemma follows. □

Lemma 3.2. $\Pi_3 = 7m - 6n + X_1$.

Proof. Using (3.1) for $k=2$ and Lemma 3.1 for $k=3$, we obtain

$$\Pi_3 = \frac{1}{2}\left(2\Pi_2 + \sum_{i,\deg(i)=3} \Pi_2(i)\right). \tag{3.2}$$

Let i be a vertex of degree 3. Let t_i be the number of its neighbors of degree 3. Clearly, $\Pi_2(i) = t_i + 3$. Therefore,

$$\sum_{i,\deg(i)=3} \Pi_2(i) = \sum_{i,\deg(i)=3} (t_i + 3) = 3n_3 + 2X_1. \tag{3.3}$$

Now we only fill up the values for Π_2 and n_3 from Lemma 2.1 into (3.2) and (3.3). □

The main result of this section is the following.

Theorem 3.1. $m_3 = \frac{1}{6}(m^3 - 27m^2 + 116m + 18mn - 96n + 6X_1)$.

Proof. We will use the formula

$$m_3 = \binom{m}{3} - (m-2)\Pi_2 + \Pi_3 + 2n_3 \tag{3.4}$$

which is obtained as follows. From the number of all 3-subsets (the first term) we subtract the number of those 3-subsets that do not represent 3-matchings. To each 2-path we add an edge that does not lie on this path. Only such subsets do not represent 3-matchings. This yields the second term. However, every 3-path $ijkl$ has been counted twice: therefore, we have to add the third term. Moreover, each subset $\{ij, ik, il\}$, where i is a vertex of degree 3, has been counted thrice: therefore, we have to add the last term.

Eventually, we insert into (3.4) the values for Π_2 and n_3 (Lemma 2.1) and for Π_3 (Lemma 3.2). □

Theorem 3.2.

$$a_6 = m_3 + 2h, \tag{3.5}$$

where $h = m - n + 1$ *is the number of hexagons in* G.

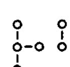
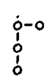
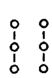
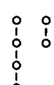
Proof. We apply Lemma 1.1. A subgraph H of G from Lemma 1.1 can only be a hexagon or three pairwise disjoint edges. The former ones have $\alpha(H) = 2$ and are therefore represented by the second term in (3.5) whereas the latter ones are precisely the 3-matchings. Note that we have also used Euler's polyhedron formula, which implies that the number of hexagons is $m - n + 1$. □

4. Matchings of order four

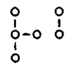
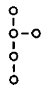
To count the number of matchings of order four, we will use the same method as developed in the proof of Theorem 3.1. Therefore, only equations will be written down and the arguments for their proof omitted.

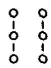
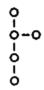
Theorem 4.1. $m_4 = m^4/24 - 9m^3/4 + 707m^2/24 + X_1 m - 329m/4 - X_2 - 10X_1 + 9n^2/2 + 147n/2 - 59mn/2 + 3m^2n/2$.

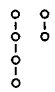
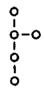
Proof.

$$m_4 = \binom{m}{4} - \binom{m-2}{2}\Pi_2 + 2\Pi_4 + 2 \cdot \#$$  $$+3 \cdot \#$$  $$+ \#$$  $$+ \#$$  . (4.1)

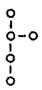
Note that the coefficient in front of each figure is equal to the number of 2-paths in that figure minus 1. This is because one pattern must remain in subtracting the second term. Similarly we get

$(m-3)\,n_3 = \sharp$  $+\,\sharp$  . (4.2)

$\binom{\Pi_2}{2} = \sharp$  $+\,\sharp$  $+\Pi_4 + \Pi_3 + 3n_3.$ (4.3)

$(m-3)\,\Pi_3 = \sharp$  $+2\cdot\sharp$  $+2\Pi_4.$ (4.4)

From each of (4.2)–(4.4) we express the first term on the right hand side and use this in (4.1). We apply expressions for Π_2, Π_3, Π_4, and n_3 from previous sections. If we also use

$\sharp$  $= \sum_{i,\deg(i)=3} \Pi_2(i) = 2X_1 + 3n_3,$ (4.5)

we get the claim. □

Theorem 4.2. $a_8 = m_4 + 2m^2 - 26m - 2mn + 30n - 24.$

Proof. Applying Lemma 1.1 and observing that appropriate subgraphs H are only hexagons together with a disjoint edge ($H = C_6 \cup P_2$) or 4-matchings, we derive

$$a_8 = 2\cdot\sharp(C_6 \cup P_2) + m_4. \tag{4.6}$$

The following relation holds:

$$(m-6)(m-n+1) = (m-6)\cdot\sharp C_6 = \sharp(C_6 \cup P_2) + \sharp(Q), \tag{4.7}$$

where Q denotes a graph consisting of a hexagon plus one pendant edge attached to a vertex of the hexagon.

To count $\sharp(Q)$, we observe that it depends only on vertices of degree 3. Let s_2 be the number of vertices of degree 3 belonging to 2 hexagons and similarly let s_3 be the number of vertices of degree 3 belonging to 3 hexagons. Observe that each vertex of degree 2 lies in exactly one hexagon. Recall that n_i is the number of vertices of degree i. Therefore we get the following system of equations:

$$n_2 + 2s_2 + 3s_3 = 6(m-n+1),$$
$$s_2 + s_3 = n_3,$$
$$n_2 + n_3 = n.$$

Now, it easily follows that $s_3 = 4m - 5n + 6$. If a vertex of degree 3 lies in 2 or 3 hexagons, then such a vertex contributes 2, 3, respectively, to the number $\sharp(Q)$. Therefore,

$$\sharp(Q) = 2n_3 + s_3 = 2n_3 + 4m - 5n + 6. \tag{4.8}$$

Putting (4.6)–(4.8) together, we get the theorem. □

5. Matchings of order five

For 5-matchings we can apply the same method as in the proof of Theorem 4.1 but the task is more tedious. The proof is rather technical and, therefore, omitted.

Theorem 5.1.

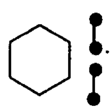
$$\begin{aligned} m_5 = {} & \frac{m^5}{120} - \frac{3m^4}{4} + \frac{475m^3}{24} - \frac{677m^2}{4} + \frac{1661m}{5} - 48n^2 - 308n + \frac{9mn^2}{2} + \frac{nm^3}{2} \\ & - \frac{43m^2n}{2} + \frac{407nm}{2} + 72X_1 - \frac{29mX_1}{2} + \frac{m^2X_1}{2} \\ & + 3nX_1 + 10X_2 - mX_2 + 2\Delta + \Pi_5. \end{aligned}$$

6. Concluding remarks

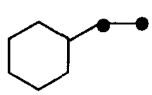
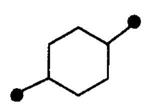
Proposition 6.1. *Let G be a hexagonal system. For $k = 1, \ldots, \lfloor n/2 \rfloor$ we have $a_{2k} \geqslant m_k$.*

Proof. For a_{2k} we have a suitable representation in Lemma 1.1. All the terms in the sum are nonnegative and some terms of the sum represent k-matchings. Since each term is nonnegative, the theorem follows. □

In Section 5 we did not relate m_5 and a_{10}. It is 'difficult' to calculate the number of subgraphs isomorphic to the graph



This number can still be expressed in terms of

 and $\sharp$ ,

where the two added edges in the last graph are allowed to be in any of the six positions on the hexagon.

What about counting the number of 6-matchings? The methods used in this paper can be applied but the formulas become much more complicated.

Example 6.1. Let us apply our results to the hexagonal system discussed in Example 1.1. All values required have already been determined ($m=35$, $n=28$). Using theorems concerning matchings, we can easily compute

$$m_1=35, \quad m_2=539,$$
$$m_3=4817, \quad m_4=27742, \quad m_5=108104.$$

According to our results expressing a_{2i} with m_i ($i=1,2,3,4$) we can in an easy way write down the first terms of the characteristic polynomial, namely, $x^{28}-35x^{26}+539x^{24}-4833x^{22}+28138x^{20}-\cdots$. □

For some examples of the matching polynomial

$$m(G;x)=x^n-m_1x^{n-2}+m_2x^{n-4}-\cdots$$

of a hexagonal system G see [3,4].

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