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On list edge-colorings of subcubic graphs

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Abstract

In this paper we study list edge-colorings of graphs with small maximal degree. In particular, we show that simple subcubic graphs are '10/3-edge choosable'. The precise meaning of this statement is that no matter how we prescribe arbitrary lists of three colors on edges of a subgraph H of G such that $\Delta(H) \leqslant 2$, and prescribe lists of four colors on $E(G) \backslash E(H)$, the subcubic graph G will have an edge-coloring with the given colors. Several consequences follow from this result.

Keywords: Coloring; List coloring; Edge coloring; Cubic graph

1. Introduction

All graphs in this paper are undirected and finite. They have no loops but they may contain multiple edges and edges with only one end, called *halfedges*. A graph is *simple* if it has no halfedges and no multiple edges. The maximal degree of G is denoted by $\Delta(G)$. A graph is *subcubic* if $\Delta(G) \leqslant 3$. A *list assignment* of G is a function L which assigns to each edge $e \in E(G)$ a *list* $L(e) \subseteq N$. The elements of the list $L(e)$ are called *admissible colors* for the edge e. An *L-edge-coloring* is a function $\lambda : E(G) \to N$ such that $\lambda(e) \in L(e)$ for $e \in E(G)$ and such that for any pair of adjacent edges e, f in G, $\lambda(e) \neq \lambda(f)$. If G admits an L-edge-coloring, it is *L-edge-colorable*. For $k \in N$, the graph is *k-edge-choosable* if it is L-edge-colorable for every list assignment L with $|L(e)| \geqslant k$ for each $e \in E(G)$.

List colorings were introduced by Vizing [5] and independently by Erdős et al. [1]. Probably, the most well-known conjecture about list colorings is the following conjecture about list-edge-chromatic numbers (see [4, Problem 12.20]). It states that every (multi)graph G is $\chi'(G)$-edge-choosable, where $\chi'(G)$ is the usual chromatic index of G. In 1979 Dinitz posed a question about a generalization of Latin squares

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which is equivalent to the assertion that every complete bipartite graph $K_{n,n}$ is n-edge-choosable. This problem became known as the Dinitz conjecture and resisted proofs up to 1995 when Galvin [2] proved the conjecture in the affirmative. More generally, Galvin established that every bipartite (multi)graph G is $\Delta(G)$-edge-choosable. Another recent result about list-edge-chromatic numbers is a result of Häggkvist and Janssen [3], who proved that every simple graph with maximal degree Δ is $(\Delta + \mathcal{O}(\Delta^{2/3}\sqrt{\log \Delta}))$-edge-choosable.

In this paper we study list edge colorings of graphs with small maximal degree. In particular, we show that simple subcubic graphs are '$\frac{10}{3}$-edge-choosable'. The precise meaning of this statement is that no matter how we prescribe arbitrary lists of three colors on edges of a subgraph H of G such that $\Delta(H) \leqslant 2$, and prescribe lists of four colors on $E(G)\backslash E(H)$, the subcubic graph G will have an edge-coloring with the given colors. Some consequences of this result are also presented.

2. Coloring paths and cycles with halfedges

Let G be a graph and H a subgraph of G. Each edge $e \in E(G)\backslash E(H)$ with both ends in H is a *chord* of H.

Let G be a graph and S its set of halfedges. If $\tau : S \to S$ is an involution, then we say that $s \in S$ is *τ-free* if $\tau(s) = s$, and *τ-constrained* otherwise. Let s and $\tau(s) \neq s$ be a τ-constrained pair. If L is a list assignment and λ an L-edge-coloring of G, we say that λ is *residually distinct* at s (and at $\tau(s)$) if $|L(s)\backslash\{\lambda(e),\lambda(f)\} \cup L(\tau(s))\backslash \{\lambda(e'),\lambda(f')\}| \geqslant 3$ whenever e, f and e', f' are edges of G adjacent to s and $\tau(s)$, respectively.

Lemma 2.1. *Let G be a subcubic graph of order $n \geqslant 2$ that is composed of a Hamilton path H, a set S of halfedges, and a set D of chords. Suppose that σ and τ are involutions of S such that no σ-constrained halfedge is τ-constrained. Suppose also that no σ- or τ-constrained halfedge is incident with an endvertex of H and that there is no chord joining the endvertices of H. Let L be a list assignment such that*

$$|L(e)| \geqslant \begin{cases} 4, & e \in D, \text{ or } e \in S \text{ is } \tau\text{-constrained}, \\ 3, & e \in E(H), \text{ or } e \in S \text{ is } \sigma\text{-constrained}, \\ 2, & e \in S \text{ is } \tau\text{-free and } \sigma\text{-free}. \end{cases} \quad (1)$$

Moreover, if each endvertex of H is incident with two halfedges, then at least one endvertex is incident with halfedges s, s' such that $|L(s) \cup L(s')| \geqslant 3$. Then G has an L-edge-coloring λ such that for each pair of distinct halfedges s, s' with $\sigma(s) = s'$ we have $\lambda(s) \neq \lambda(s')$ and such that for each τ-constrained halfedge s, λ is residually distinct at s.

Proof. Since no chord is adjacent to a constrained halfedge, multiple edges that are in D can be removed and colored at the end. Therefore, we may assume that G contains no multiple edges. We may also assume that G has only vertices of degree 3 (by adding additional halfedges with arbitrary lists of two new colors if necessary), and that we have equalities in (1).

We enumerate the vertices of H as $v_1,\dots,v_n$ as they appear on H and denote by $e_i \in E(H)$ the edge joining v_i and v_{i+1} ($1 \leqslant i < n$). By our assumptions, we may achieve that v_n is not an endvertex incident with two halfedges with the same pair of admissible colors. For $i = 1,\dots,n$, let s_i be the chord or the halfedge adjacent to v_i, and let s_1' and s_n' be the additional edges adjacent to v_1 and v_n, respectively.

Suppose first that all halfedges are τ-free and σ-free. We start coloring edges of G at vertex v_1. If both s_1 and s_1' are halfedges, then let $\lambda(s_1)$ be an arbitrary color from $L(s_1)$, let $\lambda(s_1')$ be a color from $L(s_1')\backslash\{\lambda(s_1)\}$, and let $\lambda(e_1)$ be a color from $L(e_1)\backslash\{\lambda(s_1),\lambda(s_1')\}$. If one of s_1 or s_1' is a halfedge and the other one is a chord (say s_1 is a halfedge and s_1' is a chord), then we color s_1 with a color from $L(s_1)$ and e_1 with a color from $L(e_1)\backslash\{\lambda(s_1)\}$. We shall color s_1' when encountered for the second time and then we shall regard it as a halfedge with a list of two colors from $L(s_1')\backslash\{\lambda(s_1),\lambda(e_1)\}$. If both s_1 and s_1' are chords, then we color e_1 with a color from $L(e_1)$. Let v_k and $v_{k'}$ be the other ends of s_1 and s_1', respectively. We may assume that $k < k'$. We will color s_1 when encountered at v_k, and after that we will treat s_1' in the same way as described above.

In a general step i, $1 < i < n$, we assume that we have colored e_{i-1}. If s_i is a halfedge, let $\lambda(s_i)$ be a color from $L(s_i)\backslash\{\lambda(e_{i-1})\}$ and let $\lambda(e_i)$ be a color from $L(e_i)\backslash\{\lambda(e_{i-1}),\lambda(s_i)\}$. Otherwise, s_i is a chord. If s_i is incident with v_n and v_n is incident with a halfedge, say s_n, then let $\lambda(e_i)$ be a color from $L(e_i)\backslash\{\lambda(e_{i-1})\}$ such that there exist two colors $p,q \in L(s_i)\backslash\{\lambda(e_{i-1}),\lambda(e_i)\}$ such that $\{p,q\} \neq L(s_n)$. Otherwise, color e_i arbitrarily with a color from $L(e_i)\backslash\{\lambda(e_{i-1})\}$ and choose $\{p,q\} \subseteq L(s_i)\backslash \{\lambda(e_{i-1}),\lambda(e_i)\}$. From now on, we shall regard s_i as a halfedge incident with the other endvertex having as the admissible colors the pair $\{p,q\}$.

After we have colored e_{n-1}, color s_n and s_n' with distinct colors from $L(s_n)\backslash\{\lambda(e_{n-1})\}$ and $L(s_n')\backslash\{\lambda(e_{n-1})\}$, respectively. Note that such colors exist since $|L(s_n)\cup L(s_n')| \geqslant 3$. This gives an L-edge-coloring of G.

If some halfedges are σ- or τ-constrained, we can apply the same method as above. Observe that after coloring the first halfedge of a σ-constrained pair, the second halfedge s behaves like a σ-free halfedge since it has (at least) two admissible colors left. Similar technique is used for τ-constrained halfedges with the difference that for the second halfedge s of the pair we choose a pair of colors from $L(s)$ that is disjoint from the residuum at $\tau(s)$. This assures that λ will be residually distinct at s and $\tau(s)$. □

The next lemma shows a result related to Lemma 2.1 in case of cycles instead of paths. Although similar in nature, its proof is much more involved than the proof of Lemma 2.1.

Lemma 2.2. *Let G be a subcubic graph of order $n \geqslant 3$ composed of a Hamilton cycle H, a set S of halfedges, and a set D of chords of H. Suppose that τ is an involution* *of S such that there is at most one τ-constrained pair of halfedges. Let L be a list assignment such that*

$$|L(e)| \geqslant \begin{cases} 4, & e \in D, \text{ or } e \in S \text{ is } \tau\text{-constrained}, \\ 3, & e \in E(H), \\ 2, & e \in S \text{ is } \tau\text{-free}. \end{cases} \tag{2}$$

Then G has an L-edge-coloring λ that is residually distinct at each τ-constrained halfedge unless $\tau = id$, $D = \emptyset$, *H is an odd cycle, each vertex of G has a halfedge, and there are colors a, b, c such that* $L(e) = \{a,b,c\}$ *for each* $e \in E(H)$ *and* $L(e) = \{a,b\}$ *for each* $e \in S$.

Proof. Since multiple edges can be removed and colored at the end, we assume that there are none. We may assume that G has only vertices of degree 3 (since otherwise we can add halfedges with arbitrary lists of two new colors). We may as well assume that we have equalities in (2).

Suppose first that $D \neq \emptyset$ or $\tau \neq id$. For the first subcase, suppose that G is a cubic graph without τ-free halfedges and that all edges e on H have the same list $L(e) = \{a,b,c\}$ of colors. It is easy to see that there exists an L-edge-coloring of H which is residually distinct at τ-constrained halfedges. Clearly, each chord has an admissible color distinct from a,b,c that can be used to obtain an L-edge-coloring of G.

Otherwise, let $v_1, v_2, \ldots, v_n$ be the vertices of G as they appear on H. For $i = 1, \ldots, n$, denote by e_i the edge $v_i v_{i+1} \in E(H)$ (index $i+1$ taken modulo n) and by s_i the chord or the halfedge incident with v_i. Since $D \neq \emptyset$ or $\tau \neq id$, we can assume that v_n is incident with a chord or a τ-constrained halfedge and that either v_1 is incident with a τ-free halfedge (if S contains a τ-free halfedge), or we have $L(e_1) \neq L(e_n)$. Suppose that the other endvertex of the chord at v_n is v_m $(1 < m < n-1)$. Similarly, if s_n is a τ-constrained halfedge, let v_m $(1 \leqslant m < n)$ be the endvertex of $\tau(s_n)$. If v_1 is incident with a chord, let v_k be the other end of this chord. If s_1 is a halfedge, let v_k be the end of $\tau(s_1)$. If s_n is τ-constrained, it may happen that $m = 1$. However, we can always achieve (by possibly reversing the orientation of the cycle, leaving v_1 fixed) that $m > 1$.

We will construct an L-edge-coloring λ by coloring edges of G one after another in the following order: $e_1, (s_2), e_2, (s_3), \ldots, e_n, s_1$ where the notation (s_i) means that we do not color (s_i) if it is a chord and its other end is either v_1 or v_j $(j > i)$. The exception to this rule is the chord s_k when $k < m$.

We color e_1 as follows: if s_1 is a τ-free halfedge, let $\lambda(e_1)$ be any color from $L(e_1) \backslash L(s_1)$. This is possible since $|L(e_1)| = 3$ and $|L(s_1)| = 2$. Otherwise, let $\lambda(e_1)$ be an element from $L(e_1) \backslash L(e_n)$. Note that this is possible by our assumption that $L(e_1) \neq L(e_n)$, when s_1 is a chord or a τ-constrained halfedge. In a general step $i > 1$ we assume that we have chosen a color $\lambda(e_{i-1})$ and we color (s_i) and e_i. We distinguish seven cases:

(1) $i \notin \{k, m, n\}$. In this case, if $s_i \in S$ is τ-free, let $\lambda(s_i)$ be a color from $L(s_i) \backslash \{\lambda(e_{i-1})\}$ and let $\lambda(e_i)$ be a color from $L(e_i) \backslash \{\lambda(e_{i-1}), \lambda(s_i)\}$. If $s_i \in D$ or $s_i \in S$

is τ-constrained, let $\lambda(e_i)$ be a color from $L(e_i)\backslash\{\lambda(e_{i-1})\}$. Now, the list $L(s_i)$ contains two elements, say p,q, distinct from $\lambda(e_{i-1})$ and $\lambda(e_i)$. If $s_i\in D$, we shall color s_i when encountered for the second time and we shall regard it at that time as a τ-free halfedge with admissible pair of colors $\{p,q\}$. If $s_i\in S$ is τ-constrained, then we choose $\lambda(s_i)=p$ and we shall consider $\tau(s_i)$ as a τ-free halfedge with a pair $\{\alpha,\beta\}$ of admissible colors from $L(\tau(s_i))\backslash\{p,q\}$. We say that the pair $\{\alpha,\beta\}$ is *forced* by $\lambda(e_{i-1})$ and $\lambda(e_i)$. Note that such a choice ensures that λ will be residually distinct at s_i and $\tau(s_i)$.

(2) $i=k$ and $1<k<m$. In this case $s_1=s_k$ is a chord or $\tau(s_k)=s_1$. If $s_k\in D$, we color s_k arbitrarily with a color from $L(s_k)\backslash\{\lambda(e_1),\lambda(e_{k-1})\}$ and color e_k with a color from $L(e_k)\backslash\{\lambda(e_{k-1}),\lambda(s_k)\}$. Otherwise, we color e_i and determine p, q as in (1). Then we color s_1 with a color from $L(s_1)\backslash\{p,q,\lambda(e_1)\}$ and color s_i with an admissible color. As in (1), this choice ensures residual distinctness.

(3) $i=m$ and $k<m$. Note that $L(e_n)$ contains two distinct colors a,b such that the so far constructed L-edge-coloring λ can be extended to e_n by using either of these two colors. Moreover, by selecting any of a,b as a color of e_n, we can extend the coloring also to s_1 if s_1 has not yet been colored. If $s_m\in D$, let d be a color in $L(s_n)\backslash\{a,b,\lambda(e_{m-1})\}$. Now, we color e_m by using a color from $L(e_m)\backslash\{\lambda(e_{m-1}),d\}$. We shall regard s_n as a halfedge at v_n with the list of colors $\{d,r\}\subseteq L(s_n)\backslash\{\lambda(e_{m-1}),\lambda(e_m)\}$. If s_m is τ-constrained, we color e_m and s_m so that the forced pair $\{\alpha,\beta\}$ of colors for s_n is distinct from the pair $\{a,b\}$. We shall regard s_n as a τ-free halfedge with the list of colors $\{\alpha,\beta\}$.

(4) $i=m$ and $k>m$. Color e_m with a color from $L(e_m)\backslash\{\lambda(e_{m-1})\}$. If $s_m\in D$, let x,y be two colors from $L(s_m)$ distinct from $\lambda(e_{m-1})$ and $\lambda(e_m)$. If $s_m\in S$, we color it by an available color, and let $\{x,y\}$ be a pair forced by $\lambda(e_{m-1})$ and $\lambda(e_m)$. We shall now regard s_n as a τ-free halfedge at v_n with the list of colors $L(s_n)=\{x,y\}$.

(5) $i=k$ and $k>m$. Let p be a color from $L(e_n)\backslash L(s_n)$. (Note that we regard s_n after Step (4) as a halfedge and hence $|L(s_n)|=2$.) If $s_k\in D$, choose $\lambda(s_k)$ from $L(s_k)\backslash\{\lambda(e_1),p,\lambda(e_{k-1})\}$ and let $\lambda(e_k)$ be a color from $L(e_k)\backslash\{\lambda(e_{k-1}),\lambda(s_k)\}$. If $s_k\in S$, we can choose $\lambda(e_k)\in L(e_k)\backslash\{\lambda(e_{k-1})\}$ such that the forced pair $\{\alpha,\beta\}$ on s_1 contains a color q distinct from p and $\lambda(e_1)$. We color s_1 by q and color s_k arbitrarily.

(6) $i=n$ and $k<m$. First case is when $s_m\in D$. Let a,b,d, and r be colors from Step (3). If $\lambda(e_{n-1})=d$, color s_n with r. Otherwise let $\lambda(s_n)=d$. Since $d\notin\{a,b\}$, we can color e_n using a color from $\{a,b\}\backslash\{\lambda(e_{n-1}),\lambda(s_n)\}$. If s_1 is a halfedge, we can color s_1 by a color from $L(s_1)\backslash\{\lambda(e_n)\}$, since $\lambda(e_1)\notin L(s_1)$. This gives an L-edge-coloring of the entire graph G. The second case is when s_m and s_n form the original τ-constrained pair. Let a,b,α,β be colors from (3). Since $\{a,b\}\neq\{\alpha,\beta\}$, we can color s_n and e_n by their admissible colors distinct from $\lambda(e_{n-1})$. If $k=1$, we also color s_1 and thus obtain an L-edge-coloring of G.

(7) The last possibility is when $i=n$ and $k>m$. Let x,y, and p be the colors defined in Steps (4) and (5). If $\lambda(e_{n-1})\neq p$, let $\lambda(s_n)$ be a color from $\{x,y\}\backslash\{\lambda(e_{n-1})\}$ and let $\lambda(e_n)=p$. Otherwise, let $\lambda(e_n)$ be a color from $L(e_n)\backslash\{p,\lambda(s_1)\}$, and

let $\lambda(s_n)$ be a color from $\{x,y\}\backslash\{\lambda(e_n)\}$. Again, we obtain an L-edge-coloring of G.

Suppose now that $D=\emptyset$ and $\tau=id$. We will use a similar coloring procedure as above. Let us choose v_1 such that $L(s_2)\not\subseteq L(e_1)$. If such a choice is not possible, let v_1 be such that $L(s_1)\neq L(s_2)$ or $L(e_1)\neq L(e_n)$. (If also this rule cannot be satisfied, G is as excluded by our lemma except that its length may be even. However, in that case it can easily be L-colored.)

Let us start coloring at the vertex v_1. Color e_1 with a color from $L(e_1)\backslash L(s_1)$ and proceed to the vertex v_2.

At vertex $v_i\,(2\leqslant i<n)$, we color s_i with a color from $L(s_i)\backslash\{\lambda(e_{i-1})\}$ and e_i with a color from $L(e_i)\backslash\{\lambda(e_{i-1}),\lambda(s_i)\}$ and then proceed to the next vertex. Arriving at v_n, it remains to color s_n, e_n, and s_1. By our choice of $\lambda(e_1)$, every L-edge-coloring of s_n and e_n can be extended to s_1. So, an obstruction can occur only when coloring the edge e_n. Suppose that $L(s_n)=\{c,d\}, L(s_1)=\{a,b\}, x=\lambda(e_1), y=\lambda(s_2)$, and $z=\lambda(e_2)$. If we cannot color e_n, we have: $\lambda(e_{n-1})\in\{c,d\}$ (say $\lambda(e_{n-1})=c$) and $L(e_n)=\{c,d,x\}$. If $L(e_1)\neq\{x,y,z\}$, we recolor e_1 by using a color in $L(e_1)\backslash\{x,y,z\}$, and set $\lambda(s_n)=d$, $\lambda(e_n)=x$. Since $x\notin L(s_1)$, there is also an available color for s_1. Therefore, $L(e_1)=\{x,y,z\}$. If $L(s_2)\neq\{y,z\}$, we recolor: $\lambda(s_2)\in L(s_2)\backslash\{y,z\}$, $\lambda(e_1)=y$, $\lambda(e_n)=x$, $\lambda(s_n)=d$, and $\lambda(s_1)\in L(s_1)\backslash\{y\}$.

Therefore, $L(s_2)=\{y,z\}\subseteq L(e_1)$. Then v_1 was not selected according to the first rule, and hence also $L(s_1)\subseteq L(e_n)$ and $L(s_1)\subseteq L(e_1)$. This implies that $L(s_1)=\{y,z\}$ and $L(e_n)=\{x,y,z\}$, which contradicts our choice of v_1. □

If there are more than two τ-constrained halfedges, Lemma 2.2 can be strengthened as follows.

Lemma 2.3. *Let G be a subcubic graph of order $n\geqslant 3$ composed of a Hamilton cycle H, a set S of halfedges, and a set D of chords of H. Suppose that $\tau\neq id$ is an involution of S, and let s_0 be a τ-constrained halfedge. Let L be a list assignment such that*

$$|L(e)|\geqslant\begin{cases}4, & e\in D, \text{ or } e\in S \text{ is } \tau\text{-constrained},\\ 3, & e\in E(H),\\ 2, & e\in S \text{ is } \tau\text{-free}.\end{cases}\tag{3}$$

Then G has an L-edge-coloring λ that is residually distinct at each τ-constrained halfedge distinct from s_0 and $\tau(s_0)$. If there exists a τ-free halfedge, we can also achieve that λ is residually distinct at s_0 and $\tau(s_0)$.

Proof. We may assume that there are more than two τ-constrained halfedges (otherwise Lemma 2.2 applies). If there is a τ-free halfedge, the proof of Lemma 2.2 chooses the case where s_1 is a τ-free halfedge and s_n is either in D or τ-constrained. Now, the proof of Lemma 2.2 yields the result of Lemma 2.3. If there are no τ-free halfedges, we change τ so that s_0 and $\tau(s_0)$ become τ-free and the above arguments apply. □

3. Coloring subcubic graphs

Let Y be a graph of order 4 composed of a copy of $K_{1,3}$ together with a pair of parallel edges between two vertices in the larger bipartition class (see Fig. 1).

Lemma 3.1. *Let L be a list assignment of Y which assigns to each edge of Y at least as many colors as indicated by the numbers in Fig.* 1(a). *Then Y can be L-colored.*

Lemma 3.2. *Let L be a list assignment of Y which assigns to each edge of Y at least as many colors as indicated by the numbers in Fig.* 1(b). *Then Y can be L-colored unless the admissible colors are as shown in Fig.* 1(c).

Proofs of Lemmas 3.1 and 3.2 are straightforward and are left to the reader as an easy exercise.

Let G be a subcubic graph with the set S of halfedges and let $F \subseteq E(G)$ be an edge set such that each vertex of G of degree 3 is incident with either a halfedge or an edge from F. Let L be a list assignment for G such that

$$|L(e)| \geqslant \begin{cases} 4, & e \in F, \\ 3, & e \in E(G)\backslash(F \cup S), \\ 2, & e \in S. \end{cases} \tag{4}$$

Suppose that G contains a subgraph $\tilde{Y}$ isomorphic to Y. Denote by u_0 the vertex of degree 1 in $\tilde{Y}$ and let e_0 the edge of $\tilde{Y}$ incident with u_0. Lemmas 3.1 and 3.2 show that there is at most one color $c_0 \in L(e_0)$ such that $\tilde{Y}$ cannot be $\tilde{L}$-colored where $\tilde{L}(e_0) = \{c_0\}$ and $\tilde{L}(e) = L(e)$ for $e \in E(\tilde{Y})\backslash\{e_0\}$. Let G' be the graph obtained from G by replacing $\tilde{Y}$ by a halfedge $\tilde{e}$ incident with u_0, where the admissible colors for $\tilde{e}$ are $L(e_0)\backslash\{c_0\}$ if c_0 exists, and $L(e_0)$ otherwise. Lemmas 3.1 and 3.2 guarantee that G can be L-colored if and only if G' can be L-colored. Repeating the above reduction, we can achieve that the obtained graph contains no subgraphs isomorphic to Y. Such a graph is called *reduced*. Notice that simple graphs are always reduced.

Given a reduced subcubic graph G and $F \subseteq E(G)$ as above, the subgraph $H = G - F$ of G is a disjoint union of paths (possibly of length 0) and cycles with halfedges,

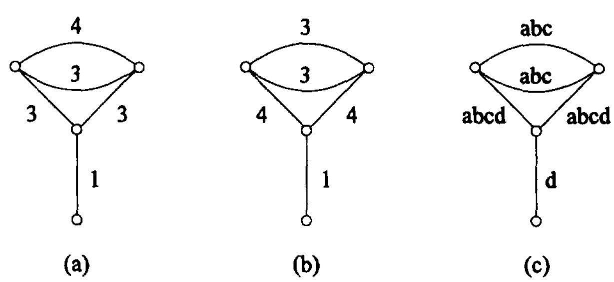

Fig. 1. Graph Y and the exceptional list assignment.

called *path components* and *cycle components* of H, respectively. A path component with one vertex is called *trivial*. A non-trivial path component Q of H is *bad* if each of its endvertices is incident with two halfedges and each pair of these halfedges has the same list of two admissible colors. If this happens at one end of Q only and the other end of Q is not incident with two halfedges having distinct pairs of admissible colors, then Q is *potentially bad*. Similarly, a cycle component Q is *bad* if it is an odd cycle whose edges all have the same list of three admissible colors, say $\{a,b,c\}$, all halfedges of Q have the same pair of admissible colors contained in $\{a,b,c\}$, and each vertex of Q is incident with a halfedge. If we replace the condition that all vertices of Q are incident with halfedges and require that Q contains at least two halfedges and a vertex not incident with a halfedge, then we say that Q is *potentially bad*. A trivial path component is *bad* if it contains three halfedges with the same pair of admissible colors on each of them. It is *potentially bad* if it has precisely two halfedges e, f and $|L(e)\cup L(f)|=2$.

Proposition 3.3. *Let G be a reduced subcubic graph and let S,F,H, and the list assignment L be as above. If H has no bad components and has at most one potentially bad component, then G is L-edge-colorable.*

Proof. We may assume that we have equalities in (4). We may also assume that all vertices have degree 3 by adding additional halfedges with new colors if necessary. Moreover, we may assume that no two endvertices of distinct path components of H are connected by an edge from F; otherwise we can remove such an edge from F. These changes can be done so that no bad components occur and no new potentially bad components arise (except that the potentially bad component may change into a larger path). Similarly, if an edge $e\in F$ joins endvertices of the same path: we can select three colors from $L(e)$ to be the new list and remove e from F, so that the path component changes into a cycle component which is neither bad nor potentially bad.

The proof proceeds by induction on the number of components of H, the base of induction being the empty graph. For the inductive step we shall first select a component Q of H. Let Q_F be the set of edges in F with one endvertex in Q and the other in $V(G)\backslash V(Q)$. Let $\bar{Q}$ be the graph obtained from Q by adding all edges from F with both endvertices in Q and by replacing each edge $e\in Q_F$ by a halfedge $\bar{e}$. We shall assign to $\bar{e}$ a list $L(\bar{e})\subseteq L(e)$ and then L-edge-color $\bar{Q}$. Moreover, some halfedges of $\bar{Q}$ will be σ- or τ-constrained in order to avoid bad and potentially bad components in the remaining graph G'. The graph G' is obtained from G by removing $V(Q)$ and replacing each edge $uv\in Q_F$, $u\in V(G')$, $v\in V(Q)$, by a halfedge incident with u whose list of admissible colors is $L(uv)$ without the colors used when coloring the edges of Q incident with v. Additionally, if Q is a path component and v its endvertex incident with two edges e,e' from Q_F, then e and e' become halfedges in G' with (at least) three admissible colors, but we must require that they receive distinct colors when coloring G'. Therefore, we regard them as σ-constrained in G'.

When selecting Q we will take care so that for each σ-constrained pair at least one of the halfedges will be on a path component of G'. Therefore, cycle components will not contain σ-constrained pairs of halfedges. Since ends of distinct path components are not adjacent, σ-constrained edges are not incident with endvertices of path components in G'. If e and e' are in the same path component of G', then Lemma 2.1 will take care that they will receive distinct colors. If they are in distinct components, one of them will become σ-free with two admissible colors left after coloring the other one. Then an L-edge-coloring of G', obtained by the induction hypothesis, and the coloring of $\bar{Q}$ give rise to an L-edge-coloring of G (where the edges in Q_F receive colors from the coloring of G').

It remains to show how to select Q, how to determine σ and τ on $\bar{Q}$, and how to color $\bar{Q}$ such that G' has at most one potentially bad path or cycle component.

If G contains a potentially bad cycle component, we select this component as Q. If two edges e, f of Q_F lead to the same endvertex of a non-trivial path component Q', then $\bar{e}$ and $\bar{f}$ are τ-constrained and $L(\bar{e}) = L(e)$, $L(\bar{f}) = L(f)$. If $e_1, \dots, e_k$ $(k \geqslant 2)$ are edges from Q_F leading to the same cycle component Q' where Q' has no halfedges, then we let $\bar{e}_1, \bar{e}_2$ be τ-constrained with admissible colors as above and for $i = 3, \dots, k$, we let $\bar{e}_i$ be τ-free halfedges with a pair $L(\bar{e}_i) \subseteq L(e_i)$ of admissible colors. We do the same as above also in the case when two or three edges of Q_F lead to a trivial path component Q'. Such choices in all of the above cases ensure that in G' the component Q' will not be bad or potentially bad whenever under the coloring of $\bar{Q}$, the τ-constrained edges are residually distinct. If $e \in Q_F$ leads to a cycle component R with at least one halfedge, say f, then we choose $L(\bar{e})$ to be a 2-element subset of $L(e)$ which is disjoint from $L(f)$. This choice guarantees that R will not become a potentially bad component in G'. Similarly, if e leads to an end of a path component which has a halfedge f at the same vertex. In other cases, $L(\bar{e})$ is an arbitrary 2-subset of $L(e)$. If $\bar{Q}$ is not the odd cycle obstruction from Lemma 2.2, then it can be L-edge-colored by Lemma 2.2 or 2.3 so that no bad or potentially bad component is introduced in G' (since Q contains halfedges). If $\bar{Q}$ is an odd cycle obstruction, then all halfedges are τ-free. Since Q is not bad (it is only potentially bad), $Q_F \neq \emptyset$. By changing the list of an edge $\bar{e}, e \in Q_F$, $\bar{Q}$ becomes colorable. The construction of L and τ guarantees that in G' only the component containing the endvertex of e not in Q may become potentially bad.

Suppose next that G contains a cycle component Q which has at least one σ-constrained halfedge e_0. Note that $|L(e_0)| \geqslant 3$. Then we apply the same method as above and select a pair of admissible colors from $L(e_0)$ such that $\bar{Q}$ is not an odd cycle obstruction. By Lemma 2.3 we can color $\bar{Q}$ such that the coloring is residually distinct at all τ-constrained pairs and, as before, we see that no new potentially bad components arise.

If G has a potentially bad trivial path component Q, we color its halfedges and remove Q. Clearly, G' has at most one potentially bad component.

Suppose now that G has a non-trivial path component R which is potentially bad. Let v be the endvertex of R which is not incident with two halfedges having the same pair of admissible colors. Let e, e' be the halfedges or edges of $E(G)\backslash E(R)$ incident with v. If $e, e' \in F$ lead, respectively, to cycle components Q, Q' (possibly $Q = Q'$), then we will choose Q to be colored next. (Otherwise e, e' might become a σ-constrained pair with both halfedges belonging to cycle components.) $\bar{Q}$, τ and admissible colors for $\bar{Q}$ are determined as above. If $Q \neq Q'$, then $\bar{e}$ is a τ-free halfedge in $\bar{Q}$. Since $L(\bar{e})$ can be chosen so that $\bar{Q}$ is not an odd cycle obstruction, no new potentially bad component is introduced in G'. Moreover, R remains (only) potentially bad in G'. If $Q = Q'$, then $\bar{e}$ and $\bar{e}'$ are τ-constrained. Since G is reduced, the order of Q is at least 3. Therefore, Lemma 2.3 (or Lemma 2.2 if $\bar{e}, \bar{e}'$ is the only τ-constrained pair in $\bar{Q}$) shows that there is a coloring of $\bar{Q}$ that is residually distinct at $\bar{e}, \bar{e}'$. Hence, R is no longer potentially bad in G', but we may obtain a new potentially bad component in G' due to the fact that the coloring is not residually distinct at one of the τ-constrained pairs. (If a component became bad, it would be a cycle component, say $\tilde{Q}$, and there would be at least three edges between Q and $\tilde{Q}$. One of them would give rise to a halfedge in $\bar{Q}$, hence, we could have taken care of all τ-constrained pairs, a contradiction.) The last case is when e or e' is a halfedge or one of them leads to a path component. (Recall that we have assumed at the beginning of the proof that none of e, e' lead to an endvertex of a path component distinct from R.) Then we select $Q = R$. We determine τ and lists of admissible colors on halfedges $\bar{e}$, $e \in Q_F$, as in the case of cycles. Note that some pairs of halfedges of Q may be σ-constrained. If $\bar{Q}$ has the same pair of admissible colors also at halfedges incident with v, we change one of the pairs. (In such a case, in G' a new potentially bad component may arise.) To color $\bar{Q}$ we apply Lemma 2.1 which also takes care of σ-constrained pairs in $\bar{Q}$. Note that no σ-constrained halfedge e of Q has its mate $\sigma(e)$ in a cycle component (by our choice of Q), and that $\sigma(e)$ is not incident with an end of a path component. Therefore, the change of $\sigma(e)$ into a σ-free halfedge in G' does not result in a new potentially bad component.

If G has no potentially bad components, we let Q be a cycle component if possible. This choice guarantees that at least one edge of each σ-constrained pair occurs in a path component of G'. If there are no cycle components, we let Q be a non-trivial path component, if possible. Otherwise Q is any (trivial) path component. This choice ensures that in G' there are no three halfedges whose colors need to be mutually distinct because of a common endvertex in Q. If Q is a cycle (path) component, we proceed as in the case when Q was a potentially bad cycle (path) component. If we succeed to color $\bar{Q}$ so that the coloring is residually distinct at each τ-constrained halfedge, then no bad or potentially bad components occur. Otherwise we get at most one potentially bad component. (We see that no component in G' is bad in the same way as above. The only exception is the graph obtained from the graph Y by removing the vertex of degree 1 of Y and replacing the adjacent edge by a halfedge incident with the other end. This graph is reduced and has to be checked separately using Lemmas 3.1 and 3.2.) This completes the proof. □

The following theorem is a straightforward consequence of Proposition 3.3.

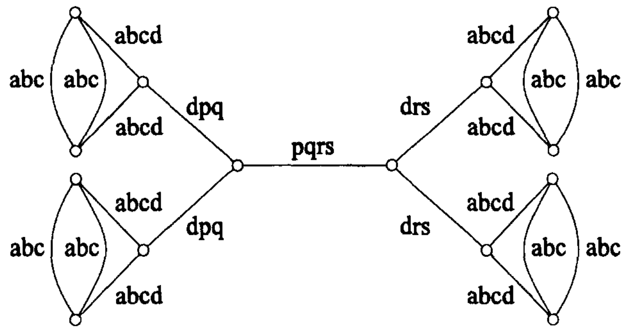

Fig. 2. Theorem 3.4 cannot be extended to non-reduced graphs.

Theorem 3.4. *Let G be a reduced subcubic graph without halfedges, H a subgraph of G such that $\Delta(H) \leqslant 2$, and L a list assignment of G such that $|L(e)| \geqslant 3$ for $e \in E(H)$ and $|L(e)| \geqslant 4$ for $e \in E(G) \setminus E(H)$. Then G is L-edge-colorable.*

Note that Theorem 3.4 does not hold if we omit the assumption that G is reduced (see Fig. 2).

Another consequence of Proposition 3.3 is:

Corollary 3.5. *Every subcubic graph is 4-edge-choosable, and there is a linear time algorithm that for every subcubic graph G and a list assignment L with $|L(e)| \geqslant 4$ $(e \in E(G))$ returns an L-edge-coloring.*

Proof. Let G' be a reduced graph obtained from G by the reduction. (Obviously, the reduction can be performed in linear time.) We first find a collection of maximal paths and cycles in G' (by a simple search) covering all vertices of G'. Then we let F be the set of edges that are not contained in these paths and cycles. Finally, we apply the construction of an L-edge-coloring from the proof of Proposition 3.3 (and Lemmas 2.1–2.3). It is easy to check that each of these steps can be accomplished in linear time. □

Let us remark that 4-edge-choosability of subcubic graphs also follows from the list version of Brooks' Theorem [5, 1].

4. Some applications

Proposition 3.3 can be used to get a simple proof of 5-edge-choosability for a large class of 4-regular graphs.

Corollary 4.1. *Let G be a graph with $\Delta(G) \leqslant 4$ that contains two disjoint 1-factors. Then G is 5-edge-choosable.*

Proof. Let L be a list assignment with $|L(e)|\geqslant 5$ for every $e\in E(G)$. Since each halfedge is adjacent to at most three other edges, halfedges can be removed and colored at the end. If M_1, M_2 are disjoint 1-factors of G, denote by H their union. Then H is a union of disjoint even cycles $C_1,\ldots,C_l$. For $i=1,\ldots,l$, consider the cycle C_i and let $e_1,\ldots,e_{2k}$ be the edges of C_i in the same order as they appear on C_i. Let $E_i=\{e_1,e_3,e_5,\ldots,e_{2k-1}\}$. We λ-color E_i as follows. If $L(e)=L(f)$ for every $e,f\in E(C_i)$, then we choose $a\in L(e_1)$ and put $\lambda(e)=a$ for every $e\in E_i$. Otherwise, we may assume that $L(e_1)\nsubseteq L(e_{2k})$. Take $\lambda(e_1)\in L(e_1)\backslash L(e_{2k})$. For $j=1,\ldots,k-1$, let A_j be a 4-element subset of $L(e_{2j})\backslash\{\lambda(e_{2j-1})\}$. Then take $\lambda(e_{2j+1})\in L(e_{2j+1})\backslash A_j$.

Consider the subcubic graph $G'=G-\bigcup_{i=1}^{l}E_i$. Define a list assignment L' on $E(G')$ as $L'(e)=L(e)\backslash\{a,b\}$, where a and b are the colors used on the already colored edges of G incident with e. Observe that $|L'(e)|\geqslant 3$ for every $e\in E(G')$ and that $|L'(e)|\geqslant 4$ for every $e\in E(G')\cap H$. Since $F=E(G')\cap H$ is a 1-factor of G', the reduced graph obtained from G by the reduction has no bad or potentially bad components. Hence, Proposition 3.3 can be used to get an L-edge-coloring, and we are done. □

The second application concerns 4-edge-colorings of cubic graphs such that the fourth color is not used too often. Note that in every 4-edge-coloring of the Petersen graph, each color is used at least twice. Therefore, there are arbitrarily large cubic graphs G where each color of a 4-edge-coloring is used at least $2|E(G)|/15$ times. Trivially, under every coloring, there is a color used on at most $|E(G)|/4$ edges. Below we give a slight improvement. Recall that the *domination number* $d(G)$ of G is the minimal cardinality of a vertex set U such that each vertex of G is either in U or adjacent to a vertex of U.

Corollary 4.2. *Let G be a subcubic graph. Then G has a 4-edge-coloring such that one of the colors is used at most $d(G)$ times.*

Proof. Let $U\subseteq V(G)$ be a dominating set with $d(G)$ vertices. Denote by F the set of edges incident with vertices in U, and let L be a list assignment with $L(e)=\{1,2,3,4\}$ if $e\in F$, and $L(e)=\{1,2,3\}$ otherwise. It is easy to see that after the reduction each halfedge of the obtained graph still has at least three admissible colors. Therefore, Proposition 3.3 can be applied to get an L-edge-coloring where color 4 is used at most $d(G)$ times. □

It is easy to see that every subcubic graph G (without isolated vertices) satisfies $|V(G)|/4\leqslant d(G)\leqslant|V(G)|/2$. Note that being close to the lower bound, Corollary 4.2 yields a bound of $|E(G)|/6$ which is not far from $2|E(G)|/15$.

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