Light paths in 4-connected graphs in the plane and other surfaces *

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Abstract

Several results concerning existence of k-paths, for which the sum of their vertex degrees is small, are presented.

1 Introduction

It is well known that every planar graph contains a vertex of degree at most 5. Kotzig [13, 14] strengthened this result by proving that every 3-connected planar graph contains an edge whose degree sum is at most 13. This result was further extended in various directions and used in deriving many properties of 3-connected planar graphs; see, e.g., Grünbaum and Shephard [7], Ivančo [9], Zaks [19], Jendrol' [10, 11], Fabrici and Jendrol' [4, 5], Harant, Jendrol', and Tkáč [8] and references therein.

In generalizing Kotzig's theorem, there are several natural directions. Two possibilities are as follows. Let $k \geq 1$ be an integer.

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(A) Find the smallest integer $w = w(k)$ such that whenever a 3-connected planar graph G contains a k-path, there is a k-path for which the sum of degrees of its vertices is at most w. (By a *k-path* we mean a path on k vertices.)

(B) Find the smallest integer $f = f(k)$ such that whenever a 3-connected planar graph G contains at least k vertices, there is a connected subgraph of G of order k whose degree sum in G is at most f.

Instead of the degree sum, one may ask about similar bounds on the maximum degree of a k-path, or a connected subgraph of order k, respectively.

Each of these problems can be formulated with further restrictions on the minimum degree, minimum face size, or the connectivity, and one may also ask about possible generalizations to graphs on more general surfaces. Several cases of such problems have been solved; cf. [5, 8, 3] and their references. We also refer to a recent survey [12] on light subgraphs.

In this note we resolve some of the open cases. First we show that a lower bound on $w(k)$ from (A) is of order $k \log k$, even if we restrict the minimum degree to be at least 4 or 5, respectively. (Such examples were constructed in the case of minimum degree 3 by Fabrici and Jendrol' [5].) Next we show that by restricting to 4-connected graphs instead of limiting the minimum degree to 4, the answer is totally different. In this case $w(k) = 6k - 1$. A similar result and its strengthening are then derived for 4-connected graphs on general surfaces. It is also shown that no connected graph other than a path occurs with bounded degrees in 4-connected planar graphs, a result which was asked by Fabrici and Jendrol' [4]. Moreover, it is proved that this result no longer holds if we exclude arbitrarily long paths of vertices of degree 4 (cf. Theorem 2.4). Finally, problem (B) is considered for 3-connected graphs embedded on general surfaces with large face-width.

2 Planar graphs

When further restricting the class of graphs in problems (A) or (B), we shall write $w(k, \text{restrictions})$ and $f(k, \text{restrictions})$, respectively, to denote the smallest upper bound on the degree sum of the restricted class of 3-connected graphs. In particular, we shall consider the following two restrictions: minimum degree at least d and d-connectivity. Then we write $w(k, \delta \geq d)$ and $w(k, \kappa \geq d)$, respectively.

Fabrici and Jendrol' [5] proved that

$$k \log_2 k \le w(k) \le 5k^2.$$

They asked if the same lower bound applies if we restrict ourselves to graphs of minimum degree 4 or 5. The following examples show that the answer is positive.

Let $T_1 = K_4$ be the complete graph on 4 vertices. For $i \ge 1$, let T_{i+1} be the plane triangulation obtained from T_i by adding a vertex of degree 3 in each of the facial triangles (including the outer one) of T_i. Let k_i be the number of vertices on a longest path in T_i. Then $k_1 = 4$, $k_2 = 8$, and $k_{i+1} = 2k_i + 1$ $(i \ge 2)$. Let w_i be the minimum degree sum in T_i on a k_i-path. It is easy to see that each path attaining this minimum starts and ends with a vertex of degree 3 (if $i \ge 3$) and that $w_1 = 12$, $w_2 = 36$, and $w_{i+1} = 2w_i + 3(k_i + 1)$ for $i \ge 2$. Solving the recurrences, we get $k_i = 9 \cdot 2^{i-2} - 1$ and $w_i = 9(3i+2)\,2^{i-3}$, $i \ge 2$.

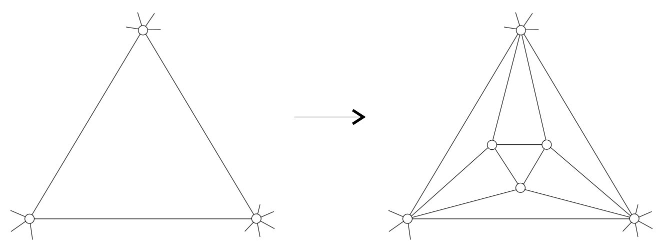

Figure 1: Adding the octahedron

Now, let T_i' (respectively T_i'') be obtained from T_i by replacing each facial triangle by a copy of the octahedron graph (respectively, icosahedron); see Figure 1. Let k_i', k_i'', w_i', w_i'' be the corresponding values in these graphs. For $i \ge 2$, we have $k_i' = 4k_i + 3 = 9 \cdot 2^i - 1$ and $w_i' = 3w_i + 12(k_i + 1) = 27(3i+10)\,2^{i-3}$. Similarly, $k_i'' = 10k_i + 9 = 45 \cdot 2^{i-1} - 1$ and $w_i'' = 9(6i+49)\,2^{i-2}$. This shows that $w(k_i', \delta \ge 4) \ge \frac{9}{8} k_i' \log_2 k_i' + \mathcal{O}(k_i')$. If $k_i' \le k < k_{i+1}'$, then $w(k, \delta \ge 4) \ge w(k_i', \delta \ge 4)$ and $k \le 2k_i'$. This implies that

$$w(k, \delta \ge 4) \ge \frac{9}{16}\, k \log_2 k + \mathcal{O}(k).$$

Similarly, we get from T_i'' that

$$w(k, \delta \ge 5) \ge \frac{3}{10}\, k \log_2 k + \mathcal{O}(k).$$

It is interesting that the restriction to 4-connected graphs brings a different behavior.

Proposition 2.1 *Every 4-connected planar graph on at least k vertices contains a k-path whose degree sum is at most $6k-1$. Consequently,*

$$w(k, \kappa \geq 4) = w(k, \kappa \geq 5) = 6k - 1.$$

Proof. Tutte [17] proved that every 4-connected planar graph contains a Hamilton cycle. Let $C = v_1 v_2 \ldots v_n$ be a Hamilton cycle of G. For $i = 1, \ldots, n$, let R_i be the k-path $v_i v_{i+1} \ldots v_{i+k-1}$ (indices modulo n). Let $w(R_i)$ denote the sum of degrees of vertices of R_i. Then

$$\sum_{i=1}^{n} w(R_i) = k \sum_{v \in V(G)} \deg(v) = 2k|E(G)| \leq 2k(3n-6).$$

The last inequality is a well known corollary of Euler's formula. Hence, one of the paths, say R_i, has its degree sum at most $2k(3n-6)/n < 6k$. This shows that $w(k, \kappa \geq 5) \leq w(k, \kappa \geq 4) \leq 6k - 1$.

Finally, there are 5-connected triangulations of the plane which contain precisely 12 vertices of degree 5, and all other vertices are of degree 6 (cf., e.g., [6, 2]). Moreover, the vertices of degree 5 are as far away from each other as we like. This shows that $w(k, \kappa \geq 5) \geq 6k - 1$ and completes the proof. □

Fabrici and Jendrol' [4] proved that if H is a connected planar graph which is not a path, then for every integer r there exists a planar 3-connected graph G containing H as a subgraph, and every subgraph of G isomorphic to H contains a vertex of degree at least r. They asked [4, Problem 4] if there is an analogue of this result for 4-connected graphs. Below we answer their question in the affirmative. Let us remark that not every planar graph is a subgraph of a 4-connected planar graph (e.g., a 3-connected planar graph with a separating triangle), and that the only 4-connected planar graph that contains a 4-connected triangulation H of the sphere is H itself.

Theorem 2.2 *Let r be an arbitrary integer and let H be a planar graph which is not a triangulation but is a subgraph of some 4-connected planar graph. If H contains a cycle or a vertex of degree more than 2, then there is a 4-connected planar graph G which contains H as a subgraph such that every subgraph of G isomorphic to H contains a vertex of degree at least r in G.*

Proof. Let e be an edge of a 4-connected planar triangulation and let v, u be the vertices in the two triangles containing e which are not the ends of e. If we subdivide e by inserting r new vertices $v_1, \ldots, v_r$ and join each v_i with v and u $(i = 1, \ldots, r)$, we get a new 4-connected triangulation. We call this operation the *r-subdivision* of e. Next, we make r-subdivisions of the new edges $v_1u, v_2v, v_3u, v_4v, \ldots$, and call the entire procedure the *dense r-subdivision* of e; see Figure 2. We also say that the endvertices of e and u, v are *involved* in the dense subdivision. Observe that all these vertices and $v_1, \ldots, v_r$ have degree greater than r.

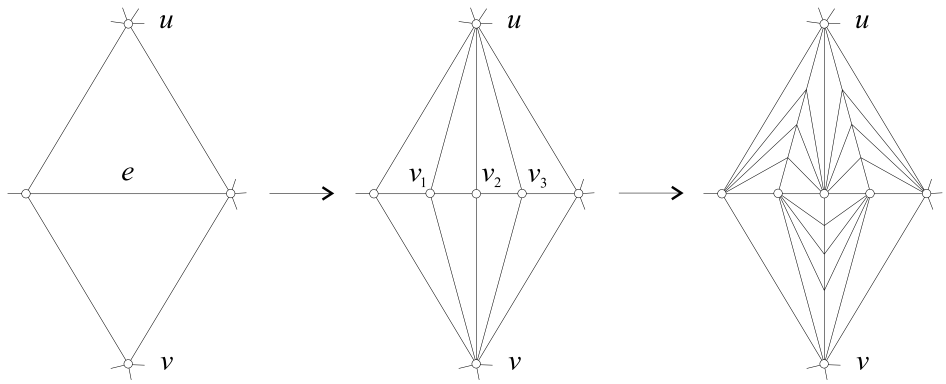

Figure 2: The dense 3-subdivision of e

Let $\tilde{H}$ be a 4-connected planar graph which contains H as a subgraph. Since adding edges does not decrease connectivity, we may assume that $\tilde{H}$ is a triangulation, and then there is at least one edge $e_0 \in E(\tilde{H}) \backslash E(H)$. Now, we successively make dense r-subdivisions in the obtained triangulations to each edge $e \in E(\tilde{H}) \backslash E(H)$. Denote by G the 4-connected planar triangulation obtained in this way. Then G contains H as a subgraph. Suppose that H_1' is a subgraph of G isomorphic to H with all vertices of degree in G less than r. Let H_1 be a connected component of H_1' which is not a path. Then H_1 contains none of the edges added in r-subdivisions and none of the vertices which were involved in dense subdivisions. This shows that each cycle of H_1 and each vertex of H_1 of degree at least 3 in H_1 are contained in H. But only those vertices x of H, for which all triangles in $\tilde{H}$ containing x are entirely in H, were not involved in dense subdivisions. Since H is not a triangulation, this implies that a vertex of H_1 has been involved in subdivisions and hence it is of degree more than r. This contradiction shows that H_1' does not exist. $\square$

Observe that all vertices on the paths corresponding to r-subdivisions in Theorem 2.2 have degree 4. Theorem 2.4 below shows that such long paths of vertices of degree 4 are necessary for such a result. In particular, Theorem 2.2 does not extend to 5-connected graphs (or even 4-connected graphs without long paths of vertices of degree 4). We will need the following lemma.

Lemma 2.3 *Let G_1 be a 2-connected outerplanar graph with outer cycle C. Suppose that each vertex of G_1 is adjacent to at most two vertices of degree 3 distinct from its neighbors on C. Let C be the outer cycle of G_1. If $G_1 \neq C$, then there is an edge $uv \in E(G_1)\backslash E(C)$ such that* $\deg(u) + \deg(v) \leq 12$.

Proof. We may assume that no two vertices of degree 2 are adjacent and that for each vertex of degree 2, its neighbors are adjacent. (Otherwise we may contract an edge incident with such a vertex.) Suppose that each edge in $E(G_1)\backslash E(C)$ has the sum of degrees at least 13. We will apply the discharging method. For each vertex $v \in V(G_1)$ we define $\phi_v = 4 - \deg(v)$. Euler's formula implies that $\sum_{v \in V(G_1)} \deg(v) \leq 4n - 6$, and hence $\sum_{v \in V(G_1)} \phi_v \geq 6$. We shall now change ϕ by redistributing the "charges" ϕ_v so that the total sum remain the same. The redistribution rules are repeated for each vertex v of degree 2 or 3 as follows:

(a) If $\deg(v) = 2$, then we first set $\phi_v = 0$. Let u, w be the neighbors of v, where $\deg(u) \geq \deg(w)$. By our assumptions, $uw \in E(G_1)\backslash E(C)$. If $\deg(u) \geq 8$, then we increase the value of ϕ_u by 2. If $\deg(u) \leq 7$, then $\deg(u) = 7$ and $\deg(w)$ is either 6 or 7. In that case we increase the values of ϕ_u and ϕ_w by 1.

(b) If $\deg(v) = 3$, let uv be the edge incident with v which is not on C. Then we set ϕ_v to 0 and increase the value ϕ_u by 1.

Let $u \in V(G_1)$. If $\deg(u) \geq 10$, its initial ϕ-value is at most -6, and it is increased by at most 6 (Rule (a) twice and Rule (b) twice). Hence it cannot become positive. If $8 \leq \deg(u) \leq 9$, then it is increased by at most 4 (Rule (a) twice), and if the degree is 6 or 7, ϕ_u increases by at most 2 (Rule (a) twice). This shows that the total sum of values cannot be positive, a contradiction. $\square$

Theorem 2.4 *Let $r \geq 0$ and $k \geq 4$ be integers, and let T_k be the graph of order k obtained from $K_{1,3}$ by replacing one of its edges by a $(k-2)$-path. If G is a 4-connected planar graph which contains T_k as a subgraph and has no r-path all of whose vertices have degree 4, then G contains a subgraph isomorphic to T_k whose degree sum is less than $96rk$.*

Proof. Let $C = v_1 v_2 \dots v_n$ be a Hamilton cycle in G. By considering the r-paths on C, we deduce that at least $\lceil n/r \rceil$ vertices have degree 5 or more. (Let us observe that we need $n \geq r$ in order to have r-paths in C, which we may assume since otherwise $w(T_k) \leq 6n \leq 6r$.) Let B_1 be the set of vertices which have two (or more) incident edges in the interior of C. We may assume that $p := |B_1| \geq \frac{1}{2}\lceil n/r \rceil$. (Otherwise we would consider the exterior of C instead.) Let $G_1 \subseteq G$ be the outerplanar graph obtained from C by adding, for each $v \in B_1$, two of the edges of G in the interior of C that are incident with v. Then G_1 has p vertices of degree 4 or more, and it satisfies conditions of Lemma 2.3. By the lemma, G_1 has an edge $e_1 = v_{j_1} v_{l_1}$ with degree sum at most 12. Let $B_2 = B_1 \setminus Q$ where Q consists of v_{j_1}, v_{l_1} and all vertices v such that $v_{j_1} v \in E(G_1) \setminus E(C)$ or $v_{l_1} v \in E(G_1) \setminus E(C)$. Since $\deg_{G_1}(v_{j_1}) + \deg_{G_1}(v_{l_1}) \leq 12$, $|B_2| \geq |B_1| - 8$. Now we define a subgraph $G_2 \subseteq G$ which is obtained from C by adding, for each $v \in B_2$, two of the edges of G in the interior of C that are incident with v. By Lemma 2.3 we get another edge $e_2 = v_{j_2} v_{l_2}$ whose degree sum in G_2 is at most 12, and we repeat the process. In this way we get a matching $e_1, \dots, e_q$ consisting of $q = \lceil p/8 \rceil \geq \frac{n}{16r}$ edges inside C. We may choose the notation so that $e_i = v_{j_i} v_{l_i}$ where the distance from v_{j_i} to v_{l_i} along the cycle C (in its positive direction $v_1 v_2 \dots$) is at most $n/2$, $i = 1, \dots, q$.

If $k \geq n/2$, then every subgraph T of G has $w(T) \leq w(G) \leq 6n \leq 12k$. Hence we may assume that $k > n/2$. For $i = 1, \dots, q$, let R_i be a copy of T_k composed of the path $v_{j_i - k+3} v_{j_i - k+4} \dots v_{j_i}$ and the edges $v_{j_i} v_{j_i + 1}$ and e_i. Each vertex of G appears in at most k of these subgraphs. Therefore,

$$\sum_{i=1}^{q} w(R_i) \leq k \sum_{i=1}^{n} \deg(v_i) < 6kn.$$

One of the subgraphs, say R_i, has

$$w(R_i) < 6kn/q \leq 96rk.$$

This completes the proof. $\square$

Theorem 2.4 can be extended to several other examples of graphs playing the role of T_k. Theorems 2.2 and 2.4 also extend to 4-connected graphs on general surfaces in the same way as shown in the next section for existence of light paths.

3 Graphs on a fixed surface

For embeddings of graphs on surfaces we refer to [16] (cf. also [15]). We consider only 2-cell embeddings in closed surfaces. If G is an embedded graph and r is the number of facial walks of this embedding, then the number

$$g = 2 - |V(G)| + |E(G)| - r$$

is called the *Euler genus* of the embedding. Since $3r \le 2|E(G)|$, we get the following bound on the number of edges in terms of the number of vertices and g:

$$|E(G)| \le 3|V(G)| - 6 + 3g.$$

The *face-width* of the embedded graph, denoted by $\mathrm{fw}(G)$, is the minimum integer k such that there exist facial walks $F_1, \ldots, F_k$ whose union contains a noncontractible cycle. The following results show that for graphs of large face-width there exist light paths of similar weight as in the case of planar graphs.

Theorem 3.1 *For every positive integer g and real number ε, $0 < \varepsilon \le 1/2$, there is a number $a(g, \varepsilon)$ such that the following holds. Let G be a 4-connected graph embedded in a surface of Euler genus g, and let $k \ge 1$ be an integer. If the face-width of the embedding is at least $a(g, \varepsilon)$ and $k \le (1-\varepsilon)n$, $n = |V(G)|$, then there exists a k-path in G whose degree sum is at most $(6 + \frac{2\varepsilon}{1-\varepsilon})k + \frac{6(g-2)}{(1-\varepsilon)n}\, k$.*

Proof. Böhme, Mohar, and Thomassen [1] proved that there is a constant $a(g, \varepsilon)$ (which is proportional to $2^g/\varepsilon$) such that G contains a cycle C of length $n' \ge (1 - \varepsilon)n$ if the face-width is at least $a(g, \varepsilon)$. Now, using the notation of the proof of Proposition 2.1, we have:

$$\begin{aligned} \sum_{i=1}^{n'} w(R_i) &= k \sum_{v \in V(C)} \deg(v) \le 2k|E(G)| - 4k(n - n') \\ &\le 2k(3n - 6 + 3g) - 4k(n - n') = 2kn + 4kn' + 6k(g - 2). \end{aligned}$$

Since $k \le n'$, R_i is a path for $i = 1, \ldots, n'$. The above inequality implies that one of the paths R_i has

$$w(R_i) \le \frac{2kn}{n'} + 4k + \frac{6k(g-2)}{n'} \le \Big(6 + \frac{2\varepsilon}{1-\varepsilon}\Big)k + \frac{6(g-2)}{(1-\varepsilon)n}\, k.$$

□

For general 4-connected graphs on a fixed surface we may apply another result of Böhme, Mohar, and Thomassen [1] which states that for each surface S, there is a constant $c_S > 0$ such that every 4-connected graph of order n embedded in S contains a cycle of length at least $c_S n$. This result implies:

Theorem 3.2 *For every positive integer g, there is a constant $c = c(g)$ such that for every integer $k \ge 1$ and every 4-connected graph G embedded in a surface of Euler genus g, if G contains a k-path, then G also contains a k-path whose degree sum is at most ck.*

Proof. Let c_S be the constant mentioned above. If G has less than k/c_S vertices, then every k-path in G has degree sum at most $2|E(G)| \le 2(3n-6+3g) \le (6/c_S)k + 6g \le (6/c_S + 6g)k$. If $n \ge k/c_S$, then by the aforementioned result of [1], G contains a cycle of length at least $c_S n \ge k$. Now, the method similar to that in the proof of Theorem 3.1 completes the proof. □

We believe that a stronger result, where the constant c is independent of g, must be true.

Conjecture 3.3 *There is a constant c, and for every positive integer g, there is a constant $c' = c'(g)$ such that for every integer $k \ge 1$ and every 4-connected graph G embedded in a surface of Euler genus g, if G contains a k-path, then G also contains a k-path whose degree sum is at most $ck + c'$.*

4 Light connected subgraphs

Let $r \ge 1$ be an integer. A walk W in a graph G is called an *r-walk* if each vertex of G appears on W at least once and at most r times. Yu [18] proved that a 3-connected graph G embedded in a surface of Euler genus g with face-width at least $48(2^g - 1)$ contains a 3-walk. This implies:

Theorem 4.1 *Let G be a 3-connected graph embedded in a surface of Euler genus g with $\mathrm{fw}(G) \geq 48(2^g - 1)$. If G has $n \geq k$ vertices, then G contains a connected subgraph of order k whose degree sum in G is at most $18k + 18k(k-3+g)/(n-k+1)$.*

Proof. Let $W = u_1 u_2 \dots u_m$ be a 3-walk in G. The vertices of G can be enumerated, $v_1, \dots, v_n$, such that there are indices $1 \leq j(1) < j(2) < \cdots < j(n) \leq m$ such that $u_{j(i)} = v_i$ for $i = 1, \dots, n$. Let R_i be the shortest subwalk of W starting at $u_{j(i)}$ such that it visits precisely k distinct vertices (some of them possibly more than once), $i = 1, \dots, n-k+1$. Then $R_1, R_{k+1}, R_{2k+1}, \dots$ are nonoverlapping subwalks of W (but may use the same vertices), and hence

$$\begin{aligned}
\sum_{j=0}^{\lfloor n/k \rfloor - 1} w(R_{jk+1}) &\leq 3 \sum_{i=1}^{n} \deg(v_i) = 6|E(G)| \\
&\leq 6(3n - 6 + 3g).
\end{aligned}$$

Let i $(1 \leq i \leq n-k+1)$ be an index with minimum $w(R_i)$. The above inequality implies that

$$w(R_i) \leq \frac{18(n-2+g)}{\lfloor n/k \rfloor} \leq \frac{18(n-2+g)}{n-k+1}\, k.$$

The induced subgraph G_i of G on vertices of R_i is connected, of order k, and $w(G_i) \leq w(R_i)$. This completes the proof. □

A special case of Theorem 4.1 restricted to planar graphs (but with a better bound) was recently obtained by Enomoto and Ota [3].

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References

[1] T. Böhme, B. Mohar, and C. Thomassen, Long cycles in graphs on a fixed surface, submitted.

[2] H. S. M. Coxeter, Virus macromolecules and geodesic domes, in "A spectrum of mathematics," J. C. Butcher, ed., Oxford Univ. Press, 1971, p. 98.

[3] H. Enomoto, K. Ota, Properties of 3-connected planar graphs, preprint,1998.

[4] I. Fabrici, S. Jendrol', Subgraphs with restricted degrees of their vertices in planar 3-connected graphs, Graphs Comb. 13 (1997) 245–250.

[5] I. Fabrici, S. Jendrol', Subgraphs with restricted degrees of their vertices in planar graphs, Discrete Math. 191 (1998) 83–90.

[6] M. Goldberg, A class of multi-symmetric polyhedra, Tôhoku Math. J. 43 (1937) 104–108.

[7] B. Grünbaum, G. C. Shephard, Analogues for tiling of Kotzig's theorem on minimal weights of edges, Ann. Discrete Math. 12 (1982) 129–140.

[8] J. Harant, S. Jendrol', and M. Tkáč, On 3-connected plane graphs without triangular faces, preprint, 1998.

[9] J. Ivančo, The weight of a graph, Ann. Discrete Math. 51 (1992) 113–116.

[10] S. Jendrol', Path with restricted degrees of their vertices in planar graphs, Czech. Math. Journal, to appear.

[11] S. Jendrol', A structural property of convex 3-polytopes, Geom. Dedicata 68 (1997) 91–99.

[12] S. Jendrol', T. Madaras, R. Soták, and Zs. Tuza, On light cycles in plane triangulations, Discrete Math. 197/198 (1999) 453–467.

[13] A. Kotzig, Contribution to the theory of Eulerian polyhedra, Mat. Čas. SAV (Math. Slovaca) 5 (1955) 111–113.

[14] A. Kotzig, Extremal polyhedral graphs, Ann. New York Acad. Sci. 319 (1979) 569–570.

[15] B. Mohar, Face-width of embedded graphs, Math. Slovaca 47 (1997) 35–63.

[16] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, to appear.

[17] W. T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99–116.

[18] X. Yu, Disjoint paths, planarizing cycles, and spanning walks, Trans. Amer. Math. Soc. 349 (1997) 1333–1358.

[19] J. Zaks, Extending Kotzig's theorem, Israel J. Math. 45 (1983) 281–296.