Labeled $K_{2,t}$ minors in plane graphs

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Abstract

Let G be a 3-connected planar graph and let $U \subseteq V(G)$. It is shown that G contains a $K_{2,t}$ minor such that t is large and each vertex of degree 2 in $K_{2,t}$ corresponds to some vertex of U if and only if there is no small face cover of U. This result cannot be extended to 2-connected planar graphs.

1 Introduction

Let G be a graph and $U \subseteq V(G)$. A subgraph H of G is called a $K_{2,t}$-*preminor* if it consists of pairwise disjoint trees Z_1, Z_2 and $T_1, \ldots, T_t$ together with edges $z_i t_j$, where $z_i \in V(Z_i)$ and $t_j \in V(T_j)$, $1 \le i \le 2$, $1 \le j \le t$. After contracting the edges in each of these trees, H becomes the complete bipartite graph $K_{2,t}$. Clearly, $K_{2,t}$ is a minor of G if and only if G contains a $K_{2,t}$-preminor. If each T_j, $1 \le j \le t$, contains a vertex of U, then H is said to be *U-labeled* and we also say that G contains a *U-labeled $K_{2,t}$-minor*.

Suppose now that G is a 3-connected planar graph. A set $\mathcal{F}$ of facial cycles of G is a *face cover* of U if each vertex of U belongs to a member of $\mathcal{F}$. The aim of this paper is to show that G contains a labeled $K_{2,t}$-minor, where t is large, if and only if there is no small face cover of U. Our original motivation for this problem came from the study of the genus of apex graphs (cf. [3]).

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Bienstock and Dean [1] proved that nonexistence of small face covers is closely related to the existence of large vertex packings, where by a *vertex packing* of U we mean a subset W of U such that no two vertices of W lie in a common facial cycle. Let $\nu(U)$ be the size of a largest packing of U, and let $\tau(U)$ be the size of the smallest face cover of U.

Theorem 1.1 (Bienstock and Dean [1]) *Let G be a plane graph and $U \subseteq V(G)$. Then*

$$\nu(U) \leq \tau(U) \leq 27\,\nu(U).$$

As noted in [1], the constant 27 in Theorem 1.1 can be improved, and there are examples which show that it cannot be replaced by anything smaller than 2.

The main result of this paper shows that the U-labeled $K_{2,t}$-minors present another obstruction for small face covers in case of 3-connected planar graphs.

Theorem 1.2 *There is a nondecreasing integer function $f : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} f(n) = \infty$ and such that the following holds. Let G be a 3-connected planar graph and let $U \subseteq V(G)$. Then G contains a U-labeled $K_{2,t}$-minor where $t \geq f(\tau(U))$. Conversely, if G contains a U-labeled $K_{2,t}$-minor, then $\tau(U) \geq t/2$.*

Theorem 1.2, whose proof is deferred to the end of Section 2, cannot be extended to the 2-connected case as the following example shows. Let G be the graph composed of n copies of the 4-cycle $Q_i = v_1^i v_2^i v_3^i v_4^i$ linked cyclically so that the vertex v_3^i is adjacent to v_1^{i+1}, together with additional vertices $U = \{u_1, \ldots, u_n\}$ where u_i is adjacent to all vertices of Q_i, $i = 1, \ldots, n$. Then $\tau(U) = \nu(U) = n$ for every embedding of G in the plane. However, G does not contain a U-labeled $K_{2,3}$-minor.

2 The proof of Theorem 1.2

We need the following definitions. Let G be a graph and let C be a cycle of G. A *C-bridge* of G is either an edge $e = xy \in E(G) \setminus E(C)$ such that $x, y \in V(C)$ or a connected component of $G - V(C)$ together with all edges from this component to C and all end vertices of these edges. If B is a C-bridge, then the vertices of $V(B) \cap V(C)$ are called the *vertices of attachment* of B.

By a *plane graph* we mean a planar graph G with a fixed embedding into the Euclidean plane. If C is a cycle of a plane graph G, then $\mathrm{Int}(G)$ (resp.,

$\text{Ext}(G)$) denotes the subgraph of G formed by C and all vertices and edges inside (resp., outside) C.

It is well known that facial cycles of a 3-connected planar graph G are (precisely) the induced nonseparating cycles of G. This implies:

Lemma 2.1 *Let G be a 3-connected planar graph, let F be a facial cycle of G, and let u, v be vertices of G that do not lie on C. Then G contains a path from u to v which is disjoint from F.*

Let G be a plane graph and $C_0, \dots, C_k$ a sequence of pairwise disjoint cycles of G such that for all indices i, j, $0 \le i < j \le k$, $C_i \subseteq \text{Int}(C_j)$. Then we say that $C_0, \dots, C_k$ is a sequence of *nested cycles*. Let $D_i = \text{Ext}(C_i) \cap \text{Int}(C_{i+1})$, $0 \le i < k$. If each D_i $(1 \le i \le k-2)$ contains a vertex of U, then we say that $C_0, \dots, C_k$ are *interlaced* with vertices of U.

Lemma 2.2 *Let G be a 3-connected plane graph and $U \subseteq V(G)$. Suppose that $C_0, \dots, C_k$ is a nested sequence of cycles that are interlaced with U. Then G contains a U-labeled $K_{2,t}$-minor where $t = \lfloor (k-3)/18 \rfloor$.*

Proof. Since G is 3-connected, there exist pairwise disjoint (C_0, C_k)-paths P_1, P_2, P_3. Select these paths so that the number of connected components of $P_i \cap (C_0 \cup \cdots \cup C_k)$, $i = 1, 2, 3$, is minimum. Let $H = C_0 \cup P_1 \cup P_2 \cup P_3 \cup C_k$.

Suppose that $v \in V(C_i) \setminus V(H)$ $(1 \le i < k)$. Since the cycles $C_0, \dots, C_k$ are nested, each of P_1, P_2, P_3 intersects C_i. Starting at v, we traverse C_i to the left and to the right until we reach one of the paths. Our choice of the paths guarantees that the path reached on the left is not the same as the one reached on the right.

Suppose that $u \in V(D_i)$, where $1 \le i \le k-2$. Let Q be a path from u to a vertex in H such that only the end vertex of Q is in H. (In particular, if $u \in V(H)$, then Q is just the trivial path.) Then we say that Q *joins* u and H. We say that u is *local* on H if every path which joins u and H ends on the same path P_j, $j \in \{1, 2, 3\}$. If Q ends on C_0 or C_k, then it intersects C_i or C_{i+1}. This implies (by the previous paragraph) that every nonlocal vertex $u \in V(D_i) \setminus V(H)$ can be joined to two distinct paths among P_1, P_2, P_3 by using paths contained in D_i.

Let $u_i \in V(D_i)$, $i = 1, \dots, k-2$, be vertices of U which interlace with the nested cycles. If $6t$ of the vertices u_i are nonlocal on H, then $2t$ of them can be joined to the same pair of the paths, say P_1 and P_2. Since u_i can be joined to P_1 and P_2 inside D_i, there is a subset of t of the vertices u_i whose paths joining u_i with P_1 and P_2 are pairwise disjoint for distinct indices

i. Then there is a U-labeled $K_{2,t}$-preminor using P_1, P_2, the corresponding vertices u_i and the paths joining u_i to P_1 and P_2 inside D_i. Therefore we may assume that at most $6t - 1$ vertices u_i ($1 \leq i \leq k-2$) are not local on H. Therefore, we may assume that at least $k/3 - 2t - 1 \geq (2k-6)/9$ vertices u_i ($2 \leq i \leq k-3$) are local on P_3, say.

Suppose now that $u_i \in V(D_i)$ is local on P_3, where $2 \leq i \leq k-3$. Take a path Q_1 joining u_i with a vertex v on P_3. Since u_i is local, $Q_1 \subseteq D_i$. Let Q_2 be the maximal segment of P_3 which contains v such that Q_2 is contained in $D_{i-1} \cup D_i \cup D_{i+1}$. Then the following holds either for $j = i$ or for $j = i+1$: $Q_2 \cap C_j$ contains a connected component S such that one of the edges of P_3 incident with an end of S is in D_{j-1} and the edge of P_3 incident with the other end of S is in D_j. Going left and right on C_j from S, we reach a path P_c on the left and P_d on the right where $c, d \neq 3$ by our choice of the paths. If $c = d$, then the traversed segment of C_j has connected intersection S with P_3. Therefore it does not cross P_3. This implies that P_3 reaches and leaves S from the same side (either from the inside of D_{j-1} or from the inside of D_j), a contradiction. This shows that u_i can be linked to both paths P_1 and P_2 using paths inside $D_{i-1} \cup D_i \cup D_{i+1}$. Therefore, the paths for every fourth index u_i (where u_i is local on P_3) are pairwise disjoint. The number of such indices i is at least $(2k-6)/36 \geq t$. Consequently, there is a U-labeled $K_{2,t}$-minor which can be obtained in the same way as above. $\square$

Lemma 2.3 *Let G be a 2-connected graph, $U \subseteq V(G)$, and let p, q be adjacent vertices of G. Let $t = \left\lceil \sqrt{|U|} \right\rceil$. Then either there is a cycle through the edge pq which contains t vertices of U, or G contains a U-labeled $K_{2,t}$-minor.*

Proof. Each ear decomposition of G starting with a cycle containing the edge pq determines an st-numbering $s : V(G) \to \{1, \ldots, |V(G)|\}$ with $s(p) = 1$ and $s(q) = |V(G)|$ (cf. [2]). In that numbering, every vertex distinct from p and q has a neighbor with a smaller number and a vertex with larger number. This gives rise to a partial order $\preceq$ on $V(G)$ where $v \preceq u$ if there is an s-monotone increasing path in G whose initial vertex is v and terminal vertex is u. Consider the induced partial order on U. By the Dilworth Theorem, the size of a maximal antichain in this partial order is equal to the minimum number of chains covering U. This implies that there is either an antichain of cardinality t, or there is a chain containing at least t elements of U. In the first case, the set of s-monotone paths from the vertices in the antichain to q together with the set of s-monotone paths from p to these vertices contain a U-labeled $K_{2,t}$-minor. In the latter case, the chain gives rise to a (p, q)-path containing the chain. Together with the edge pq we have the required cycle. $\square$

Let C be a cycle of a graph G, and let B be a C-bridge. Two vertices $x, y \in V(C)$ are *separated* by B if there are vertices $a, b \in V(B) \cap (V(C) \setminus \{x, y\})$ such that they appear on C in the cyclic order a, x, b, y. Two distinct C-bridges B_1, B_2 are *separated* by a C-bridge B_3 if $B_3 \neq B_1, B_2$ and there are vertices $x \in V(B_1) \cap V(C)$ and $y \in V(B_2) \cap V(C)$ such that B_3 separates x and y on C.

Lemma 2.4 *There is a nondecreasing integer function $g : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} g(n) = \infty$ and such that the following holds. Let C be a cycle of a 2-connected plane graph G and let $U = \{u_1, ..., u_k\}$ be a subset of $V(C)$ such that the vertices in U appear on C in the cyclic order $u_1, ..., u_k$ and no facial cycle of* Int(C) *except C contains more than one vertex in U. Then there is a subsequence $1 \leq i_1 < i_2 < \cdots < i_s \leq k$, where $s = g(k)$, for which one of the following holds:*

(a) *Let $v_j = u_{i_j}$, $1 \leq j \leq s$. Denote by S_j the open segment of C from v_j to v_{j+1}, $1 \leq j \leq s$. Then there is a C-bridge B in* Int(C) *which has a vertex of attachment in each S_j, $1 \leq j \leq s$.*

(b) *There is an index $i_0 \in \{1, ..., s\}$ and $s - 1$ distinct C-bridges B_i, $i \in \{1, ..., s\} \setminus \{i_0\}$ such that each B_i has a vertex of attachment in S_{i_0} and in S_i.*

(c) *There is a facial cycle F in* Int(C) *which has a vertex in each segment S_j, $1 \leq j \leq s$, and does not contain any of the vertices v_j, $1 \leq j \leq s$.*

Proof. A C-bridge B is called *U-essential* if it separates two vertices in U. Let T_i denote the open segment of C from u_i to u_{i+1}, $1 \leq i \leq k$. If B is a C-bridge, $I(B)$ denotes the set of all indices i such that T_i contains a vertex of attachment of B. Obviously, a C-bridge B is essential if and only if $|I(B)| > 1$. A C-bridge B_1 *covers* a C-bridge B_2 if $I(B_1) \supseteq I(B_2)$. Let $\mathcal{B}$ denote a minimal set of U-essential C-bridges such that every U-essential C-bridge is covered by one in $\mathcal{B}$, and let $d = \max\{|I(B)| \mid B \in \mathcal{B}\}$. Since no two vertices in U belong to the same facial cycle of Int(C) distinct from C, each T_i, $1 \leq i \leq k$, contains a vertex of attachment of some C-bridge in $\mathcal{B}$. Consequently, $d|\mathcal{B}| \geq k$.

Let $\mathcal{A} \subseteq \mathcal{B}$ be a largest set of C-bridges such that no two C-bridges in $\mathcal{A}$ are separated by a C-bridge in $\mathcal{B}$ and let $l = |\mathcal{A}|$. Then it is not hard to see, that no two C-bridges in $\mathcal{A}$ are separated by any C-bridge of $\mathrm{Int}(C)$. Consequently, there is a facial cycle F of $\mathrm{Int}(C)$ such that $F \neq C$ and F contains at least two vertices of attachment of each C-bridge in $\mathcal{A}$. Since any C-bridge in $\mathcal{A}$ separates two vertices in U there is a subsequence $1 \leq i_1 < \cdots < i_l \leq k$ such that F has a vertex in each segment of C from u_{i_j} to $u_{i_{j+1}}$ and F does not contain any vertex u_{i_j}, $1 \leq j \leq l$.

Let $\mathcal{B}_1$ denote the set of all C-bridges $B \in \mathcal{B}$ such that B does not separate any two C-bridges in $\mathcal{B}$, and for $i \geq 2$, let $\mathcal{B}_i$ be the set of all C-bridges $B \in \mathcal{B} \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_j$ such that B does not separate any two C-bridges in $B \in \mathcal{B} \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_j$. Let e denote the largest integer such that $\mathcal{B}_e \neq \emptyset$. A simple induction on e shows that, after possibly changing the indices, there is a subsequence $1 \leq i_1 < \cdots < i_{e+1} \leq k$ and a subset $\{B_1, \ldots, B_{s-1}\}$ of $\mathcal{B}$ such that $B_j \in \mathcal{B}_j$ and each B_j has a vertex of attachment in the open segment of C from u_{i_j} to $u_{i_{j+1}}$ and one in the open segment of C from u_e to u_{e+1}.

Now we wish to prove that $|\mathcal{B}|$ is bounded by a function of d, l and e. Obviously, $|\mathcal{B}| = |\mathcal{B}_1| + \cdots + |\mathcal{B}_e|$. Since no two C-bridges in $\mathcal{B}_e$ are separated by any other C-bridge, $|\mathcal{B}_e| \leq l$. Let $1 \leq i < e$, and call two C-bridges in $\mathcal{B}_i$ *similar* if they are not separated by any C-bridge in $\mathcal{B}_{i+1} \cup \cdots \cup \mathcal{B}_e$. It is not hard to see, that similarity is an equivalence relation on $\mathcal{B}_i$. It follows from the definition of $\mathcal{B}_i$, that no two similar C-bridges in $\mathcal{B}_i$ are separated by any other C-bridge of $\mathrm{Int}(C)$. Consequently, an equivalence class with respect to similarity consists of at most l C-bridges. There are at most $d \sum_{j=i+1}^{e} |\mathcal{B}_j|$ pairwise nonsimilar C-bridges in $\mathcal{B}_i$ (if $i < e$). A simple inductive proof shows that

$$|\mathcal{B}_i| \leq l(ld+1)^{e-i}.$$

This implies

$$|\mathcal{B}| \leq \sum_{i=1}^{e} l(ld+1)^{e-i} \leq \frac{1}{d}(ld+1)^e.$$

Let s be an integer such that for every subsequence $1 \leq i_1 < i_2 < \cdots < i_s \leq k$, none of (a), (b) and (c) holds. Then $s > \max\{d, l, e\}$ and, consequently,

$$k \leq s|\mathcal{B}| \leq z(s) = s^{2s}. \tag{1}$$

Since $z(s)$ is an increasing function this proves the lemma. $\square$

Let us observe that (1) shows that the function $g(k)$ in Lemma 2.4 is of order $\log(k)/\log\log(k)$.

Lemma 2.5 *There is a nondecreasing integer function $h : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} h(n) = \infty$ and such that the following holds. Let C be a cycle of an arbitrary 3-connected plane graph G and let $u_1, \ldots, u_k$ be vertices which appear on C in that order such that no two of them belong to the same facial cycle. Then G contains a $\{u_1, \ldots, u_k\}$-labeled $K_{2,t}$-minor, where $t = h(k)$.*

Proof. By Lemma 2.4, there is a subsequence $v_1, \ldots, v_{g(k)}$ of $u_1, \ldots, u_k$ satisfying one of the cases (a)–(c) of that lemma. Repeating the same in $\mathrm{Ext}(C)$ with vertices $v_1, \ldots, v_{g(k)}$, we get a subsequence $z_1, \ldots, z_r$, $r = g(g(k))$, such that in each of $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$, one of the cases (a)–(c) occurs. Considering $\mathrm{Int}(C)$, we denote by w_j a vertex of attachment of B (or a vertex of B_j, or a vertex of F in cases (b) and (c), respectively) which belongs to the segment S_j, $j = 1, \ldots, r$. For $\mathrm{Ext}(C)$, we denote the corresponding bridge(s) or face by B' or B'_j, or F' (respectively), and define corresponding vertices w'_j on S_j. If case (b) occurs in $\mathrm{Int}(C)$ (resp., $\mathrm{Ext}(C)$) we denote by i_0 (resp., i'_0) the index i_0 from Lemma 2.4. Because of symmetry, we distinguish 6 cases. We will use the notation (b|c) to denote the case where (b) occurs in $\mathrm{Int}(C)$ and (c) in $\mathrm{Ext}(C)$, and similarly for the other cases.

Case (a|a): Let Z_1 be a spanning tree in $B - V(C)$ together with an edge joining this tree with w_j for each odd index j. Similarly, let Z_2 be a spanning tree in $B' - V(C)$ together with an edge joining this tree with w'_j for each even index j. Now we get a U-labeled $K_{2,t}$-preminor in G, where $t = \lfloor r/2 \rfloor$, by adding segments of C joining vertices w_j and w'_{j+1}, $j = 1, 3, 5, \ldots$.

Case (a|b): This case is similar to the above, except that the tree Z_2 is obtained as follows. We may assume that $i_0 = r$. Now, start with spanning trees in interiors of bridges $B'_j - V(C)$ together with edges from these trees to w'_j, $j = 2, 4, 6, \ldots$. Finally, add the segment S_r and edges from these trees to S_r. Then we get a U-labeled $K_{2,t}$-minor in G, where $t = \lfloor (r-1)/2 \rfloor$.

Case (a|c): For $i \in \{1, \ldots, r\}$, let α' be the vertex of $F' \cap S_{i-1}$ which is closest to u_i on C. Similarly, let β' be the vertex of $F' \cap S_i$ as close as possible to u_i on C. Then the segment A' of F' from α' to β' is internally disjoint from C. Let α and β be attachments of B on S_{i-1} and S_i, respectively, chosen as close as possible to u_i. Then there is a facial cycle R in $\mathrm{Int}(C)$ which contains an edge e of B incident with α and contains an edge f of B incident with β. Let $R_B \subseteq B$ be the segment of R from e to f.

By Lemma 2.1, there is a path joining u_i and $B - V(C)$ which is disjoint from F'. It is easy to see that such a path P_i can be chosen so that it is contained in the disk bounded by A', R_B, and the segments of C joining α, α' and β, β'. In particular, the paths P_i, P_j are internally disjoint if $|i - j| \geq 2$.

Let R_i be the union of P_i and the segment of C from α' to β'. Let T_i be a spanning tree in $R_i - (V(B) \cup \{\alpha', \beta'\})$, let e_i be the edge of P_i connecting T_i with $B - V(C)$, and let f_i be an edge of C joining T_i and $F' - w'_r$. Now, we get a U-labeled $K_{2,t}$-preminor in G, $t = \lfloor r/2 \rfloor$, by taking a spanning tree Z_1 in $B - V(C)$, the path $Z_2 = C - w'_r$, the trees T_i and the connecting edges e_i, f_i, $i = 1, 3, 5, \ldots$.

Case (b|c): This case is similar to the case (a|c) above except that we consider the union of S_r and the bridges $B_1, \ldots, B_{r-1}$ to play the role of the bridge B.

Case (b|b): We assume that $i_0 = r$. If $i_0 \neq i'_0$, we can proceed similarly to the case (a|b) above except that we consider the union of $S_{i'_0}$ and the bridges B'_i, $i \in \{1, ..., r\} \setminus \{i'_0\}$ to play the role of the bridge B'. Thus we may assume that $i_0 = i'_0 = r$.

Let $q = \lfloor (r-1)^{1/3} \rfloor$. Let z_j (z'_j) be a vertex of B_j (resp. B'_j) in S_r. If $x, y \in V(S_r)$, we write $x \preceq y$ if x is closer to u_1 on S_r than y. Clearly, $z_1 \preceq z_2 \preceq \cdots \preceq z_{r-1}$ and $z'_1 \preceq z'_2 \preceq \cdots \preceq z'_{r-1}$. We distinguish three subcases.

(i) There is an index i such that $z_i = z'_i = \cdots = z_{i+q} = z'_{i+q}$: In this case we remove all edges of $B'_{i+1}, \ldots, B'_{i+q-1}$ incident with z_i. The resulting graph G' is 2-connected (since $G - z_i$ is 2-connected). Let F' be the new facial cycle of G'. Now, a proof similar to the case (a|c) shows that there is a U-labeled $K_{2,t}$-minor, where $t = \lfloor (q-3)/2 \rfloor$.

(ii) There is an index i such that $z_{i+q} \prec z'_i$: This case is similar to the case (a|a) where the union of the segment of S_r from z_i to z_{i+q} and bridges $B_i, \ldots, B_{i+q}$ play the role of B, while the union of the segment of S_r from z'_i to z'_{i+q} and bridges $B'_i, \ldots, B'_{i+q}$ play the role of B'.

A similar proof works if $z'_{i+q} \prec z_i$.

(iii) Otherwise: In this case, there are indices $1 \leq i_1 < i_2 < \cdots < i_q \leq r$ such that for $j = 1, \ldots, q$ we have $z_{i_j} \prec z'_{i_{j+2}} \prec z_{i_{j+4}} \prec z'_{i_{j+6}}$. Let Q_j be a cycle contained in $B_{i_j} \cup B'_{i_j} \cup S_j$ and the segment of S_r from z_{i_j} to z'_{i_j}. The cycles for $j = 1, 5, 9, \ldots$ are pairwise disjoint and nested and are interlaced with U. Now, Lemma 2.2 applies.

Case (c|c): Since G is 3-connected, $F \cap F'$ is connected. Therefore, we may assume that $F \cap F' \subseteq S_r$. Suppose that $3 \leq i \leq r-2$. By Lemma 2.1, there is a path P_i joining u_i and F which is disjoint from F'. Let A' be the segment of F' defined as in case (a|c), and let A be the segment of F defined in the same way. It is easy to see that we may assume that P_i is contained in the disk bounded by $A \cup A'$ and the two segments of C joining the ends of A and A'.

We define similarly P_i', a path joining u_i and F' which is contained in the same disk as P_i and is disjoint from F. Now, we have a U-labeled $K_{2,t}$-minor, $t = \lfloor (r-4)/2 \rfloor$, in the similar way as in previous cases, where we take $Z_1 = F - S_r$, $Z_2 = F' - S_r$, and T_i a spanning tree in $(P_i \cup P_i') - (F \cup F')$, $i = 3, 5, 7, \ldots$. □

Proof. (of Theorem 1.2). Let U' be a subset of U such that $|U'| = \tau(U)$ and no two vertices in U' belong to the same facial cycle of G. Furthermore, let $t = \lceil \sqrt{\tau(U)} \rceil$. By Lemma 2.3 either G contains a U-labeled $K_{2,t}$-minor or there is a cycle C of G which contains t vertices of U'. In the latter case it follows from Lemma 2.5 that G contains a U-labeled $K_{2,h(t)}$-minor. This proves the first part of Theorem 1.2.

To prove the second part, let H be a plane embedding of a $K_{2,t}$ and let A, B denote the color classes of H such that $|A| = t$ and $|B| = 2$. For an arbitrary embedding of H in the plane, every facial cycle of H has length four and contains precisely two vertices in each color class. Consequently, each face cover of A in H contains at least $t/2$ facial cycles. It follows that if G contains a U-labeled $K_{2,t}$-minor, then $\tau(U) \geq t/2$.

□

It is worth mentioning that the proof of Theorem 1.2 is constructive and yields a polynomial time algorithm such that, given a 3-connected planar graph G and vertex set $U \subseteq V(G)$, no two vertices of which are in the same facial cycle, finds a U-labeled $K_{2,f(|U|)}$-preminor in G.

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