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Abstract

Is it possible to label the edges of K_n with distinct integer weights so that every Hamilton cycle has the same total weight? We give a local condition characterizing the labellings that witness this question's perhaps surprising affirmative answer. More generally, we address the question that arises when "Hamilton cycle" is replaced by "k-factor" for nonnegative integers k. Such edge-labellings are in correspondence with certain vertex-labellings, and the link allows us to determine the growth rate of the maximum edge-label in a "most efficient" injective metric trivial-TSP labelling.

1 Introduction

Recall the Travelling Salesman Problem (TSP): given a labelling $\lambda : E(K_n) \to \mathbb{Z}^+$ of the edges of K_n, determine a Hamilton cycle H (a *TSP-tour*) minimizing $\sum_{A\in E(H)} \lambda(A)$. Of course, TSP is notoriously difficult; its decision version is NP-complete—see [6]—and even the restricted case MTSP for metric λ (definitions to follow) is intractable. In this paper we focus on the other extreme, when all TSP-tours, or MTSP-tours, have equal length.

Any constant function on the edge set provides a simple example of a labelling with this property; a more complicated example appears in Fig. 1. We are primarily interested

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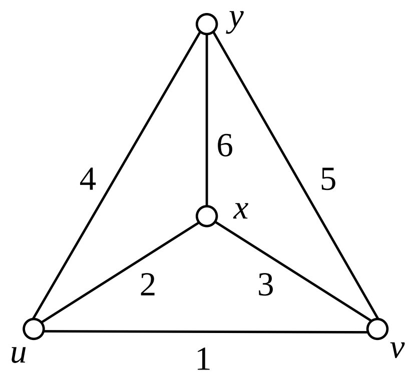

Figure 1: An injective trivial-MTSP edge-labelling of K_4.

in labellings with distinct edge-labels, i.e., ones for which λ is injective, as in Fig. 1. But most of our results apply to non-injective λ as well.

We call the function $\lambda : E(K_n) \to \mathbb{Z}$ *trivial-TSP* whenever the value of $\sum_{A \in E(H)} \lambda(A)$ is independent of the Hamilton cycle H. Being trivial-TSP is a global property of λ in the sense that naive verification requires inspection of every Hamilton cycle, each of which spans K_n. A main contribution of this paper is the identification of a local property, called C_4-matching, that characterizes the trivial-TSP edge-labellings.

Using the C_4-matching property allows us to establish a connection between those λ which are trivial-TSP and certain vertex-labellings ν; namely, there is a function F such that each edge ij of K_n satisfies $\lambda(ij) = F(\nu_i, \nu_j)$. That such a connection exists brings our study into the (overwhelming) realm of graph labellings; the extensive survey [5] contains over 400 references. Our graph labellings are related to, but different from, several other labelling methods studied previously. We explore a few of these connections after introducing the basic definitions.

Notation and terminology

Sets

We write $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$ and $\mathbb{R}^+$, respectively, for the sets of integers, positive integers, nonnegative integers and positive real numbers. For $n \in \mathbb{Z}^+$, we use $[n]$ to denote the set $\{1, \ldots, n\}$, and $\mathbb{Z}_n$ to denote the ring of integers modulo n.

Graphs

Most of our graph-theoretic notation and terminology is relatively standard; see, e.g., [2] or [24] for any omitted definitions. For graphs G, H, we write $H \cong G$ when H is isomorphic to G and $H \leq G$ when H is a subgraph of G. If G and H have identical vertex sets and disjoint edge sets, then $G \oplus H$ denotes the graph on the common vertex set with edge set $E(G) \cup E(H)$. If A is an edge with ends x, y, then we write $A = xy$. The vertex set of K_n is usually $[n]$. We use $\delta = \delta(G)$ for the minimum degree of a graph G. A cycle visiting the vertices $x_1, x_2, \ldots, x_r$ in this order and then returning to x_1 is denoted by $(x_1, x_2, \ldots, x_r)$. For a nonnegative integer k, a *k-factor* of G is a k-regular spanning subgraph of G. A 1-factor is often called a *perfect matching*. See [14, 22] for more specifics on the theory of matchings and factorizations.

Labellings

An *edge-labelling* (resp. *vertex-labelling*) of a graph $G = (V, E)$ is a function $\lambda : E \to S$ (resp. $\nu : V \to S$) into some set S of *labels*. For edges, we use the label sets $S = \mathbb{Z}$ and $\mathbb{Z}^+$; for vertices, we use variously $S = \mathbb{Z}$, $\mathbb{N}$, $\frac{1}{2}\mathbb{Z}$ and $\frac{1}{2}\mathbb{N}$. If λ is an edge-labelling and $A \in E$, then $\lambda(A)$ is called the *label* of A. We use analogous terminology for vertex-labellings ν, but the label of a vertex i is always denoted by ν_i. In discussing edge-labellings, it is often convenient to view the edge labels as "weights". The (*total*) *weight* of a subgraph H of G means simply the sum $\lambda(H) := \sum_{A \in E(H)} \lambda(A)$.

We say that an edge-labelling λ of G has *constant-weight on k-factors* if each k-factor of G has the same total weight. For $G = K_n$, we call λ *metric* if it satisfies the triangle-inequality: $\lambda(xy) \leq \lambda(xz) + \lambda(zy)$ for every triple $x, y, z \in V(K_n)$. For trivial-TSP λ, we call the common weight of all Hamilton cycles the *Hamilton-weight* of λ. If λ is both metric and trivial-TSP, then λ is *trivial-MTSP*.

As suggested above, we enter the realm of *graph labelling* when some function F connects a pair λ, ν of edge- and vertex-labellings of G via

$$\lambda(ij) = F(\nu_i, \nu_j) \text{ for each } ij \in E.$$

In this case we say that λ is *induced* from ν (via F). The two examples of such functions under study in this paper are $F(x, y) = x + y$ and $F(x, y) = (x + y)/2$. Starting from a vertex-labelling $\nu : V \to \mathbb{R}^+$, the first of these was considered by Deuber and Zhu [4] in their study of circular colourings of weighted graphs. The following subsection compares trivial-TSP labelling with three other common labelling notions.

Sequences

A (finite or infinite) sequence (x_i) of integers has *constant-parity* if $x_i \equiv x_j \pmod 2$ for all i, j. Following Kotzig [11], we call (x_i) *well-spread* if all the pairwise sums $x_i + x_j$, for $i < j$, are different; see also [16]. Finally, (x_i) is a *Sidon sequence* if all the sums $x_i + x_j$, for $i \leq j$, are distinct. In connection with his studies in Fourier theory, Sidon [18, 19] considered these sequences under the name *B_2-sequence*. Every Sidon sequence is well-spread, but not conversely: $(1, 2, 3)$ is well-spread but not Sidon. See [10] for a basic reference on Sidon sequences.

Other graph labelling notions

To put the present paper into context, we compare trivial-MTSP labelling with three other labelling schemes that have received considerable attention: graceful, harmonious, and magic labellings. See [5] for details. A *graceful* labelling of $G = (V, E)$ is an injective vertex-labelling $\nu : V \to \{0, 1, \ldots, |E|\}$ such that the edge-labelling induced from ν via $F(x, y) = |x - y|$ is also injective. This term was suggested by Golomb [7], though the idea was introduced by Rosa a few years earlier. Since $|x - y| \neq |z - w|$ implies that $x + w \neq y + z$, every graceful labelling of K_n is a well-spread, $\mathbb{N}$-sequence.

Graham and Sloane [8] called a graph G *harmonious* if it admits a vertex-labelling $\nu : V \to \mathbb{Z}_{|E|}$ such that both ν and the edge-labelling induced from ν via $F(x, y) = x + y \pmod{|E|}$ are injective. For example, in Fig. 1, if we label the vertices u, v, x and y respectively with 0, 1, 2 and 4 (and reduce the edge labels modulo 6), then we obtain a harmonious labelling of K_4. Note that the vertex labels of harmonious complete graphs are also well-spread, $\mathbb{N}$-sequences.

Kotzig and Rosa [12] introduced the notion of a *magic* labelling of G, i.e., a bijection $\lambda : V \cup E \to [|V \cup E|]$ such that for each $ij \in E$, the value $\lambda(i) + \lambda(ij) + \lambda(j)$ is the same, say κ. (These are now called *edge-magic total* labellings; see [23] for a short survey and some recent results.) It is illusory if this labelling scheme appears ill-fitted for the present framework. As observed in [23], since $\sum_{ij\in E} \lambda(ij) + \sum_{i\in V} \lambda(i) = (|V|+|E|)(|V|+|E|+1)/2$ and $\lambda(ij) = \kappa - \lambda(i) - \lambda(j)$, we see that κ is determined by the vertex labels, so that $\lambda|_V$ induces $\lambda|_E$. Again, it is easy to see that the vertex labels of a magic labelling of K_n comprise a well-spread, $\mathbb{N}$-sequence.

As we shall see (Corollary 4.2), the injective trivial-MTSP edge-labellings λ are induced via $F(x, y) = (x + y)/2$ from constant-parity, well-spread, $\mathbb{N}$-sequences of vertex labels. Comparing these properties with those observed for the vertex labels of graceful, harmonious, and magic labellings of complete graphs, one might expect a strong connection between these labellings and injective trivial-MTSP λ. Indeed, each scheme labels the vertices of K_n with a well-spread, $\mathbb{N}$-sequence, say $\nu_1, \ldots, \nu_n$. So the sequence $2\nu_1, \ldots, 2\nu_n$ satisfies the requirements of a vertex-labelling inducing our desired λ. If we now replace the graceful, harmonious, or magic edge-labels by $\lambda(ij) = \nu_i + \nu_j$, then λ is injective and trivial-MTSP. The simplest case of the transformation just described was illustrated in reverse when we indicated how to convert the labelling in Fig. 1 to a harmonious labelling.

Unfortunately, the connection discussed in the preceding paragraph is rather limited because the definitions of graceful, harmonious, and magic labellings are too restrictive to allow many complete graphs to enjoy these properties. The results are easily summarized: K_n is graceful if and only if $n \leq 4$ ([7], [20]); K_n is harmonious if and only if $n \leq 4$ ([8]); K_n is magic if and only if $n \in \{1, 2, 3, 5, 6\}$ ([13]; see also [23] for a listing of all magic labellings of K_n). On the other hand, since the constant-parity, well-spread, $\mathbb{N}$-sequences may be extended indefinitely, we easily obtain injective trivial-MTSP edge-labellings of K_n for all $n \geq 1$.

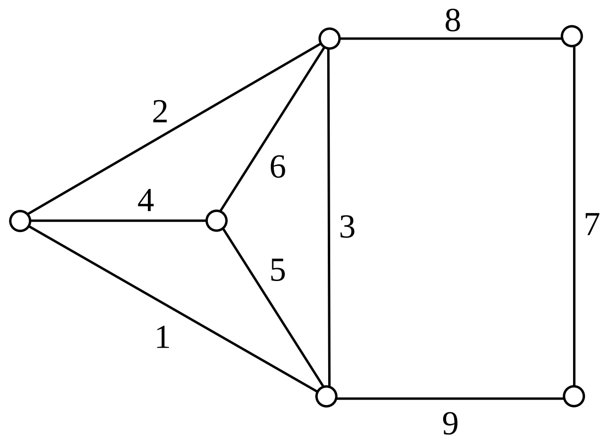

Figure 2: An edge-labelled graph satisfying the K_4-MP but not the C_4-MP.

Outline

The rest of the paper is organized as follows. The next section characterizes the trivial-TSP edge-labellings by the C_4-matching property. There, we also prove that such labellings have constant-weight on 2-factors. Section 3 re-proves the latter result and extends it to 1-factors (provided n is even) without reference to the C_4-matching property. In Section 4, we establish our fundamental connection between these edge-labellings and vertex-labellings. The main result (Theorem 4.1) and its corollaries tie together some of the earlier results and—as suggested above—provide an essential link between edge-labellings and well-spread sequences. This link eventually allows us (in Section 5) to determine the growth-rate of the maximum label in the "most efficient" injective trivial-MTSP edge-labelling scheme.

2 Local conditions

An edge-labelling $\lambda : E(G) \to \mathbb{Z}$ of a graph G has the *C_4-matching property* if, for each 4-cycle in G, say with consecutive edges A, B, C, D, the relation $\lambda(A)+\lambda(C) = \lambda(B)+\lambda(D)$ holds. We shall abbreviate this property by C_4-MP.

Another way to formulate the C_4-MP for λ is to require that in each 4-cycle H of G, the total λ-weight of every perfect matching of H is the same. With this view in mind, we introduce a related local property. An edge-labelling $\lambda : E(G) \to \mathbb{Z}$ has the *K_4-matching property* (K_4-MP) if, for each 4-clique H of G, the total weight assigned by λ to each perfect matching of H is identical.

If an edge-labelling λ of a general graph G satisfies the C_4-MP, then it necessarily satisfies the K_4-MP, but the converse is not true. Fig. 2 depicts a graph with edge labels $\{1, 2, \dots, 9\}$ satisfying the K_4-MP but not the C_4-MP. We are mainly interested in edge-labellings of complete graphs, for which the two local properties are easily seen to be equivalent:

Proposition 2.1 *An edge-labelling of K_n satisfies the C_4-MP if and only if it satisfies the K_4-MP.* ∎

It is perhaps surprising that the edge-labellings of K_n that are trivial-TSP can be recognized by verifying local conditions only.

Theorem 2.2 *An edge-labelling of K_n is trivial-TSP if and only if it satisfies the C_4-matching property.*

Proof. The result is vacuously true for $n = 1, 2$ and trivial for $n = 3$, so we will assume that $n \geq 4$.

For the necessity of the C_4-MP, suppose that λ is a trivial-TSP edge-labelling of $G = K_n$, and consider a 4-cycle of G with consecutive edges $A = xu$, $B = uv$, $C = vw$ and $D = wx$. Let H_1 denote a Hamilton cycle of G that visits the vertices x, u, w, v consecutively in this order; thus A, C are edges of H_1 while B, D are not. Let H_2 be obtained from H_1 by deleting the edges A, C and adding the edges B, D; clearly H_2 is also a Hamilton cycle of G. Since $H_1 \smallsetminus \{A, C\} = H_2 \smallsetminus \{B, D\}$ and λ is trivial-TSP, we must have $\lambda(A) + \lambda(C) = \lambda(B) + \lambda(D)$, and since the 4-cycle was arbitrary, this shows that λ satisfies the C_4-MP.

In proving the converse, it is convenient to consider more carefully the operation leading from H_1 to H_2, which we call a *C_4-exchange*. Notice that the C_4-exchange described above transposes the adjacent vertices u, w in the visiting order of the initial Hamilton cycle, while preserving the visiting order of the remaining vertices. It is clear that any given pair of adjacent vertices on a Hamilton cycle of G can be transposed by a C_4-exchange.

Now suppose that λ satisfies the C_4-MP. We will argue that $\lambda(H_1) = \lambda(H_2)$ for any two Hamilton cycles H_1, H_2 of G. If H_1 visits the vertices of G in the order $v_1, v_2, \ldots, v_n$, then H_2 visits them in the order $\sigma(v_1), \sigma(v_2), \ldots, \sigma(v_n)$ for some permutation σ of V. By a sequence of transpositions of adjacent vertices, it is possible to shuffle the H_1-order of V into the H_2-order. Since this transposition sequence corresponds to a sequence of C_4-exchanges, each preserving the total weight of the resulting Hamilton cycle (by the C_4-MP), it follows that $\lambda(H_1) = \lambda(H_2)$. ∎

Theorem 2.2 reduces the complexity of the problem of recognizing the trivial-TSP edge-labellings of K_n from what on the surface appears to be super-exponential in n (check all TSP-tours) to polynomial in n (verifying the C_4-MP requires only $O(n^4)$ time). In Section 4, we outline an $O(n^2)$—hence optimal—algorithm for this recognition problem.

As we shall see (cf. Theorems 3.2, 3.4, and all of Section 4), besides being trivial-TSP, there are a number of other equivalent properties of edge-labellings of K_n which can therefore be recognized via the C_4-MP. As a first illustration, we offer the next result. Although it is a special case of Corollary 4.3—which itself has a short proof—we provide a separate proof here because of its completely different flavour.

Theorem 2.3 *An edge-labelling λ of K_n has constant-weight on 2-factors if and only if it satisfies the C_4-matching property.*

Proof. The necessity of the C_4-MP is immediate from Theorem 2.2 since Hamilton cycles are 2-factors. For the sufficiency, suppose that λ satisfies the C_4-MP. Theorem 2.2 shows that we need only establish that each 2-factor F in K_n has the same weight as some Hamilton cycle. Write F as

$$(x_1, \ldots, x_{m_1})(x_{m_1+1}, \ldots, x_{m_2}) \cdots (x_{m_{k-1}+1}, \ldots, x_n),$$

with each cycle of length at least three.

Given a cycle $C = (y_1, y_2, \ldots, y_m)$, with $m \geq 6$ and $3 \leq i \leq m-3$, the *split* of C at y_i yields the disjoint cycles $C_1 = (y_1, \ldots, y_i)$ and $C_2 = (y_{i+1}, \ldots, y_m)$. The total weight of the new cycles is $\lambda(C_1)+\lambda(C_2) = \lambda(C)+[\lambda(y_iy_1)+\lambda(y_my_{i+1})-\lambda(y_iy_{i+1})-\lambda(y_my_1)]$. Since the C_4-MP implies that the bracketed expression is zero, we see that a split preserves the λ-weight of C.

Starting with the Hamilton cycle $H := (x_1, \ldots, x_n)$—the concatenation of the cycles of F—and successively applying the split operation at $x_{m_1}, x_{m_2}, \ldots, x_{m_{k-1}}$ yields F, and we have $\lambda(F) = \lambda(H)$. ∎

3 One-factors and two-factors

Theorems 2.2 and 2.3 together establish the equivalence of an edge-labelling of K_n being trivial-TSP and having constant-weight on 2-factors. Eventually we will extend the scope of this equivalence to replace '2' by 'k', for all—and indeed *any*—$k \in [n-2]$; see Corollary 4.3 and Theorem 4.4. In this section, we prove the special case $k = 1$ of the general result and take another look at the $k = 2$ case. Though these results are subsumed in Section 4, the proofs here may be of independent interest.

Lemma 3.1 *For any two 1-factors F, G of K_n, there exists a 1-factor H of K_n such that both $F \cup H$ and $G \cup H$ are Hamilton cycles in K_n.*

Proof. Suppose the components of $F \cup G$ are $C_1, C_2, \ldots, C_t$. Then the C_i are disjoint subgraphs whose union spans K_n, and each is either an edge (common to F and G) or an even cycle (with edges alternately in F and G).

If C_i is an edge, call one endpoint x_i and the other y_i.

If C_i is a cycle of length $2m$, label its vertices sequentially as $a_{i,1}, a_{i,2}, \ldots, a_{i,2m}$, where $a_{i,1}a_{i,2}, a_{i,3}a_{i,4}, \ldots$ are in F and $a_{i,2}a_{i,3}, a_{i,4}a_{i,5}, \ldots$ are in G; then $a_{i,1}$ is labelled x_i and $a_{i,m+1}$ is labelled y_i. For each such cycle C_i, all the edges $a_{i,2}a_{i,2m}, a_{i,3}a_{i,2m-1}, \ldots, a_{i,m}a_{i,m+2}$ of K_n are allocated to H. Adding the edges y_1x_2, y_2x_3,..., $y_{t-1}x_t$ to H yields a suitable 1-factor. ∎

Theorem 3.2 *For every even positive integer n, an edge-labelling λ of K_n is trivial-TSP (with Hamilton-weight κ) if and only if it has constant-weight on 1-factors (with weight $\kappa/2$).*

Proof. First suppose that λ is trivial-TSP, and let F be a 1-factor of K_n. Select any 1-factor G of K_n, and find a 1-factor H such that both $F \cup H$ and $G \cup H$ are Hamilton cycles in K_n. Then

$$\lambda(F) + \lambda(H) = \kappa = \lambda(G) + \lambda(H),$$

so $\lambda(F) = \lambda(G)$; i.e., λ has constant-weight on 1-factors. In particular, $\lambda(F) = \lambda(H)$, so $2\lambda(F) = \kappa$.

The converse is trivial since, with n even, each Hamilton cycle is a disjoint union of two 1-factors. ∎

Lemma 3.3 *If G is a union of two disjoint cycles of length m and $m+t$, with $0 \leq t < m$, then there exists a Hamilton cycle H in $K_{2m+t} \setminus E(G)$ such that $G \oplus H$ can be factored into two Hamilton cycles.*

Proof. Suppose $G = (x_1, x_2, \ldots, x_m) \cup (y_1, y_2, \ldots, y_{m+t})$. If $t = 0$, let

$$H := (x_1, y_2, x_2, \ldots, x_{i-1}, y_i, x_i, \ldots, x_m, y_1).$$

If $t = 1$, let

$$H := (x_1, y_m, y_2, x_2, \ldots, x_{i-1}, y_i, x_i, \ldots, x_{m-1}, y_{m+1}, x_m, y_1).$$

Finally, if $t \geq 2$, let

$$H := (x_1, y_2, y_{2+t}, x_2, y_3, y_{3+t}, \ldots, x_t, y_{t+1}, y_{2t+1}, x_{t+1}, y_{2t+2}, x_{t+2}, \ldots, x_{m-1}, y_{m+t}, x_m, y_1).$$

In each case, define

$$\begin{array}{lcl} L & := & G \cup x_1y_1 \cup x_my_{m+t} \setminus x_1x_m \setminus y_1y_{m+t}; \\ M & := & H \cup x_1x_m \cup y_1y_{m+t} \setminus x_1y_1 \setminus x_my_{m+t}. \end{array} \tag{1}$$

Then L and M are Hamilton cycles, and $G \oplus H = L \oplus M$. ∎

Now we are ready to give the promised second proof of the equivalence established by Theorems 2.2 and 2.3.

Theorem 3.4 *An edge-labelling λ of K_n is trivial-TSP (with Hamilton-weight κ) if and only if it has constant-weight on 2-factors (with weight κ).*

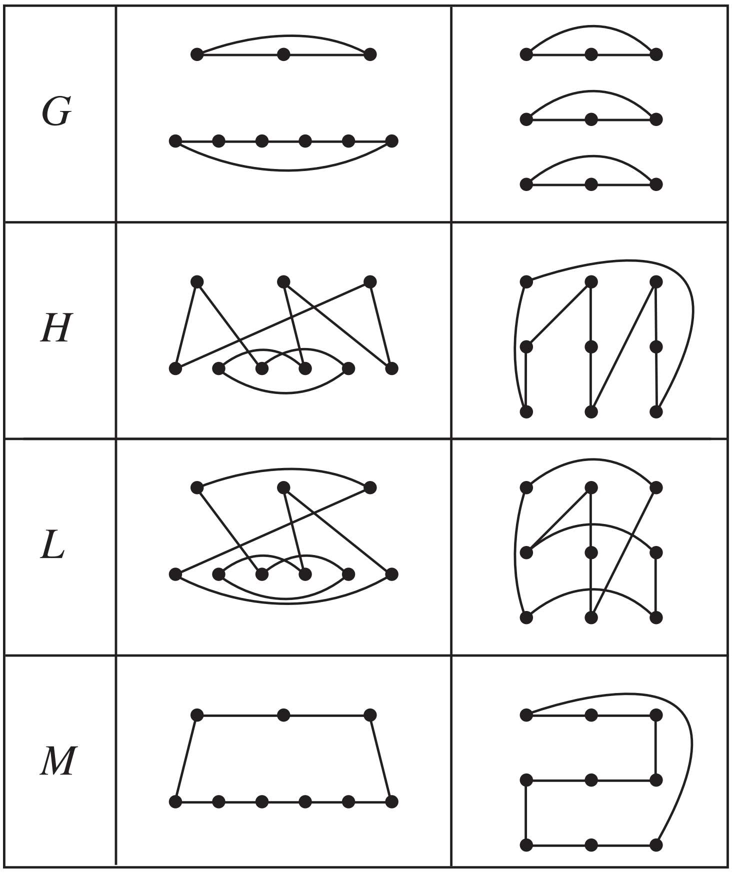

Figure 3: Two small cases for Theorem 3.4

Proof. The sufficiency is immediate since Hamilton cycles are 2-factors. For the necessity, suppose λ is trivial-TSP, and let G be a 2-factor of K_n. If G is Hamiltonian, there is nothing to prove. So assume that G consists of at least two cycles. We shall prove that for every such G there exist a 2-factor L such that $\lambda(L) = \kappa$ and two Hamilton cycles H, M such that $G \oplus H = L \oplus M$. It will then follow from $\lambda(G) + \lambda(H) = \lambda(L) + \lambda(M)$ that $\lambda(G) = \kappa$.

Assume the result is true for all 2-factors with fewer components (cycles) than G. Denote by $m = m(G)$ the size of the smallest cycle of G. If $n - m < 6$, then G consists of two cycles, the larger being of size $n - m < 2m$ (since $m \geq 3$), and the required H, L and M exist by Lemma 3.3. If $n - m = 6$, Lemma 3.3 applies in every case except $G = C_3 \cup C_6$ or $C_3 \cup C_3 \cup C_3$, and suitable H, L and M are shown in Fig. 3. (In these cases $\lambda(L) = \kappa$ because L is Hamiltonian.)

Now we assume $n - m > 6$. Denote by $C_1 = (x_1, x_2, \ldots, x_m)$ a component of G of length m, and write G' for the graph derived from K_n by deleting all the edges of G and all the vertices of C_1. Then G' has $n - m$ vertices, is regular of degree $n - m - 3$, and hence satisfies

$$\delta(G') \geq \frac{1}{2}(|V(G')| + 1). \tag{2}$$

From a theorem of Ore [15], a graph satisfying (2) has a spanning path whose endpoints are any specified pair of vertices. Select two vertices, y_1, y_{n-m}, that are adjacent in G; say the path $y_1, y_2, \ldots, y_{n-m}$ is a Hamilton path in G'. Then consider the Hamilton cycle (in K_n)

$$H := (y_1, x_1, y_2, \ldots, y_{m-1}, x_{m-1}, y_m, y_{m+1}, y_{m+2}, \ldots, y_{n-m}, x_m);$$

notice that G and H are edge-disjoint.

Now define L, M from G, H by the construction (1), with $n - m$ in the role of $m + t$. Then L and M are edge-disjoint, and M is a Hamilton cycle. Since L has fewer components than G, by hypothesis we have $\lambda(L) = \kappa$. Moreover $G \oplus H = L \oplus M$. ∎

Remark. Theorem 3.4 also follows from the fact that, given any 2-factor G of K_n, there exist Hamilton cycles H, L and M such that $G \oplus H = L \oplus M$ (with the obvious small exceptions). However, a proof of that fact would be longer than the proof given. ∎

4 Edge labels from vertex labels

Theorem 4.1 and Corollary 4.2 below establish the connection between trivial-TSP edge-labellings and vertex-labellings mentioned in the introduction. This link provides the key to generalizing Theorems 2.3, 3.2 and 3.4 to include k-factors for $k \geq 0$; see Corollary 4.3 and Theorem 4.4. It also brings constant-parity and well-spread sequences into the fold, gives an easy algorithm for producing trivial-TSP edge-labellings of K_n, and finally yields an optimal algorithm for recognizing these labellings.

Theorem 4.1 *For $n \geq 3$ and $G \cong K_n$, an edge-labelling $\lambda : E(G) \to \mathbb{Z}$ satisfies the C_4-matching property if and only if there is a vertex-labelling $\nu : V(G) \to \frac{1}{2}\mathbb{Z}$ such that*

$$\lambda(ij) = \nu_i + \nu_j \text{ for each edge } ij \text{ of } G. \tag{3}$$

The sequence $(\nu_i)_{i=1}^n$ is uniquely determined by λ, is nonnegative if and only if λ is metric, and is well-spread if and only if λ is injective.

Proof. If such a vertex-labelling exists, then each Hamilton cycle H of G satisfies

$$\sum_{ij \in E(H)} \lambda(ij) = \sum_{ij \in E(H)} (\nu_i + \nu_j) = 2 \sum_{i=1}^{n} \nu_i,$$

since H is a 2-factor of G. Thus λ is a trivial-TSP labelling, and Theorem 2.2 implies that λ satisfies the C_4-MP.

We prove the converse by induction on n.

Any edge-labelling λ of K_3 vacuously satisfies the C_4-MP, so we must establish the existence of a unique half-integer vertex-labelling ν satisfying (3). In this case ($n = 3$), this system takes the form

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \lambda(12) \\ \lambda(13) \\ \lambda(23) \end{pmatrix},$$

and since this coefficient matrix is nonsingular with inverse $\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$, we see that (ν_1, ν_2, ν_3) is indeed uniquely determined by (3) and has half-integer entries.

Now fix $n > 3$, assume the result is true in case $G \cong K_{n-1}$, and suppose that $\lambda : E(K_n) \to \mathbb{Z}$ satisfies the C_4-MP. Let G be the subgraph of K_n induced by the vertices in $[n-1]$. Then $G \cong K_{n-1}$ and $\lambda|_{E(G)}$ satisfies the C_4-MP for G, so our inductive hypothesis implies that there is a unique vertex-labelling $\nu : V(G) \to \frac{1}{2}\mathbb{Z}$ such that

$$\lambda(ij) = \nu_i + \nu_j \text{ for each edge } ij \text{ of } G. \tag{4}$$

We complete the proof by arguing that ν extends uniquely and unambiguously to $[n]$, subject to (3). For an appropriate choice of ν_n, the equations in (3) still to be satisfied are

$$\lambda(in) = \nu_i + \nu_n \text{ for } 1 \leq i \leq n-1. \tag{5}$$

The only way to satisfy the first of these is to set $\nu_n := \lambda(1n) - \nu_1$. To show that this value satisfies the remaining equations, we fix i, $1 < i < n$, and derive the ith equation in (5). Since $n > 3$, there is an index $j \in [n] \setminus \{1, i, n\}$, so that $(1, j, i, n)$ is a 4-cycle. Since λ satisfies the C_4-MP, we have

$$\lambda(1j) + \lambda(in) = \lambda(ij) + \lambda(1n),$$

which by (4) yields

$$(\nu_1 + \nu_j) + \lambda(in) = (\nu_i + \nu_j) + \lambda(1n),$$

or

$$\lambda(in) = \nu_i + (\lambda(1n) - \nu_1) = \nu_i + \nu_n.$$

Therefore, our choice of ν_n indeed satisfies (5).

Finally, notice that nonnegative vertex-labels correspond exactly to trivial-MTSP edge-labellings, since, for any three vertices x, y, z, we have

$$\lambda(xy) \leq \lambda(xz) + \lambda(zy) \iff \nu_z \geq 0. \qquad \blacksquare$$

Corollary 4.2 *For $n \geq 3$, an edge-labelling $\lambda : E(K_n) \to \mathbb{Z}$ satisfies the C_4-matching property if and only if there is a vertex-labelling $\nu : V(K_n) \to \mathbb{Z}$ such that*

$$\lambda(ij) = \frac{\nu_i + \nu_j}{2} \text{ for each edge } ij \text{ of } K_n. \tag{6}$$

The sequence $(\nu_i)_{i=1}^n$ is uniquely determined by λ, has constant-parity, is nonnegative if and only if λ is metric, and is well-spread if and only if λ is injective.

Proof. Double the vertex labels in Theorem 4.1. ∎

Remarks. Corollary 4.2 (or Theorem 4.1) suggests an algorithm for producing trivial-TSP edge-labellings: start with a constant-parity integral sequence $(\nu_i)_{i=1}^n$ for which the mean of any two terms is positive, and define $\lambda : E(K_n) \to \mathbb{Z}$ by (6). We can arrange for λ to be injective (or metric) by starting with a well-spread (or nonnegative) ν.

With one further observation, we can use these results to obtain the algorithm alluded to following the proof of Theorem 2.2, namely, an *optimal* algorithm to check if a given edge-labelling λ of K_n is trivial-TSP. Notice that any fixed spanning tree T of K_n allows us to obtain, in $O(n)$ time, solutions $(\nu_i)_{i=1}^n$ to (6)—with T in place of K_n—with one degree of freedom. For any edge $A \in K_n \setminus T$, the value of $\lambda(A)$ then uniquely determines all the ν_i. By Corollary 4.2 (and Theorem 2.2), to decide whether λ is trivial-TSP, it remains only to verify (6) for all remaining edges. Since this can be done in $O(n^2)$ time, and this decision problem obviously requires examining every edge of K_n, this algorithm is indeed optimal. ∎

The next result generalizes Theorems 2.3, 3.2 and 3.4.

Corollary 4.3 *For $n \geq 3$, an edge-labelling λ of K_n satisfies the C_4-matching property if and only if it has constant-weight on k-factors, for all $k \geq 0$.*

Proof. For the sufficiency of the k-factor condition, take $k = 2$ and apply Theorem 2.3 (or Theorem 2.2). For the necessity, suppose that λ satisfies the C_4-MP, and fix an integer $k \geq 0$. By Theorem 4.1, there is a vertex-labelling ν satisfying (3). Now any k-factor F of K_n, provided it exists, satisfies

$$\sum_{ij \in E(F)} \lambda(ij) = \sum_{ij \in E(F)} (\nu_i + \nu_j) = k \sum_{i=1}^{n} \nu_i.$$ ∎

We can weaken the condition in Corollary 4.3 considerably, provided n and k are restricted to avoid trivially satisfying the weakened condition. This statement is made precise in part (e) of the following result, which also summarizes our various characterizations of trivial-TSP edge-labellings.

Theorem 4.4 *If $n \geq 4$ and λ is an edge-labelling of K_n, then the following statements are equivalent:*

(a) λ is trivial-TSP;

(b) λ satisfies the C_4-matching property;

(c) λ satisfies the K_4-matching property;

(d) for every k, $0 \le k \le n-1$, the labelling λ has constant-weight on k-factors;

(e) there exists an integer k, $1 \le k \le n-2$, such that λ has constant-weight on k-factors, and k is even if n is odd.

Proof. We know (cf. Proposition 2.1, Theorem 2.2 and Corollary 4.3) that (a)–(d) are equivalent. Moreover, Theorem 2.3 shows that (b) implies (e), with $k = 2$.

To see that (e) implies (b), fix $k \in [n-2]$, and assume that λ has constant-weight on k-factors. Since k is even if n is odd, there exists a k-factor F of K_n. Since the complement $\overline{F}$ of F is an $(n-k-1)$-factor, and $\lambda(\overline{F}) = \lambda(K_n) - \lambda(F)$, we see that λ has constant-weight on $(n-k-1)$-factors. Therefore, after possibly interchanging the roles of k and $n-k-1$, we may assume that $k \le (n-1)/2$.

Since $k \le (n-1)/2 \le n-2$, there exist vertices x, y that are nonadjacent in F. Let x_1 be a neighbour of x in F. Since y and x_1 both have degree k in F, and since x_1 is adjacent to x while y is not, there exists a neighbour y_1 of y in F that is different from, and nonadjacent with x_1. Now, a C_4-exchange (see the proof of Theorem 2.2) on the 4-cycle (x, x_1, y_1, y) produces another k-factor F'. Since $\lambda(F) = \lambda(F')$, we have $\lambda(xx_1) + \lambda(yy_1) = \lambda(xy) + \lambda(x_1y_1)$.

Now let $C = (u, u_1, v_1, v)$ be any 4-cycle of K_n, and let π be a permutation of $[n]$ with $\pi(x) = u$, $\pi(y) = v$, $\pi(x_1) = u_1$ and $\pi(y_1) = v_1$. Then $\pi(F)$ and $\pi(F')$—defined in the natural way—are k-factors which differ by a C_4-exchange on C. As in the preceding paragraph, this implies that C does not violate the C_4-MP, and since C was arbitrary, we conclude that (b) holds. ∎

5 Edge label growth-rate

Recall from Theorem 4.1 that an injective, metric edge-labelling corresponds to a well-spread, nonnegative, half-integer sequence of vertex labels. With its first term deleted, the Fibonacci sequence furnishes one example of such a sequence; see, e.g., [3] for related background.

Now we consider the rate of growth of the maximum label of the most efficient injective trivial-MTSP edge-labelling scheme. We shall prove that the function

$$\Psi(n) := \min_{\lambda} \max_{A \in E(K_n)} \lambda(A)$$

(the minimum being taken over all injective trivial-MTSP edge-labellings λ) exhibits quadratic growth. This should be compared with the growth rate of the edge labels induced by the Fibonacci numbers as vertex labels. Here, if φ is the golden ratio, then $\max_{A\in E(K_n)} \lambda(A) \in \Theta(\varphi^n)$, so these labels grow exponentially.

Define S, W, $W_{cp} : \mathbb{N} \to \mathbb{Z}^+$ and ψ_{cp}, $\sigma_{cp} : \mathbb{Z}^+ \to \mathbb{N}$ by

$$
\begin{aligned}
S(N) &:= \max\{n : \exists \text{ Sidon sequence } 0 \le x_1 < \cdots < x_n \le N\};\\
W(N) &:= \max\{n : \exists \text{ well-spread sequence } 0 \le x_1 < \cdots < x_n \le N\};\\
W_{cp}(N) &:= \max\{n : \exists \text{ constant-parity well-spread sequence } 0 \le x_1 < \cdots < x_n \le N\};\\
\psi_{cp}(n) &:= \min\{x_{n-1} + x_n : \exists \text{ constant-parity well-spread } \mathbb{N}\text{-sequence } x_1 < \cdots < x_n\};\\
\sigma_{cp}(n) &:= \min\{x_n : \exists \text{ constant-parity well-spread } \mathbb{N}\text{-sequence } x_1 < \cdots < x_n\}.
\end{aligned}
$$

A celebrated result of Erdős and others is that $S(N) \sim \sqrt{N}$; i.e.,

$$\Big(1 - o(1)\Big)\sqrt{N} \le S(N) \le \Big(1 + o(1)\Big)\sqrt{N} \quad \text{as } N \to \infty. \tag{7}$$

Remarks. The upper bound in (7) was proved by Erdős and Turán, who also established the lower bound $(1/\sqrt{2} - o(1))\sqrt{N}$; later Erdős and Chowla applied a theorem of Singer to improve the lower bound to that in (7). See [1, 21] for further discussion and references. It remains open—and was given a price tag by Erdős—to decide whether, for every $\varepsilon > 0$, the inequality $S(N) \le \sqrt{N} + o(N^\varepsilon)$ holds; see [9] for related material.

Recall (Corollary 4.2) that the set of edge labels of an injective trivial-TSP labelling takes the form $\{(\nu_i + \nu_j)/2 \mid i \ne j\}$ for some constant-parity, well-spread, integer sequence $(\nu_i)_{i=1}^n$. For Sidon sequences (x_i) with $x_i \in [N]$, similar "sum-sets" $\{x_i + x_j \mid i \le j\}$ have been studied considerably; see [17] for recent results and further references. ∎

Notice that W_{cp} is surjective and nondecreasing, while σ_{cp} is increasing; thus σ_{cp}^{-1} : $\text{range}(\sigma_{cp}) \to \mathbb{Z}^+$ exists, as does the following approximate inverse for W_{cp}:

$$W_{cp}^-(n) := \min\{N : W_{cp}(N) = n\}, \quad \text{for } n \in \mathbb{Z}^+.$$

Then W_{cp}^- is a right inverse for W_{cp}, but when composed on the left yields the weaker

$$W_{cp}^- \circ W_{cp}(N) \le N.$$

Since every Sidon sequence is well-spread, we have

$$W(N) \ge S(N) \text{ for each } N \in \mathbb{N}. \tag{8}$$

Of the myriad connections between the seven functions just defined, we shall need only a few more, enumerated as Lemmas 5.1–5.5.

Lemma 5.1 *Every $n \in \mathbb{Z}^+$ satisfies $\psi_{cp}(n) \le 2W_{cp}^-(n)$.*

Proof. Since $\psi_{cp}(1) = W_{cp}^-(1) = 0$, the assertion holds for $n = 1$. For $n \geq 2$, let $N = W_{cp}^-(n)$. Since $W_{cp}(N) = n$, we can choose a constant-parity well-spread sequence $0 \leq x_1 < \cdots < x_n \leq N$. By definition, ψ_{cp} satisfies

$$\psi_{cp}(n) \leq x_{n-1} + x_n \leq 2N - 2 = 2W_{cp}^-(n) - 2. \qquad \blacksquare$$

Lemma 5.2 *Each $N \in range(\sigma_{cp})$ satisfies $W_{cp}(N) \geq \sigma_{cp}^{-1}(N)$.*

Proof. Let $n = \sigma_{cp}^{-1}(N)$. Since $\sigma_{cp}(n) = N$, there exists a constant-parity well-spread sequence $0 \leq x_1 < \cdots < x_n = N$. Hence, $W_{cp}(N) \geq n = \sigma_{cp}^{-1}(N)$. ■

Lemma 5.3 *For every $N \in \mathbb{N}$, if $k = W_{cp}(N)$, then $\binom{k}{2} \leq N$.*

Proof. If $0 \leq x_1 < \cdots < x_k \leq N$ is a constant-parity well-spread sequence, then the $\binom{k}{2}$ sums $x_i + x_j$, $i < j$, are distinct and belong to the set $\{0, 2, \ldots, 2(N-1)\}$. ■

Lemma 5.4 *Each $n \geq 2$ satisfies $\psi_{cp}(n) \geq \sigma_{cp}(n) + \sigma_{cp}(n-1)$.*

Proof. Choose a constant-parity well-spread sequence $x_1 < \cdots < x_n$ so that $\psi_{cp}(n) = x_{n-1} + x_n$. Since $\sigma_{cp}(n) \leq x_n$ and $\sigma_{cp}(n-1) \leq x_{n-1}$, the assertion follows. ■

Lemma 5.5 *Every $N \in \mathbb{N}$ satisfies $W_{cp}(N) \geq W(\lfloor N/2 \rfloor)$.*

Proof. If $n = W(N)$ and $0 \leq x_1 < \cdots < x_n \leq N$ is well-spread, then $y_i := 2x_i$ defines a constant-parity well-spread sequence of length n contained in $\{0, 1, \ldots, 2N\}$. Thus, $W_{cp}(2N+1) \geq W_{cp}(2N) \geq n = W(N)$. ■

Theorem 5.6 $\Psi(n) \in \Theta(n^2)$*; in particular, we have*

$$\Psi(n) \geq \frac{(n-1)^2}{2} \qquad \text{for } n \geq 2, \tag{9}$$

and

$$\Psi(n) \leq 2n^2\Big(1 + o(1)\Big) \qquad \text{as } n \to \infty. \tag{10}$$

Proof. For the lower bound, let $n \in \mathbb{N}$, $N = \sigma_{cp}(n)$, and $k = W_{cp}(N)$. Lemma 5.2 shows that $n = \sigma_{cp}^{-1}(N) \leq k$, while Lemma 5.3 gives $\binom{k}{2} \leq N$, so that

$$\sigma_{cp}(n) \geq \frac{n(n-1)}{2}.$$

If $n \geq 2$, then Lemma 5.4 gives $\psi_{cp}(n) \geq \sigma_{cp}(n) + \sigma_{cp}(n-1) \geq (n-1)^2$. Now Corollary 4.2 shows that

$$\Psi(n) = \frac{\psi_{cp}(n)}{2}, \tag{11}$$

yielding (9).

For the upper bound, given a (large) $n \in \mathbb{N}$, let $N = W_{cp}^{-}(n)$. Lemma 5.5, (8) and (7) give

$$n = W_{cp}(N) \geq W(\lfloor N/2 \rfloor) \geq S(\lfloor N/2 \rfloor) \geq \lfloor N/2 \rfloor^{1/2}\Big(1 - o(1)\Big),$$

whence $N \leq 2n^2(1 + o(1))$ as $n \to \infty$. Now Lemma 5.1 shows that

$$\psi_{cp}(n) \leq 2W_{cp}^{-}(n) = 2N \leq 4n^2\Big(1 + o(1)\Big),$$

and (11) gives (10). ∎

With the upper and lower bounds on $\Psi(n)$ differing only by a factor of four, Theorem 5.6 goes a long way in determining the growth-rate of $\Psi(n)$. In the spirit of (7), we close with

Conjecture 5.7 *The function $\Psi(n) = \min_{\lambda} \max_{A \in E(K_n)} \lambda(A)$, the minimum being taken over all injective trivial-MTSP edge-labellings λ of K_n, satisfies*

$$\Psi(n) \sim 2n^2 \quad \textit{as } n \to \infty.$$

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