LIGHT SUBGRAPHS IN PLANAR GRAPHS OF MINIMUM DEGREE 4 AND EDGE-DEGREE 9

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Abstract. Let $\mathcal{G}$ be the class of simple planar graphs of minimum degree ≥ 4 in which no two vertices of degree 4 are adjacent. A graph H is light in $\mathcal{G}$ if there is a constant w such that every graph in $\mathcal{G}$ which has a subgraph isomorphic to H also has a subgraph isomorphic to H whose sum of degrees in G is $\leq w$. Then we also write $w(H) \leq w$. It is proved that the cycle C_s is light if and only if $3 \leq s \leq 6$, where $w(C_3) = 21$ and $w(C_4) \leq 35$. The 4-cycle with one diagonal is not light in $\mathcal{G}$, but it is light in the subclass of all triangulations. The star $K_{1,s}$ is light if and only if $s \leq 4$. In particular, $w(K_{1,3}) = 23$. The paths P_s ($s \geq 1$) are light, and $w(P_3) = 17$ and $w(P_4) = 23$.

1. Introduction

The *weight* of a subgraph H of a graph G is the sum of the valences (in G) of its vertices. Let $\mathcal{G}$ be a class of graphs and let H be a connected graph such that infinitely many members of $\mathcal{G}$ contain a subgraph isomorphic to H. Then we define $w(H, \mathcal{G})$ to be the smallest integer w such that each graph $G \in \mathcal{G}$ which contains a subgraph isomorphic to H has a subgraph isomorphic to H of weight at most w. If $w(H, \mathcal{G})$ exists then H is called *light* in $\mathcal{G}$, otherwise H is *heavy* in $\mathcal{G}$. For brevity, we write $w(H)$ if $\mathcal{G}$ is known from the context.

Fabrici and Jendrol' [6] showed that all paths are light in the class of all 3-connected planar graphs. They further showed that no other connected graphs are light in the class of all 3-connected planar graphs. Fabrici, Hexel, Jendrol' and Walther [7] proved that the situation remains unchanged if the minimum degree is raised to four, i.e. in this class of graphs only the paths are light. Mohar [15] showed that the same is true for 4-connected planar graphs.

Borodin [3] proved that the 3-cycle C_3 is light in the class of plane triangulations without vertices of degree 4. Moreover, C_3 is light in the class of all plane triangulations containing no path of k degree 4

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vertices. But for arbitrary 3-connected plane graphs without vertices of degree 4, the triangle is not light. This is shown by the pyramids. So, we shall suppress vertices of degree 3 and consider the class of (simple) planar graphs of minimum degree ≥ 4 in which no two vertices of degree 4 are adjacent. More generally, the latter condition can be relaxed by requiring that there are no k-paths ($k \geq 1$) consisting of degree-4 vertices. One of our side remarks is that 3-connectivity is not needed at all.

Lebesgue [12] showed that every 3-connected plane graph of minimum degree at least four contains a 3-face with one of the following valency triples: $\langle 4,4,j \rangle$, $j \in [4,+\infty)$; $\langle 4,5,j \rangle$, $j \in [5,19]$; $\langle 4,6,j \rangle$, $j \in [6,11]$; $\langle 4,7,j \rangle$, $j \in [7,9]$; $\langle 5,5,j \rangle$, $j \in [5,9]$; and $\langle 5,6,j \rangle$, $j \in [6,7]$. This implies that C_3 is light with $w(C_3) \leq 28$ if there are no adjacent 4-vertices. We show, in particular, that $w(C_3) = 21$.

1. Let us consider plane graphs of minimum degree 5. In this class $w(C_3) = 17$ by Borodin [2]. More is known for triangulations: C_4 and C_5 are light by Jendrol' and Madaras [10] and $w(C_4) = 25$ and $w(C_5) = 30$ by Borodin and Woodall [4]; $C_6, \ldots, C_{10}$ are light by Jendrol' et al. [11] and Madaras and Soták [14]. The cycles C_s ($s \geq 11$), are not light [11]. By our results, the cycles C_4, C_5, C_6 are also light for arbitrary plane graphs of minimum degree 5. For the cycle lengths $7, \ldots, 10$ the problem remains to be open.

2. In our paper we mainly consider plane graphs of minimum degree ≥ 4 which contain no adjacent 4-vertices. It is shown that the cycles C_s ($3 \leq s \leq 6$) are light, and for the 3-cycle the precise weight is $w(C_3) = 21$. The cycles C_s, $s \geq 7$, are not light (not even in triangulations). This is shown by the graph obtained from $K_{2,n}$ by replacing each face of $K_{2,n}$ by the graph shown in Fig. 1(a) such that the top and the bottom vertex are identified with vertices of degree n in $K_{2,n}$.

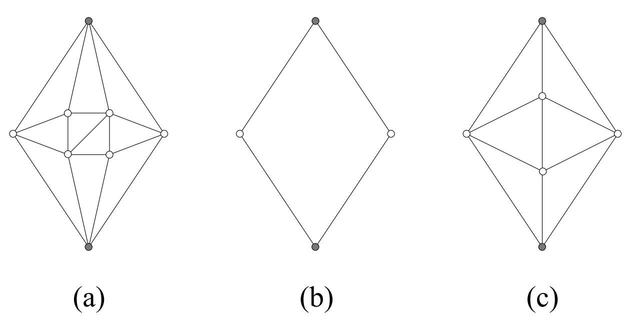

FIGURE 1. Long cycles are not light

3. Consider plane graphs of minimum degree ≥ 4 that contain no path with k degree-4 vertices ($k \geq 3$ is fixed). For triangulations, the lightness of C_3 was proved by Borodin [3] with $w(C_3) \leq \max(29, 5k+8)$. For arbitrary graphs in this class, C_3 is also light. One can prove that there is always a 3-cycle with maximum vertex degree $\leq 12k$. The cycle C_4 is light for $k \in \{3, 4\}$; there is always a C_4 with maximum degree ≤ 48. For $k \geq 23$, the 4-cycle is not light. This is shown by the graph obtained from the $K_{2,n}$ by inserting the line graph of the dodecahedron into each 4-face F and adding some new edges as shown in Fig. 1(b). This graph does not contain a path with 23 degree-4 vertices. For $5 \leq k \leq 22$, the lightness of C_4 is an open problem.

The cycle C_s is not light for any $s \geq 5$ and $k \geq 3$. This is shown by the graph obtained from $K_{2,n}$ by inserting two adjacent 4-vertices into each face as shown in Fig. 1(c).

4. As mentioned above, the only light subgraphs in the class of all 4-connected plane graphs are the paths [15]. Hence, we consider again the plane graphs of minimum degree ≥ 4 containing no path with k degree-4 vertices. If $k = 1$ the graphs have minimum degree ≥ 5. In this class the star $K_{1,s}$ is light if and only if $s \leq 4$ by Jendrol' and Madaras [10] (they also require 3-connectivity of graphs). Moreover, $w(K_{1,3}) = 23$ by [10] and $w(K_{1,4}) = 30$ by Borodin and Woodall [4]. For $k = 2$ we shall prove that $w(K_{1,3}) = 23$ and that $K_{1,4}$ is light. The star $K_{1,3}$ is light for any $k \geq 3$; we prove that there is always a $K_{1,3}$ with maximum degree $\leq 12k$. For $s \geq 4$ the star $K_{1,s}$ is not light: Consider the graph obtained from $K_{2,2n}$ by replacing every second face by the graph shown in Fig. 1(c) and by adding a diagonal into each other face of $K_{2,2n}$.

5. As mentioned above, the s-path P_s is light in the class of all 3-connected planar graphs. Little is known about the precise weight of P_s. Only for small values of s the exact weight of P_s has been determined: $w(P_1) = 5$, $w(P_2) = 13$ by Kotzig [13], and $w(P_3) = 21$ by Ando, Iwasaki, and Kaneko [1]. proved $w(P_1) = 5$, $w(P_2) = 11$, and $w(P_3) = 19$, respectively. For triangulations, $w(P_4) = 23$ by Jendrol' and Madaras [10]. In the class of all 3-connected plane graphs of minimum degree ≥ 5 Wernicke [16] and Franklin [8] proved that $w(P_2) = 11$ and $w(P_3) = 17$, respectively. Here we investigate the class of all plane graphs of minimum degree ≥ 4 containing no two adjacent 4-vertices. For P_3, the weight is $w(P_3) = 17$; and again $w(P_1) = 5$, and $w(P_2) = 11$. Our results are summarized in the following theorem. Observe that we do not require 3-connectivity of graphs (while for non-lightness we may require 3-connectivity).

Theorem 1.1. *Let $\mathcal{G}$ be the class of simple planar graphs of minimum degree ≥ 4 having no adjacent 4-vertices.*

(i) *The cycle C_s is light if and only if $3 \leq s \leq 6$, where $w(C_3) = 21$, and $w(C_4) \leq 35$.*
(ii) *The graph K_4^- (K_4 minus an edge) is not light in the whole class; but it is light in the subclass of all triangulations.*
(iii) *The star $K_{1,s}$ is light if and only if $s \leq 4$. In particular, $w(K_{1,3}) = 23$.*
(iv) *The paths P_s ($s \geq 1$) are light, and $w(P_3) = 17$ and $w(P_4) = 23$.*

For each of the graphs H whose lightness is proved in Theorem 1.1 (i.e., $C_3, C_4, C_5, C_6, K_{1,3}, K_{1,4}, P_3$, and P_4), except for the long paths, we actually prove that every $G \in \mathcal{G}$ contains a subgraph isomorphic to H.

In the proof of Theorem 1.1, we will use the *discharging method* which works as follows. Let G be a plane graph. Denote by $F(G)$ the set of faces of G. Let $d(v)$ denote the degree of the vertex $v \in V(G)$, and let $r(f)$ denote the size of the face $f \in F(G)$. Now, assign the *charge* $c : V(G) \cup F(G) \to \mathbb{R}$ to the vertices and faces of G as follows. For $v \in V(G)$, let $c(v) = d(v) - 6$ and for $f \in F(G)$, let $c(f) = 2\,r(f) - 6$. We can rewrite the Euler formula in the following form:

$$\sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} (2\,r(f) - 6) = -12. \tag{1}$$

This shows that the total charge of vertices and faces of G is negative. Next, we redistribute the charge of vertices and faces by applying some *rules* so that the total charge remains the same. The charge of $x \in V(G) \cup F(G)$ after applying the rules, will be denoted by $c^*(x)$. It will also be called the *final charge* of x. In each claim, after applying the rules, we will prove that each face and vertex of G has non-negative final charge if G does not have a light copy of the considered subgraph. This will contradict (1) and complete the proof.

In order to make proofs easier, we shall allow multiple edges and loops (where each loop counts 2 in the degree of its endvertex). However, some restrictions will be imposed. Let α and β be fixed integers (depending on the considered case). The class denoted by $\mathcal{G}(\alpha, \beta)$ consists of all plane (multi)graphs satisfying the following conditions:

(i) There are no faces of size ≤ 2.
(ii) No multiple edge is incident with a vertex of degree 4, and if it is incident with a vertex of degree 5, then the other endvertex has degree $\geq \alpha$.
(iii) The endvertices of loops are of degree $\geq \beta$.

Considering the class $\mathcal{G}(\alpha, \beta)$ enables us to prove, in each specific case, that vertices of "large" degree (usually $\geq \alpha - 1$) in extreme counterexamples are incident only with triangular faces. Roughly speaking, these conditions enable us to dismiss the 3-connectivity assumption used in some related works (e.g., [1, 3, 6, 7]).

We will prove in the sequel that $w(C_3) \leq 21$, $w(K_{1,3}) \leq 23$, $w(P_3) \leq 17$, and $w(P_4) \leq 21$. Equalities in all these cases are shown by the following examples. It is well known that there exists a 5-connected triangulation G_d of the plane which contains precisely 12 vertices of degree 5, all other vertices are of degree 6, and any two vertices of degree 5 are at distance at least d (cf., e.g., [9, 5]). These examples show that $w(P_s) \geq 6s - 1$ $(s \geq 1)$ and $w(K_{1,3}) \geq 23$. By taking the barycentric subdivision of G_d and removing all edges joining vertices of degree 5 in G_d with the vertices corresponding to their incident faces, we get an example which shows that $w(C_3) \geq 21$.

In what follows, we will use the following definitions. A vertex of degree k is said to be a *k-vertex* and a face of size k is a *k-face*. Denote by $m_k(v)$ the number of k-vertices adjacent to the vertex v, and let $r_k(f)$ be the number of k-vertices incident with the face f. (In these definitions, multiple adjacency is considered.) Let u, w be consecutive neighbors of the vertex v in the clockwise orientation around v. Then we say that u is a *predecessor* of w and w is a *successor* of u with respect to v. We say that two vertices v and u incident with a face f are *f-adjacent*, if it is not possible to add an edge uv in f without obtaining a face of size ≤ 2.

2. The lightness of P_3

Theorem 2.1. *$w(P_3) \leq 17$.*

Proof. In this proof, we work with the class $\mathcal{G}(9, 8)$. Suppose that the claim is false and $G \in \mathcal{G}(9, 8)$ is a counterexample on $|V(G)|$ vertices with $|E(G)|$ as large as possible. Suppose that $f = x_1 \cdots x_k x_1$ $(k \geq 4)$ is a face of G of size at least 4. Without loss of generality we may assume that $d(x_1) \geq 5$. We claim that $d(x_1) + d(x_3) \leq 11$. Otherwise, let G' be the graph obtained from G by adding the edge x_1x_3 in f. It is easy to see that $G' \in \mathcal{G}(9, 8)$. If P is a 3-path in G' of weight at most 17, then it must contain the new edge. But the sum of degrees of x_1 and x_3 in G' is at least 14, so P does not exist. Therefore, G' contradicts the maximality of $|E(G)|$. Similarly, $d(x_2) + d(x_4) \leq 11$, etc. This implies that $d(x_i) \leq 7$ for $i = 1, \ldots, k$. Consequently, $x_i \neq x_{i+1}$. Also, $x_2 \neq x_4$. (Otherwise we would have multiple edges joining x_2 and x_3.

Moreover, $d(x_2) \le 5$ and $d(x_3) \le 7$ which contradicts property (ii) of $\mathcal{G}(9,8)$.) Thus, $P = x_2x_3x_4$ is a path with $w(P) \le 11 + 11 - 5 = 17$, a contradiction. This proves that G is a triangulation.

Discharging Rule R. Suppose that v is a vertex with $d(v) \ge 7$ and that u is a 4- or 5-vertex adjacent to v. If $d(v) = 7$ then v sends 1 to u. If $d(v) > 7$, then v sends $\frac{2}{3}$ (if $d(u) = 4$) or $\frac{2}{5}$ (if $d(u) = 5$) to u.

We claim that after applying Rule R, $c^*(x) \ge 0$ for every $x \in V(G) \cup F(G)$. This is clear for $x \in F(G)$. Suppose now that v is a vertex of G. Let $d = d(v)$. We consider the following cases.

$d = 4$: Vertex v has at least three neighbors of degree ≥ 7. Thus, $c^*(v) \ge -2 + 3 \cdot \frac{2}{3} = 0$.

$d = 5$: Obviously, v has at least 4 neighbors of degree ≥ 7. From these neighbors, v receives at least $\frac{8}{5}$. So, it has positive final charge.

$d = 6$: In this case, v neither sends nor receives any charge. So, $c^*(v) = c(v) = 0$.

$d = 7$: At most one neighbor of v is of degree ≤ 5. Hence, $c^*(v) \ge 1 - 1 = 0$.

$d = 8$: If v is incident with a 4-vertex, then it is not incident with a 5-vertex. In this case, $c^*(v) \ge 2 - \frac{2}{3} > 0$. If v is not incident with a 4-vertex, then it is incident with at most five 5-vertices. Then $c^*(v) \ge 2 - 5 \cdot \frac{2}{5} = 0$.

$d = 9$: Note that v has at most one neighbor of degree 4, at most six neighbors of degree ≤ 5, and at least three neighbors of degree ≥ 6. Hence, $c^*(v) \ge 3 - \frac{2}{3} - 5 \cdot \frac{2}{5} > 0$.

$d \ge 10$: Observe that v has at most $\lfloor \frac{2d}{3} \rfloor$ neighbors of degree ≤ 5. So, $c^*(v) \ge d - 6 - \lfloor \frac{2d}{3} \rfloor \frac{2}{3} \ge 0$. This completes the proof. □

3. The lightness of P_4

Theorem 3.1. *$w(P_4) \le 23$.*

Proof. We shall prove the theorem for the class $\mathcal{G}(9,9)$. Suppose that the claim is false and G is a counterexample on $|V(G)|$ vertices with $|E(G)|$ maximum.

(1) *Let f be a face with $r(f) \ge 4$ and let x and y be vertices on f, which are not f-adjacent. Then $d(x) + d(y) \le 13$. Moreover, if $d(x) = 4$ or $d(y) = 4$, then $d(x) + d(y) \le 12$. In particular, every vertex v with $d(v) \ge 9$ is incident only with 3-faces.*

Suppose that (1) is false. Then we can add the edge xy in f. Denote by G' the resulting graph. Observe that $G' \in \mathcal{G}(9,9)$ and that every

4-path in G' which contains the new edge xy has weight at least 24. Thus, G' contradicts the maximality of $|E(G)|$. This proves (1).

(2) *No face of size ≥ 5 is incident with a 4-vertex.*

Suppose that (2) is false and let $f = x_1x_2\cdots x_kx_1$ be a face such that x_3 is a 4-vertex and $k \geq 5$. Then, $x_2 \neq x_4$, $x_1 \neq x_3$, and $x_3 \neq x_5$. By (1), $d(x_2)+d(x_4) \leq 13$ and $d(x_i) \leq 8$ for every $i \in \{1,\ldots,k\}$. Because of planarity, either $x_1 \neq x_4$ or $x_2 \neq x_5$. We may assume that $x_1 \neq x_4$. Since $d(x_1) \leq 8$ and $G \in \mathcal{G}(9,9)$, $x_1 \neq x_2$. Hence, $P = x_1x_2x_3x_4$ is a 4-path. By (1), $d(x_1) + d(x_4) \leq 13$. Since P is not light, $d(x_2) \geq 7$. Consequently, $x_2 \neq x_5$ (otherwise, we could add the loop joining x_2 and x_5 in f, contradicting maximality of $|E(G)|$). Therefore, we may apply the same arguments to the path $x_5x_4x_3x_2$ to conclude that $d(x_4) \geq 7$. Then $d(x_2) + d(x_4) \geq 14$, a contradiction.

(3) *Let f be a 4-face incident with a 4-vertex x. Then, every vertex of f distinct from x is of degree ≥ 6.*

For, suppose that (3) is false. Let $f = x_1x_2x_3x_4x_1$, where $x = x_1$. Since $G \in \mathcal{G}(9,9)$ and since $d(x_2) + d(x_4) \leq 13$ by (1), it is easy to see that all vertices on f are distinct. If $d(x_3) = 4$ or 5, then x_2 and x_4 contradict (1) (or the path $P = x_1x_2x_3x_4$ is light). So, we may assume that $d(x_2) = 5$. Since P is not light, it follows by (1) that the degrees of x_3 and x_4 are 7 or 8 but not both equal to 7. Let $G' = G + x_2x_4$. It is easy to see that if G' has a light P_4, then G also has a light P_4. This contradicts the maximality of G.

(4) *Let v be a 7- or 8-vertex. Then $m_4(v) + m_5(v) \leq \lfloor \frac{d(v)}{2} \rfloor$.*

Suppose that (4) is false. Let $x_1, x_2, \ldots, x_{d(v)}$ be the neighbors of v in the clockwise order around v. We may assume that $d(x_1) \leq 5$ and $d(x_2) \leq 5$. Vertices x_1 and x_2 are not adjacent. (Otherwise we would obtain a light P_4 in G.) Denote by f the face incident with the walk x_1vx_2. Thus, $r(f) \geq 4$. Let x be a neighbor of x_1 on f different from v. (We can always choose x. Otherwise, x_1 is a 4- or 5-vertex which has two common edges with the 7- or 8-vertex v, a contradiction.) Since x_1 is not adjacent to x_2, $x \neq x_2$. Since the path xx_1vx_2 is not light, $d(v) + d(x) \geq 14$, a contradiction to (1).

Let us now introduce the discharging rules.

Rule R1. If f is a face with $r(f) \geq 5$ and u is a 5-vertex incident with f, then f sends 1 to u.

Rule R2. Suppose that f is a 4-face and u is a 4- or 5-vertex incident with f.

(a) If $d(u) = 5$, then f sends $2/r_5(f)$ to u.

(b) If $d(u) = 4$, then f sends 2 to u.

Rule R3. Suppose that v is a 7- or 8-vertex. Then v sends $\frac{1}{3}$ to every adjacent 5-vertex. The remaining charge is then equally redistributed between the adjacent 4-vertices.

Rule R4. Suppose that v is a vertex with $d(v) \geq 9$ and suppose that u is a 4- or 5-vertex adjacent to v. Let $\alpha = c(v)/d(v)$. Let u^- and u^+ be the predecessor and the successor of u with respect to v. Also, let u^{--} be the predecessor of u^- and u^{++} be the successor of u^+ with respect to v.

(a) Suppose that $d(u) = 4$. Vertex v sends 1 to u if one of the following conditions is satisfied:
(1) u^-, u^+, u^{++} are of degree ≥ 6;
(2) u^{--}, u^-, u^+ are of degree ≥ 6;
(3) u^+ is a 5-vertex and u^{--}, u^-, u^{++} are of degree ≥ 6;
(4) u^- is a 5-vertex and u^{--}, u^+, u^{++} are of degree ≥ 6.
Otherwise v sends 2α to u.

(b) Suppose that $d(u) = 5$ and u^-, u^+ are not both 4-vertices. If u^-, u^+, u^{++} are of degree ≥ 6 or u^{--}, u^-, u^+ are of degree ≥ 6 then v sends 2α to u. Otherwise v sends α to u.

Rule R5. Suppose that u is 5-vertex adjacent to a 4-vertex v. Suppose also that other three neighbors of v are of degree at least 11. Then, v sends $\frac{1}{11}$ to u.

Rule R6. Suppose that v is a 5-vertex adjacent to a 4-vertex u and $m_4(v) = 1$. Suppose that v has at most one neighbor x distinct from u which has degree ≤ 8, and if x exists, then u and x are adjacent and $d(x) = 6$. Then v sends $\frac{1}{3}$ to u. Otherwise, if v has at most two neighbors of degree ≤ 9, then it sends $\frac{1}{5}$ to u.

We claim that after applying Rules R1–R6, $c^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$. Suppose first that f is a face of G. If $r(f) = 3$ then it neither receives nor sends any charge. So, $c^*(f) = c(f) = 0$. If $r(f) = 4$ then by Rule R2 and (3), $c^*(f) = 0$ or 2. Finally, assume that $r(f) \geq 5$. By (2), no 4-vertex is incident with f. Observe also that there are no four consecutive 5-vertices $v_1v_2v_3v_4$ on the boundary of f. If not, $v_1v_2v_3v_4$ would be a light path or we would have $v_1 = v_4$. In the latter case, let w be a vertex on f adjacent to v_1 and different from v_2 and v_3. Note that $d(w) \geq 9$ (otherwise $wv_1v_2v_3$ is a light path). But then w contradicts (1). Hence, $c^*(f) \geq 2r(f) - 6 - \lfloor \frac{3r(f)}{4} \rfloor > 0$.

Suppose now that v is a vertex of G with $c^*(v) < 0$. Let $d = d(v)$. Enumerate the neighbors of v by $x_1, x_2, \ldots, x_d$ in the clockwise order around v. Consider the following cases.

$d = 4$: Suppose first that v is incident with a face f of size at least 4. By (2), $r(f) = 4$. Observe that Rule R5 does not apply to v. (Otherwise, v has three neighbors of degree ≥ 11, which implies that all faces incident with v are triangles.) By Rule R2, f sends 2 to v, so $c^*(v) \geq 0$. Now, we may assume that all faces incident with v are triangles. Consider the following subcases.

$m_7(v) \geq 2$: Let x_i and x_j be distinct neighbors of v of degree 7. Then v is the only neighbor of x_i of degree ≤ 5. Otherwise, G contains a light P_4. Similarly for x_j. By R3, each of x_i and x_j sends 1 to v. Thus $c^*(v) \geq 0$.

$m_7(v) = 1$: Let x_1 be a 7-vertex. Suppose that one of x_2, x_3, x_4 is of degree ≤ 6. Then, the other two are of degree ≥ 7. If $d(x_2) = 5$ then v and x_2 are the only neighbors of x_1 of degree ≤ 5. Hence x_1 sends $\frac{2}{3}$ to v. If $d(x_3) \geq 9$ then v receives at least $\frac{2}{3}$ from x_3. Suppose now that $d(x_3) = 8$. If x_3 has a neighbor of degree ≤ 5 distinct from v and x_2, then we have a light P_4. Hence, x_3 sends $\frac{5}{3}$ to v by R3. Same arguments apply at x_4, and so $c^*(v) \geq 0$. Similar arguments work if $d(x_3) = 5$ or $d(x_4) = 5$. Hence we may assume that the neighbor of v of degree ≤ 6 has degree equal to 6. By (4) and Rules R3 and R4, each of the two neighbors of v of degree ≥ 8 sends at least $\frac{1}{2}$ to v. Note that v is the only neighbor of x_1 of degree ≤ 5 (otherwise we obtain a light P_4). So, x_1 sends 1 to v by R3. Thus, $c^*(v) \geq -2 + 1 + 2 \cdot \frac{1}{2} = 0$, a contradiction. Suppose now that x_2, x_3, x_4 are all of degree ≥ 8. If some of these three vertices is an 8-vertex, then it has no neighbor of degree 4 different from v (otherwise, G has a light P_4). So, by Rule R3 (and (4)), each neighbor of v of degree 8 sends at least 1 to v. And, if some of x_2, x_3, x_4 is of degree ≥ 9, then it sends at least $\frac{2}{3}$ to v. Thus, $c^*(v) \geq -2 + \frac{1}{3} + 3 \cdot \frac{2}{3} > 0$, a contradiction.

$m_7(v) = 0$: We may assume that at least one of x_1, x_2, x_3, x_4 is of degree ≤ 6. Otherwise, each of them sends at least $1/2$ to v and hence $c^*(v) \geq 0$. If three neighbors of v have degree ≤ 6, the there is a light P_4. Suppose now that precisely one of them, say x_1, has degree ≤ 6. If x_i $(i \geq 2)$ is of degree 8, then only v and possibly x_1 are its neighbors of degree ≤ 5. Hence x_i sends to v at least $\frac{4}{3}$ by Rule R3. If $d(x_i) \geq 9$, then x_i sends $\geq \frac{2}{3}$ to v. If Rule R5 is applied at v, then $d(x_i) \geq 11$ $(i \geq 2)$ and hence x_i sends $\geq \frac{10}{11}$ to v by R4. This implies that $c^*(v) \geq 0$. Therefore, v has precisely two neighbors of degree ≤ 6. Say x_i and x_j are vertices of degree ≥ 8 and x_k and x_l are of degree ≤ 6. If at least one of x_k, x_l is of degree 5 and one of x_i, x_j is of degree 8, then we have a light P_4. If x_k, x_l are 6-vertices and x_i is an 8-vertex, then v is the only neighbor of x_i of degree ≤ 5. By Rule R3, x_i sends 2 to v, so $c^*(v) \geq 0$. Therefore, we may assume that x_i, x_j both have degree ≥ 9. By (1), all faces containing x_i and x_j are of size 3. Now we consider several possibilities. If $d(x_k) = d(x_l) = 6$, then each of x_i and x_j sends 1 to v by Rule R4(a) (otherwise, we obtain a light path which contains x_k, v, x_l, and a neighbor of x_i or x_j of degree ≤ 5.) Suppose that $d(x_k) = 5$ and $d(x_l) = 6$. If x_k and x_l are not consecutive neighbors of v, then each of x_i and x_j sends 1 to v by Rule R4(a). So, we may assume that $k = 1$, $l = 2$, $i = 3$, and $j = 4$. Then, by R4(a) and R6, x_i sends 1 to v, x_j sends $\geq \frac{2}{3}$ and x_k sends $\frac{1}{3}$ to v. Suppose now that $d(x_k) = d(x_l) = 5$. Then $d(x_i) \geq 10$ and $d(x_j) \geq 10$. By Rules R4 and R6, it follows that each of x_k and x_l sends $\geq \frac{1}{5}$ and each of x_i and x_j sends $\geq \frac{4}{5}$ to v. So, $c^*(v) \geq 0$.

$d = 5$: If v sends a charge to some 4-vertex by Rule R6, then each neighbor of v of degree ≥ 9 sends $\geq \frac{1}{3}$ to v and each neighbor of degree ≥ 10 sends $\geq \frac{2}{5}$ to v by Rule R4. If v sends $\frac{1}{5}$ by Rule R6, then $c^*(v) \geq -1 + 3 \cdot \frac{2}{5} - \frac{1}{5} \geq 0$. Suppose now that v sends $\frac{1}{3}$ by Rule R6. If the vertex x of R6 does not exist, then $c^*(v) \geq -1 + 4 \cdot \frac{1}{3} - \frac{1}{3} \geq 0$. So, x exists. Let x' be the neighbor of v which is the successor of x. Since there is not light P_4 in G, x' sends $\geq \frac{2}{3}$ to v by R4(b). Consequently, $c^*(v) \geq 0$. So, assume that Rule R6 does not apply to v.

Suppose first that v is incident with a face f of size at least 4. Because of Rule R1 we may assume that f is of size 4. Let vx_1wx_2 be the boundary of f. By (3), f is not incident with a 4-vertex. If f is incident with no more than two 5-vertices, then v receives at least 1 from f by R2. Note that $r_5(f) \neq 4$. So, we may assume that $r_5(f) = 3$. Hence, f sends $\frac{2}{3}$ to v. This implies that the other faces incident with v are triangles. Note that x_3 and x_5 are of degree ≥ 9 (otherwise, we obtain a light P_4 in G). Each of these vertices sends at least $\frac{1}{3}$ to v, by Rule R4. Thus, the final charge of v is positive, a contradiction.

Now, we may assume that all faces incident with v are triangles. Note that v has at least three neighbors of degree at least 7 (else there would be a light P_4). With one exception, each of these neighbors sends at least $\frac{1}{3}$ to v, so $c^*(v) \geq 0$. The exceptional case is when v is adjacent to two 4-vertices, say x_1 and x_3, and $d(x_2) \geq 9$. In this case, x_2 sends no charge to v. Since there is no light P_4 in G, x_2, x_4, and x_5 are all of degree at least 11. By R4, each of x_4 and x_5 sends $\geq \frac{5}{11}$ to v. The fourth neighbor of x_1 is of degree ≥ 11 (otherwise, we obtain a light P_4). Hence, by R5, v also receives $\frac{1}{11}$ from x_1. Similarly, v receives $\frac{1}{11}$ from x_3. Thus, $c^*(v) \geq -1 + 2 \cdot \frac{5}{11} + 2 \cdot \frac{1}{11} > 0$.

$6 \leq d \leq 8$: If $d = 6$ then v neither sends nor receives any charge, i.e. $c^*(v) = c(v) = 0$. By (3) and R3 it follows that for $d = 7$ and 8, v has nonnegative final charge.

$d \geq 9$: Let $\alpha = \frac{d-6}{d}$. We are interested in the minimal possible value of $c^*(v)$. Observe that if there are four consecutive neighbors $x_{i-2}, x_{i-1}, x_i, x_{i+1}$ of v whose degrees are ≥ 6, ≥ 6, 4, ≥ 6, then we may reset $d(x_{i-1}) = 5$. After this resetting the value of $c^*(v)$ is unchanged or decreased. We argue similarly, if the degrees of $x_{i-2}, x_{i-1}, x_i, x_{i+1}$ are ≥ 6, 4, ≥ 6, ≥ 6, respectively. If there are five consecutive vertices $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ whose degrees are $\geq 6, 5, 4, \geq 6, \geq 6$, then we may reset $d(x_i) = 5$. After this the value of $v^*(v)$ do not increase. We argue similarly, if the degrees of $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$ are $\geq 6, \geq 6, 4, 5, \geq 6$. Similarly, if $x_{i-2}, x_{i-1}, x_i, x_{i+1}$ are of degrees ≥ 6, ≥ 6, 5, ≥ 6 or ≥ 6, 5, ≥ 6,≥ 6, then set $d(x_{i-1}) = 5$ or $d(x_i) = 5$, respectively. If there are four consecutive neighbors of v which all have degree ≤ 5, then the first and the fourth are 5-vertices which coincide. Observe also that there are no five consecutive neighbors of v all of degree ≤ 5. (Otherwise, in both cases, we obtain a light P_4.)

Above observations imply the following. Denote by k_i ($1 \leq i \leq 4$) the number of maximal subwalks with $i - 1$ edges of the walk $x_1 x_2 \cdots x_d x_1$ after deleting the vertices of degree ≥ 6. Denote by k the total number of such maximal subwalks. (Since G may have parallel edges, it is possible that two different maximal subwalks have a common vertex.) Then, $k = k_1 + k_2 + k_3 + k_4$ and $k_1 + 2k_2 + 3k_3 + 4k_4 + k \leq d$. Finally, after applying Rule R4,

$$c^*(v) \geq d - 6 - 2\alpha k_1 - 3\alpha k_2 - 4\alpha k_3 - 5\alpha k_4 \geq d - 6 - \alpha d = 0.$$

□

4. The lightness of C_3

Theorem 4.1. *$w(C_3) \leq 21$.*

Proof. We shall prove the theorem for the class $\mathcal{G}(12, 13)$. Suppose that the claim is false and G is a counterexample on $|V(G)|$ vertices with $|E(G)|$ as large as possible.

We claim that every vertex $v \in V(G)$ with $d(v) \geq 11$ is incident only with 3-faces. Suppose not. Then there exists a face f of size ≥ 4 which is incident with v. Let w be a vertex on f which is not f-adjacent with

v. It is easy to see that the graph $G' = G + vw$ belongs to $\mathcal{G}(12, 13)$ and that every 3-cycle in G' has weight ≥ 22. Thus, G' contradicts the maximality of $|E(G)|$.

We have five discharging rules.

Rule R1. From each r-face f ($r \geq 4$) send $\frac{3}{4}$ to each incident 4-vertex. After that, send the remaining charge equally distributed to all incident vertices of degree ≥ 5. There is one exception to this rule. For further reference it is described as Rule R1' below.

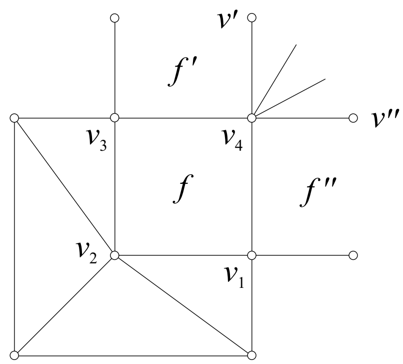

FIGURE 2. The Rule R1'

Rule R1'. Suppose that $f = v_1v_2v_3v_4$ is a 4-face where $d(v_1) = d(v_3) = 4$, and $d(v_2) = 5$. Suppose also that the other faces containing v_2 are of size 3. Let f' and f'' be the faces distinct from f containing the edges v_3v_4 and v_1v_4, respectively. Let $v' \neq v_3$ and $v'' \neq v_1$ be the neighbors of v_4 incident to f' and f'', respectively. (See Figure 2.) If f' is a 4-face and $d(v') = 4$, then we say that the pair (f, f') is *admissible*. Similarly, (f, f'') is admissible if f'' is a 4-face and $d(v'') = 4$. Now, send charge $3/4$ to each of v_1 and v_3. If both pairs (f, f') and (f, f'') are admissible, send $12/40$ to v_2 and $8/40$ to v_4. If only one of (f, f') and (f, f'') is admissible, send $11/40$ to v_2 and $9/40$ to v_4. Otherwise, apply R1 to f, i.e., send $10/40$ to each of v_2 and v_4.

It is easy to see that each ≥ 5-vertex incident with a ≥ 4-face f receives at least $\frac{5}{12}$ from f except when f is a 4-face incident with two vertices of degree 4. In that case, each $\geq$5-vertex on f receives at least $\frac{1}{5}$ from f. If f is not a 4-face exceptional in Rule R1 (so that R1' does not apply), then each $\geq$5-vertex on f receives at least $\frac{1}{4}$ from f.

In Rule R2 below we use the following function $\phi : \mathbb{Z} \to \mathbb{R}$. We let $\phi(d) = 0$ if $d \leq 6$ or $d \geq 12$, and we put $\phi(7) = \frac{1}{4}$, $\phi(8) = \phi(9) = \frac{1}{2}$, $\phi(10) = \frac{3}{4}$, $\phi(11) = \frac{4}{5}$. Then we have for any $d_1 \leq d_2 \leq 11$:

(O1) $\phi(d_1) + \phi(d_2) \geq 1$, if $d_1 + d_2 \geq 18$.
(O2) $\phi(d_1) + \phi(d_2) \geq \frac{4}{5}$, if $d_1 + d_2 \geq 17$.
(O3) $\phi(d_1) + \phi(d_2) \geq 1$, if $d_1 + d_2 \geq 17$ and $(d_1, d_2) \neq (6, 11)$.

Rule R2. If v is a vertex with $d(v) \leq 11$, then v sends charge $\phi(d(v))$ to each 4- or 5-vertex u adjacent to v such that the edge uv is contained in at least one 3-face.

Rule R3. If v is a d-vertex with $d \geq 12$, then v sends charge determined below to each neighbor u of degree 4.

(a) If u is incident with precisely two 3-faces, then v sends $1/2$ to u.
(b) If u is incident with precisely three 3-faces uvu_1, uvu_2, and uu_2u_3, and $d(u_2) \leq 11$, then v sends $1/4$ to u.
(c) If u is incident with precisely three 3-faces uvu_1, uvu_2, and uu_2u_3, and $d(u_2) \geq 12$, then v sends $5/8$ to u.
(d) If u is incident with four 3-faces, then v sends 1 to u.

Rule R4. If v is a d-vertex with $d \geq 12$, then v sends charge determined below to each neighbor u of degree 5.

(a) Suppose that u is incident with precisely two 3-faces, uvu_1 and uvu_2. If all neighbors of u distinct from v are of degree 4, send $7/20$ from v to u. If all neighbors of u except v and one of u_1, u_2 are of degree 4, send $6/20$. Otherwise, send $1/4$ from v to u.
(b) Suppose that u is incident with precisely three 3-faces uvu_1, uvu_2, and uu_2u_3, and $d(u_2) \geq 12$. If all neighbors of u except v and u_2 are of degree 4 and if the other two faces incident with u are 4-faces, then v sends $11/40$ to u; otherwise, v sends $1/4$ to u.
(c) Suppose that u is incident with precisely four 3-faces uvu_1, uvu_2, uu_2u_3, and either uu_1u_4 or uu_3u_4. Suppose also that $d(u_2) \geq 12$, and $d(u_1) \geq 12$ (if uu_1u_4 is a face), $d(u_3) \geq 12$ (if uu_3u_4 is a face). Then v sends $1/4$ to u.
(d) Suppose that u is incident with precisely four 3-faces uvu_1, uvu_2, uu_2u_3, and uu_3u_4, where $d(u_2) \leq 11$. Let f be the ≥ 4-face containing u. If $d(u_1) = d(u_4) = 4$ and Rule R1' applies in f where u corresponds to the vertex v_2 in R1' and u_1 corresponds to v_3 (if (f, f') is admissible) or to v_1 (if (f, f'') is admissible), then send $7/20$ from v to u. If $d(u_1) = d(u_4) = 4$, f is a 4-face and R1' does not apply in f as stated above, then send $3/8$ from v to u. If $d(u_1) \neq 4$ or $d(u_4) \neq 4$, or f is not a 4-face, send $7/24$ from v to u.
(e) If u is incident with five 3-faces and at least one of the two vertices sharing a 3-face with the edge uv has degree ≥ 11, then v sends $1/2$ to u.

After applying rules R1 and R1', all faces have charge 0 and this charge remains unchanged by other discharging rules. Rule R2 sends nonzero charge only from vertices of degrees $7, \ldots, 11$ to vertices of degree 4 or 5. Rules R3–R4 send charge from ≥ 12-vertices to 4- and 5-vertices, respectively. Along any edge, nonzero charge is sent by at most one of the rules (or their subcases).

Let $v \in V(G)$ and $d = d(v)$. Let d' denote the number of 3-faces containing v. Let $v_1, \ldots, v_d$ be the neighbors of v enumerated in the clockwise order around v, and let $d_i = d(v_i)$, $i = 1, \ldots, d$. We claim that $c^*(v) \geq 0$, and this is proved depending on the value of d.

$d = 4:$ If $c^*(v) < 0$, then $d' \geq 2$ because of Rules R1 and R1'. We distinguish the following four cases:

(a) $d' = 2$ and vv_1v_2 and vv_2v_3 are 3-faces.

(b) $d' = 2$ and vv_1v_2 and vv_3v_4 are 3-faces.

(c) $d' = 3$ and vv_1v_2, vv_2v_3, and vv_3v_4 are 3-faces.

(d) $d' = 4$.

The charge at v after applying R1 and R1' is equal to $-\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{5}{4}$, and -2, respectively. Suppose that v_iv_{i+1} $(1 \leq i \leq 3)$ is an edge of G. Since vv_iv_{i+1} is not light, $d_i + d_{i+1} \geq 18$. If $d_i \leq 11$ and $d_{i+1} \leq 11$, then (O1) implies that v_i and v_{i+1} together send charge ≥ 1 to v. (In such a case we say that (O1) *applies*.) Since the vertices on faces of size ≥ 4 all have degree ≤ 10, case (b) is settled. Similarly in case (a) if $d_2 \leq 11$. If $d_2 \geq 12$, then R3(a) applies. Similarly in case (c): either (O1) applies twice, (O1) once and R3(b) once, or R3(c) twice. Finally, in case (d), R3(d) or (O1) are applied at least twice. In each case we get nonnegative final charge at v.

$d = 5:$ If $c^*(v) < 0$, then $d' \geq 1$ by the remark after Rule R1'. If $d' = 1$, then (O2) applies (in addition to the charge received from four faces), so $c^*(v) > 0$. We are left with the following six cases:

(a) $d' = 2$ and vv_5v_1 and vv_1v_2 are 3-faces. If $d_1 \leq 11$, then (O2) applies to v_1v_2 and this easily implies that $c^*(v) \geq 0$. Otherwise, $d_1 \geq 12$ and the rule R4(a) is used on the edge v_1v. Let f_1, f_2, f_3 be the $\geq$ 4-faces incident with v (in this clockwise order). If Rule R1' is not used in any of them, then v receives charge $\geq 1/4$ from each of them and receives $1/4$ by R4(a) from v_1. So, $c^*(v) \geq 0$. If Rule R1' was applied at v, then v played the role of v_4 in R1', and faces f_1, f_2, f_3 have played the role(s) of f, f', f''. In particular, $d_3 = d_4 = 4$ and at least one of d_2, d_5 is also equal to 4. If $d_2 \neq 4$ or $d_5 \neq 4$, then at most two ordered pairs (f_i, f_j) were admissible in applications of R1', and the total charge received at v from f_1, f_2, f_3 is $\geq \frac{3}{4} - \frac{2}{40} = \frac{7}{10}$. In this case, the remaining $\frac{3}{10}$ come from v_1 by R4(a). If $d_2 = d_5 = 4$, then there were at most four admissible ordered pairs (f_i, f_j), and the total charge received at v from f_1, f_2, f_3 is $\geq \frac{3}{4} - \frac{4}{40} = \frac{13}{20}$. In this case, the remaining $\frac{7}{20}$ come from v_1 by R4(a), again.

(b) $d' = 2$ and vv_2v_3 and vv_4v_5 are 3-faces. In this case all neighbors of v have degree ≤ 10 and (O2) applies twice.

(c) $d' = 3$ and vv_2v_3, vv_3v_4, and vv_4v_5 are 3-faces. We are done if (O2) applies at least once. Otherwise, $d(v_3) \geq 12$ and $d(v_4) \geq 12$. Now, R4(b) applies at v_3 and at v_4. Denote by f and f' the two faces of size ≥ 4 incident with v. If v receives $\geq \frac{1}{4}$ from each of f and f', then $c^*(v) \geq 0$. Otherwise, R1' has been used in f and f', sending $\frac{9}{40}$ to v. In that case, $d_1 = d_2 = d_5 = 4$ and f, f' are 4-faces. By R4(b), v receives $\frac{22}{40}$ from v_3 and v_4 and $\frac{18}{40}$ from f and f', so $c^*(v) = 0$.

(d) $d' = 3$ and vv_5v_1, vv_1v_2, and vv_3v_4 are 3-faces. Similarly as in case (b), (O2) applies to v_3v_4.

(e) $d' = 4$ and vv_4v_5, vv_5v_1, vv_1v_2, and vv_2v_3 are 3-faces. Denote by f the $\geq$ 4-face incident with v. For each of the 3-faces vv_iv_{i+1}, we have $d_i + d_{i+1} \geq 17$ (indices modulo 5). By (O2) we may assume that either $d_i \geq 12$ or $d_{i+1} \geq 12$ in such a case. Since v_3 and v_4 are of degree ≤ 10, we have $d_2 \geq 12$ and $d_5 \geq 12$. If $d_1 \geq 12$, then v receives charge $1/4$ from each of v_1, v_2, v_5 by R4(c), and receives $\geq 1/4$ from f by R1 or R1'. If $d_1 \leq 11$, then R4(d) was used at v_2 and v_5. If v receives at least $5/12$ from f, then we are done. Otherwise, by the remark after Rule R1', $d_3 = d_4 = 4$ and f is a 4-face. If R1' applies in f by sending $\frac{12}{40}$ (respectively $\frac{11}{40}$) to v, then v receives from v_2 and v_5 total charge $\frac{7}{20} + \frac{7}{20} = \frac{28}{40}$ (respectively $\frac{7}{20} + \frac{3}{8} = \frac{29}{40}$), by R4(d), so $c^*(v) \geq 0$. Similarly, if R1' does not apply in f, then v_2 and v_5 each send $\frac{3}{8}$ to v by R4(d), so $c^*(v) = 0$.

(f) $d' = 5$. If two consecutive neighbors of v have degree ≥ 12, then R4(e) applies twice, and we are done. Otherwise, we may assume that $d_1 \leq d_2 \leq 11$. By observation (O3), we may assume that $d_1 = 6$ and $d_2 = 11$. Now, v_2 sends $4/5$ to v by R2. If R2 applies at some other vertex of degree between 7 and 11 then, clearly, $c^*(v) \geq 0$. Otherwise, since vv_5v_1 is not light, $d_5 \geq 12$. Consequently, $d_4 \leq 11$ (and hence $d_4 \leq 6$), and so $d_3 \geq 12$. Therefore, Rule 4(e) applies at v_3 as well.

$d = 6:$ Vertices of degree 6 retain their original charge 0.

$7 \leq d \leq 11:$ These vertices may lose charge only by R2. If vv_iv_{i+1} is a 3-face, then one of v_i, v_{i+1} has degree > 5, so R2 applies at most once. Otherwise, R2 may apply at v_i and v_{i+1}, but v receives $\geq \frac{1}{5}$ from the face containing these vertices. We may assume that each edge vv_i is contained in at least one 3-face. Using these facts, it is easy to see that the charge at v remains nonnegative. (The "worst case" for $d \geq 8$ is when R2 is used on vv_i and vv_{i+2} and the faces containing $v_{i-1}vv_i$

and $v_{i+2}vv_{i+3}$ (indices modulo d) are of size ≥ 4.) In the extreme case for $d = 10$, one also has to observe that charge $\geq 1/4$ is sent from the two faces of size ≥ 4. The details are left to the reader.

$d = 12$: By the claim at the beginning of the proof, v is incident with 3-faces only. Therefore, $d_i + d_{i+1} \geq 10$ for $i = 1, \ldots, 12$ (all indices modulo 12). In particular, if $d_i = 4$, then $d_{i+1} \geq 6$. If $d_i = 5$, then v sends $\leq 1/2$ to v_i. Since v sends charge only to vertices of degrees 4 or 5, this implies that v sends at most 1/2 on average, and hence its charge does not become negative.

$d \geq 13$: Denote by ϕ_i the charge sent from v to v_i, $i = 1, \ldots, d$. We claim that, for each i, there exists an integer t, $1 \leq t \leq d$, such that

$$\phi_i + \phi_{i+1} + \cdots + \phi_{i+t-1} \leq \begin{cases} \frac{t}{2} + \frac{t}{26}\,, & \text{if } d = 13 \\ \frac{t}{2} + \frac{t}{14}\,, & \text{if } d > 13 \end{cases} \tag{2}$$

where indices are taken modulo d. In fact, we shall need a strengthening of (2) when $d = 13$. We shall prove that one can get $t \leq 6$ and

$$\phi_i + \phi_{i+1} + \cdots + \phi_{i+t-1} \leq \frac{t}{2} + \frac{t - 0.6}{26} \tag{3}$$

unless $d_i = d_{i+2} = d_{i+4} = 4$, $d_{i+1} = d_{i+3} = 5$, and $\phi_i = \phi_{i+4} = 1$, $\phi_{i+2} = \frac{1}{2}$, $\phi_{i+1} + \phi_{i+3} \leq \frac{7}{10}$.

The claim is trivial if $d_i \neq 4$ (take $t = 1$). So we may assume that $i = 1$ and $d_1 = 4$. The claim is also obvious if R3(a) or R3(b) is used for ϕ_1 ($t = 1$) or if $d_2 \geq 6$ ($t = 2$). Hence we may assume that $d_2 = 5$. If $d_3 \geq 6$, then $t = 3$ will do. So, $d_3 \in \{4, 5\}$. In particular, R4(b), R4(c), R4(e) were not used for ϕ_2. Hence $\phi_2 \leq \frac{3}{8}$, so we may also assume that R3(d) was used for ϕ_1 (otherwise $t = 2$ would do). In particular, v_1 is contained in four 3-faces. This implies that R4(d) was used for ϕ_2. In particular, the edge v_2v_3 is contained in an r-face f, where $r \geq 4$, so neither R3(d) nor R4(e) was used for ϕ_3.

Suppose now that $d_3 = 5$. Then $\phi_2 = 7/24$. If R4(a) was used for ϕ_3, then $\phi_3 \leq \frac{6}{20}$ since $d_2 = 5$. Then $\phi_1 + \phi_2 + \phi_3 \leq 1 + \frac{7}{24} + \frac{6}{20} < \frac{3}{2} + \frac{2.4}{26}$ yields (3). Same holds if $\phi_3 \leq \frac{11}{40}$. Hence, we may assume that R4(d) was used for ϕ_3. Since $d_2 \neq 4$, $\phi_3 = 7/24$. So, $\phi_1 + \phi_2 + \phi_3 = 3/2 + 1/12$, and $t = 3$ will do.

It remains to consider the case when $d_3 = 4$. If R3(c) is used for ϕ_3, then $d_4 \geq 12$. Hence, $\phi_4 = 0$, and $\phi_1 + \phi_2 + \phi_3 + \phi_4 \leq 1 + 3/8 + 5/8 = 2$, so $t = 4$ will do. If R3(b) is used for ϕ_3, then $\phi_1 + \phi_2 + \phi_3 + \phi_4 \leq 1 + 3/8 + 1/4 + 1/2 = 2 + 1/8$, so $t = 4$ proves (3). Otherwise, R3(a) is used for ϕ_3. Then R4(e) is not used for ϕ_4. If $\phi_4 \leq 1/4$, then $t = 4$ verifies (3). If R4(b) is used for ϕ_4, then $d_5 \geq 12$, so $\phi_5 = 0$ and $t = 5$ works. Hence, we may assume that $d_4 = 5$, $\phi_4 > \frac{1}{4}$, and that R4(d) or R4(a) is used for ϕ_4. Then we have $\phi_1+\phi_2+\phi_3+\phi_4 \le 1+3/8+1/2+3/8 = 2+4/16$, and we are done if $d \ge 14$.

From now on we may w.l.o.g. assume that $d = 13$. Suppose first that R4(a) is used for $\phi_4 > \frac{1}{4}$. If $\phi_4 = \frac{6}{20}$, then $d_5 \neq 4$. If $\phi_5 = \frac{1}{2}$ (Rule R4(e)), then $d_6 \ge 11$, so $\phi_6 = 0$ and $\phi_1+\cdots+\phi_6 \le 1+3/8+1/2+6/20+1/2$ which yields (3). If $\phi_5 \le \frac{3}{8}$, then $t = 5$ will do. The remaining case is when $\phi_4 = \frac{7}{20}$. Then $d_5 = 4$. If R3(c) is used for ϕ_5, then $d_6 \ge 12$, so $t = 6$ gives (3). If R3(b) is used, then $t = 5$ works. Finally, having R3(a) for ϕ_5 implies that $\phi_6 \le \frac{3}{8}$. Hence $\phi_1 + \cdots + \phi_6 \le 3 + \frac{2}{20}$ proves (3).

From now on we may assume that R4(d) is used for ϕ_4. If $\phi_2, \phi_4 \in \{\frac{3}{8}, \frac{7}{20}\}$, then Rule R1' has been used in f and in the $\ge$4-face f' containing v_3v_4. Therefore $\phi_2 = \phi_4 = \frac{7}{20}$. Otherwise, $\phi_2+\phi_4 \le \frac{3}{8}+\frac{7}{24} < \frac{7}{10}$. In any case, $\phi_1 + \phi_2 + \phi_3 + \phi_4 \le 1 + 1/2 + 7/10 = 2 + 1/5$. If $\phi_5 \le \frac{3}{8}$, we take $t = 5$. If R4(e) is used for ϕ_5, then $\phi_5 = \frac{1}{2}$ and $\phi_6 = 0$, so $t = 6$ works. Similarly if R3(c) is used for ϕ_5. Since R3(a) cannot be used for ϕ_5, the only remaining possibility is that $d_5 = 4$ and $\phi_5 = 1$. This completes the proof of (3) and characterizes the only exception.

Now we continue with the proof of (2). We apply (3) to $i = 5$. Let t_1 be the corresponding value of t. (If the exception to (3) occurs, we take $t_1 = 4$ and say that t_1 is *exceptional*.) Next, we repeat the same with $i = 5 + t_1$. Let t_2 be the corresponding value of t. If neither of t_1, t_2 is exceptional, then we let $t = 4 + t_1 + t_2$ and (3) implies

$$\phi_1 + \cdots + \phi_t \le 2 + \frac{1}{5} + \frac{t_1 + t_2}{2} + \frac{t_1 + t_2 - 1.2}{26} \le \frac{t}{2} + \frac{t}{26}.$$

Let us observe that any of the cases satisfying (3) uses R3(d) only at its first edge. Therefore $t \le 13$.

We may assume henceforth that t_1 or t_2 is exceptional. Suppose first that t_1 is exceptional. Then $\phi_1+\cdots+\phi_8 \le 4+\frac{2}{5}$ and $\phi_9 = 1$. It suffices to prove (by taking $t = 13$) that

$$\phi_{10} + \phi_{11} + \phi_{12} + \phi_{13} \le \frac{8}{5}. \tag{4}$$

We shall make use of the following claims:

(1.1) If $d_i \le 5$, $d_{i+3} \le 5$, $d_{i+1} \ge 5$, and $d_{i+2} \ge 5$, then $\phi_{i+1} + \phi_{i+2} \le \frac{7}{12}$.

This is clear if $d_{i+1} \ge 11$ or $d_{i+2} \ge 11$. Otherwise, neither R4(e) nor R4(d) with $\frac{3}{8}$ or $\frac{7}{20}$ is used for ϕ_{i+1}, ϕ_{i+2}. This implies (1.1). The next claim is also obvious by inspection.

(1.2) If $d_i \le 5$, $d_{i+2} \le 5$, $d_{i+1} \ge 5$, and all four faces containing the edges v_iv_{i+1} and $v_{i+1}v_{i+2}$ are 3-faces, then $\phi_{i+1} = 0$.

Returning to the proof of (4), assume first that $d_{11} = 4$. If $\phi_{11} = 1$, then $\phi_{10} = 0$ by (1.2) and $\phi_{12} + \phi_{13} \le \frac{7}{12}$ by (1.1). This implies (4). Otherwise, $\phi_{11} \le \frac{5}{8}$ and $\phi_{10} \le \frac{3}{8}$. Hence (4) follows in the same way as before. Similar estimates prove (4) if $d_{12} = 4$.

We assume henceforth that $d_i \ge 5$ for $i = 10, 11, 12, 13$. If $\phi_i = 0$ for some i, then $\phi_{10} + \phi_{11} + \phi_{12} + \phi_{13} \le 3 \cdot \frac{1}{2}$, so (4) follows. Otherwise, $\phi_i \le \frac{7}{24}$ for $i = 10, 11, 12, 13$. This implies (4) as well.

It remains to consider the case when t_2 is exceptional. Then $t_2 = 4$ and $t_1 \le 5$. If $t_1 = 5$, then the above proof of (4) shows that $t = 13$ works for (2). Next, $t_1 \ne 4$ since that would imply that $d_{13} = 4$ (which is not possible since $d_1 = 4$ and v_1 and v_{13} are adjacent). Since $d_9 = 4$, we have $t_1 \ge 2$. If $t_1 = 2$, then $\phi_6 = 0$ by (1.2) and $t = 6$ works. So, $t_1 = 3$. Then $\phi_{13} = 0$ by (1.2), and $\phi_6 + \phi_7 \le \frac{7}{12}$ by (1.1). Hence

$$\phi_1 + \cdots + \phi_{13} \le 2\,(2 + \tfrac{1}{5}) + 1 + \tfrac{7}{12} + 1 < 7,$$

so $t = 13$ gives (2). This completes the proof of (2).

The following averaging argument using (2) shows that $c^*(v) \ge 0$. For $i = 1, \ldots, d$, let t_i be the integer t in (2). Let $n_1 = 1$ and $n_{j+1} = n_j + t_{n_j}$, $j = 1, 2, \ldots$. Let r be an integer, and let $q = n_r - 1$ and $s = \lfloor q/d \rfloor$. It follows by (2) that

$$\sum_{i=1}^{q} \phi_i \le \frac{q}{2} + \frac{q}{\alpha}$$

where $\alpha = 26$ if $d = 13$, and $\alpha = 14$ if $d \ge 14$. Let $\varphi = \sum_{i=1}^{d} \phi_i$. Then $\sum_{i=1}^{q} \phi_i \ge s\varphi$, so

$$\frac{s}{s+1} \cdot \varphi \le \frac{q/(s+1)}{2} + \frac{q/(s+1)}{\alpha} \le \frac{d}{2} + \frac{d}{\alpha} \le \frac{d}{2} + \frac{d-12}{2} = c(v).$$

Since r and hence s may be arbitrarily large, this shows that $\varphi \le c(v)$ and, consequently, $c^*(v) \ge 0$. □

5. The lightness of $K_{1,3}$

Theorem 5.1. $w(K_{1,3}) \le 23$.

Proof. This proof is given for the class $\mathcal{G}(11, 13)$. Suppose that the claim is false and G is a counterexample on $|V(G)|$ vertices with $|E(G)|$ as large as possible.

We claim that every vertex $v \in V(G)$ with $d(v) \ge 11$ is incident only with 3-faces. Suppose not. Then there exists a face f of size ≥ 4 which is incident with v. Let w be a vertex on f which is not f-adjacent with v. In the graph $G' = G + vw$, every subgraph $H \cong K_{1,3}$ which contains the edge vw has weight $w(H) \geq 12+5+4+4 = 25$. Thus, G' contradicts the maximality of $|E(G)|$.

Rule R1. Suppose that f is a face with $r(f) \geq 4$ and u is a 4- or 5-vertex incident with f. Then f sends $\frac{1}{2}$ to u.

Rule R2. Suppose that u is a 4-vertex adjacent to a vertex v with $d(v) \geq 7$.

(a) If $d(v) = 7$ and $m_4(v) = 1$, then v sends $\frac{2}{3}$ to u.
(b) If $d(v) = 7$ and $m_4(v) = 2$, then v sends $\frac{1}{2}$ to u.
(c) In all other cases, v sends 1 to u.

Rule R3. Suppose that u is a 5-vertex adjacent to a vertex v with $7 \leq d(v) \leq 10$.

(a) If $d(v) = 7$ and $m_4(v) = 1$, then v sends $\frac{1}{3}$ to u.
(b) In all other cases, v sends $\frac{c(v)-m_4(v)}{m_5(v)}$ to u.

Rule R4. Suppose that u is a 5-vertex adjacent to an 11-vertex v. Let u^- and u^+ be the predecessor and successor of u with respect to v.

(a) If $d(u^-) \leq 5$ and $d(u^+) \geq 6$ or vice-versa, then v sends $\frac{1}{2}$ to u.
(b) In all other cases, v sends $\frac{1}{3}$ to u.

Rule R5. Suppose that u is a 5-vertex adjacent to a vertex v with $d(v) \geq 12$. Let u^- and u^+ be the predecessor and successor of u with respect to v.

(a) If $d(u^-) = 4$ and $d(u^+) = 5$ or vice-versa, then v sends $\frac{1}{4}$ to u.
(b) If $d(u^-) = d(u^+) = 4$, then v sends no charge to u.
(c) In all other cases, v sends $\frac{1}{2}$ to u.

In Rules R4 and R5, multiple adjacency is allowed. In other words, a 5-vertex receives a charge as many times as it is adjacent to a vertex of degree ≥ 11. Since G has no light $K_{1,3}$, the following holds for any vertex v of G:

(a) If $d(v) \leq 11$ then $m_4(v) \leq 2$.
(b) If $d(v) \leq 10$ and $m_4(v) = 2$, then $m_5(v) = 0$.
(c) If $d(v) \leq 9$ and $m_4(v) = 1$, then $m_5(v) \leq 1$.
(d) If $d(v) \leq 8$ then $m_4(v) + m_5(v) \leq 2$.

Using (a)–(d) and Rule R3, we can calculate $\bar{c}$, the minimal possible charge which a 5-vertex receives from a neighbor v with $7 \leq d(v) \leq 10$. The values of $\bar{c}$ depending on $d(v)$ and $m_4(v)$ are given in Table 1.

Next, we claim that after applying Rules R1–R5, $c^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$. Suppose first that $f \in F(G)$. If $r(f) = 3$ then it neither receives nor sends any charge. So, $c^*(f) = c(f) = 0$. And, if

$d(v)$	7	7	8	8	9	9	10	10
$m_4(v)$	0	1	0	1	0	1	0	1
$\bar{c}$	1/2	1/3	1	1	1/3	2	2/5	1/3

TABLE 1. The minimal charge $\bar{c}$

$r(f) \geq 4$, then $c^*(f) \geq 2r(f) - 6 - \frac{r(f)}{2} \geq 0$ (by R1). This proves the claim.

Suppose now that v is a vertex of G with $c^*(v) < 0$. Under this assumption, we will obtain a contradiction. Let $d = d(v)$ and let $x_1, x_2, \ldots, x_d$ be the neighbors of v in the clockwise order around v. Consider the following cases.

$d = 4$: By Rule R2, v has at most one neighbor of degree ≥ 8 and at most three neighbors of degree 7. Otherwise, its final charge is nonnegative. If $m_7(v) \leq 1$ then G has a light $K_{1,3}$ whose central vertex is v. If $m_7(v) = 2$, then v has exactly one neighbor of degree ≥ 8. Otherwise, v is a central vertex of a light $K_{1,3}$. Now, by Rule R2, $c^*(v) \geq -2 + \frac{1}{2} + \frac{1}{2} + 1 = 0$. Finally, if $m_7(v) = 3$ then the fourth neighbor of v, say x_4, has degree 6. We may assume that v is incident only with 3-faces. Otherwise, by Rule R1, $c^*(v) \geq 0$. By Rule R2, at least one of x_1, x_2, x_3 sends $\frac{1}{2}$ to v (otherwise, $c^*(v) = -2 + 3 \cdot \frac{2}{3} = 0$). Denote one such vertex by x. By Rule R2(b), x is a 7-vertex and it has another neighbor of degree 4. Hence, we obtain that G has a light $K_{1,3}$ whose central vertex is x, a contradiction.

$d = 5$: Note that v has at least three neighbors of degree ≥ 7 (multiplicity is considered) and at most two neighbors of degree 4. We may assume that v is incident only with 3-faces. Otherwise, v receives $\frac{1}{2}$ from some face and it receives from adjacent vertices totally $\geq \frac{1}{2}$ by Rules R3–R5. If $m_4(v) = 0$ then each neighbor of v of degree at least 7 sends at least $\frac{1}{3}$ to v. Thus, $c^*(v) \geq -1 + 3 \cdot \frac{1}{3}$.

If $m_4(v) = 1$ then we may assume that $d(x_1) = 4$. It is easy to see that v receives $\frac{1}{4}$ or $\geq \frac{1}{3}$ from each neighbor of degree ≥ 7. So, assume that some neighbor sends $\frac{1}{4}$ to v. By R5, let this neighbor be x_2. Hence, $d(x_2) \geq 12$ and $d(x_3) = 5$. Now, we see that $d(x_4) \geq 10$ and $d(x_5) \geq 10$. By Rules R3–R5, each of x_4 and x_5 sends at least $\frac{3}{8}$ to v. So, $c^*(v) \geq -1 + \frac{1}{4} + \frac{6}{8} = 0$. (The minimum $\frac{3}{8}$ of charge, which x_5 sends to v, is obtained by R3(b) when $d(x_5) = 10$, $m_4(x_5) = 1$. Note that in this case $m_5(x_5) \leq 8$ since x_4 is a neighbor of x_5 of degree > 5. Similar conclusion holds for x_4.)

Finally, if $m_4(v) = 2$ then we may assume that x_1 and x_3 are 4-vetices and x_2, x_4, x_5 are vertices of degree at least 11. Now, by Rules R4 and R5, each of x_4 and x_5 sends $\frac{1}{2}$ to v. This implies that $c^*(v) \geq 0$.

$6 \leq d \leq 10$: If $d = 6$ then it neither sends nor receives charge. So, $c^*(v) = c(v) = 0$. If $d = 7$ then $m_4(v) + m_5(v) \leq 2$. It is easy to verify that $c^*(v) \geq 0$. Suppose now that $8 \leq d \leq 10$. If $m_5(v) = 0$ then $m_4(v) \leq 2$ and hence $c^*(v) = d - 6 - m_4(v) \geq 0$. And, if $m_5(v) > 0$, then using Rule R3(b) it is easy to show that $c^*(v) = 0$.

$d = 11$: We are looking for the minimum possible value of $c^*(v)$. Denote by m_5^- the number of vertices which receive $\frac{1}{2}$ from v and denote by m_5^+ the number of vertices which receive $\frac{1}{3}$ from v (multiplicity is considered). Then, $c^*(v) = 5 - m_4(v) - \frac{1}{2}m_5^- - \frac{1}{3}m_5^+$.

We may assume that $m_5^- = 0$ by the following observation. Suppose that x_i receives $\frac{1}{2}$ from v. Then, we may assume that $d(x_{i+1}) \geq 6$, $d(x_i) = 5$, and $d(x_{i-1}) \leq 5$. Reset $d(x_{i+1}) = 5$. It is easy to verify using Rule R4 that after this resetting $c^*(v)$ cannot increase. Finally, $m_4(v) \leq 2$ implies that

$$c^*(v) = 5 - m_4(v) - \frac{m_5^+(v)}{3} \geq 5 - m_4(v) - \frac{11 - m_4(v)}{3} \geq 0.$$

$d \geq 12$: We are interested in the minimum possible value of $c^*(v)$. Denote by m_5^- the number of vertices which receive $\frac{1}{2}$ from v and denote by m_5^+ the number of vertices which receive $\frac{1}{4}$ from v (multiplicity is considered). Then, $c^*(v) = d - 6 - m_4(v) - \frac{1}{2}m_5^- - \frac{1}{4}m_5^+$. We may assume that for each x_i, $d(x_i) \leq 5$. Otherwise, we can reset $d(x_i) = 5$. By Rule R5, it is not hard to check that after this resetting, $c^*(v)$ remains unchanged or decreases. We may also assume that there are no three consecutive neighbors x_{i-1}, x_i, x_{i+1} of v all of degree 5. Otherwise, we can reset $d(x_i) = 4$. Observe that after this $c^*(v)$ never increases. Finally, we may assume that v has two consecutive neighbors both of degree 5, say x_1 and x_d. Otherwise, every second neighbor of v is a 4-vertex and by Rules R2 and R5(b), $c^*(v) = d - 6 - \frac{d}{2} \geq 0$.

A $(5, 4)$-*chain* is a walk $y_1, y_2, \ldots, y_{2k+1}$ whose vertices have degrees $5, 4, 5, 4, \ldots, 5$, respectively. The possibility that some vertex can have multiple appearance in a $(5, 4)$-chain is not excluded. By the above assumptions, we can split $x_1 x_2 \cdots x_d$ into k different $(5, 4)$-chains $P_1, \ldots, P_k$. Suppose that the length of P_i is $2l_i + 1$, $i = 1, \ldots, k$. Then

$$c^*(v) = d - 6 - \sum_{i=1}^{k} \left(l_i + \frac{1}{2} \right) = d - 6 - \frac{d}{2} \geq 0.$$

□

6. The lightness of C_4

Theorem 6.1. *$w(C_4) \leq 35$.*

Proof. This proof is given for the class $\mathcal{G}(22, 23)$. Suppose that the claim is false and G is a counterexample on $|V(G)|$ vertices with $|E(G)|$ as large as possible. First, we claim that every vertex $v \in V(G)$ with $d(v) \geq 21$ is incident only with 3-faces. Suppose not. Then there exists a face f of size ≥ 4 which is incident with v. Let w be a vertex on f which is not f-adjacent with v. In the graph $G' = G + vw$ every 4-cycle which contains the edge vw has weight $\geq 22 + 5 + 4 + 5 = 36$. Since $G' \in \mathcal{G}(22, 23)$, this implies that G' is also a counterexample, a contradiction to the maximality of $|E(G)|$.

The discharging rules are as follows.

Rule R1. Suppose that f is a face and u is a vertex incident with f.

(a) If $d(u) = 4$ and $r(f) \geq 4$, then f sends 1 to u.
(b) If $d(u) = 5$ and $r(f) \geq 6$, then f sends 1 to u.
(c) If $d(u) = 5$ and $4 \leq r(f) \leq 5$, then f sends $\frac{1}{2}$ to u.

Rule R2. Suppose that u is a 5-vertex adjacent to a vertex v of degree 10. Then v sends $\frac{4}{10}$ to u.

Rule R3. Suppose that u is a 4- or 5-vertex adjacent to a vertex v of degree 11.

(a) If $d(u) = 4$ and uv is in two 3-faces, then v sends $\frac{10}{11}$ to u.
(b) If $d(u) = 4$ and uv is in precisely one 3-face, then v sends $\frac{5}{11}$ to u.
(c) If $d(u) = 5$ and uv is in two 3-faces, then v sends $\frac{5}{11}$ to u.

Rule R4. Suppose that u is a 4- or 5-vertex adjacent to a vertex v of degree ≥ 12.

(a) If $d(u) = 4$ and uv is in two 3-faces, then v sends 1 to u.
(b) If $d(u) = 4$ and uv is in precisely one 3-face, then v sends $\frac{1}{2}$ to u.
(c) If $d(u) = 5$ and uv is in two 3-faces, then f sends $\frac{1}{2}$ to u.

Now, we shall prove that after applying R1–R4, $c^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$.

Let f be a face of G. If f is a 3-face, then $c^*(f) = c(f) = 0$. If $r(f) \geq 6$, then f has charge $2(r(f) - 3) \geq r(f)$. By Rule R1 the face f sends to each incident vertex a charge ≤ 1, hence $c^*(f) \geq 0$. If $r(f) = 5$, then $c(f) = 4$. The face f has at most two 4-vertices. So, two of its vertices receive ≤ 1 and three receive $\leq \frac{1}{2}$. Hence, $c^*(f) \geq 0$. Finally, suppose that $r(f) = 4$. Then, f has charge 2. All vertices on f have degree ≤ 21. Therefore, no edge on f is a loop, and if v is a ≤ 5-vertex on f, its two neighbors on f are distinct vertices of G. In particular, if f has two 4-vertices, then f is bounded by a 4-cycle. Since its weight is ≥ 36, the other two vertices are of degree ≥ 6. Similarly, if f has a 4-vertex and (at least) two vertices of degree 5. This implies that f sends a charge ≤ 2 to its neighbors, and so $c^*(f) \geq 0$.

Let v be a vertex of G and let $d = d(v)$. Denote by $v_1, \dots, v_d$ the neighbors of v in the clockwise order around v. We consider the following cases.

$d = 4$**:** Suppose first that v is incident with two ≥ 4-faces. Then by Rule R1(a) these two faces send 2 to v, and so $c^*(v) \geq 0$.

Let v be incident with precisely one face f of size ≥ 4. Assume that f contains v_1 and v_2. If $d(v_2) \geq 12$ or $d(v_3) \geq 12$ or v_1 and v_2 are both of degree ≥ 12, then the vertices of degree ≥ 12 send a charge ≥ 1 to v according to Rule R4. By R1(a), f sends 1 to v. Consequently, $c^*(v) \geq 0$.

Suppose now that v has at most one neighbor (say v_1) of degree ≥ 12 incident with f. Since $vv_2v_3v_4v$ is a 4-cycle in G, two vertices among v_2, v_3, v_4 have degree 11 (and the third one has degree 10 or 11). By Rules R3(a) and R3(b), the 11-neighbors of v send a charge $\geq \frac{15}{11}$ to v. By R1(a), f sends 1 to v, and thus $c^*(v) \geq 0$.

Let v be incident with four 3-faces. If v is adjacent with two vertices of degree ≥ 12 then by Rule R4(a) these vertices send 2 to v, and $c^*(v) \geq 0$. If v is adjacent with at most one vertex of degree ≥ 12, then v is incident with at least three vertices of degree ≥ 11 (otherwise there would be a light 4-cycle). By Rules R3 and R4, these vertices send a charge $\geq 3 \cdot \frac{10}{11} > 2$ to v, and $c^*(v) \geq 0$.

$d = 5$**:** If v is incident with one face of size ≥ 6 then by Rule R1(b) this face sends 1 to v, and $c^*(v) \geq 0$. If v is incident with two faces of size ≥ 4, then these faces send ≥ 1 to v, and so $c^*(v) \geq 0$.

Let v be incident with precisely one face f of size 4 or 5. Suppose that f contains v_1 and v_2. If one of v_3, v_4, v_5 has degree ≥ 12, then $c^*(v) \geq 0$ by R1 and R4(c). Otherwise, $C = vv_2v_3v_4v$ is a 4-cycle. Then, C contains an 11-vertex w and a vertex w' of degree ≥ 10. Then v receives a charge $\geq \frac{1}{2} + \frac{5}{11} + \frac{4}{10} > 1$ from f, w, and w', so $c^*(v) \geq 0$.

Suppose that v is incident only with 3-faces. If two neighbors of v have degree ≥ 12, then by R4(c) these vertices send 1 to v, and so $c^*(v) \geq 0$. If at most one neighbor of v has degree ≥ 12 then two neighbors have degree ≥ 11 and a third neighbor has degree ≥ 10. By Rules R2 and R3 these three neighbors send a charge $\geq 2 \cdot \frac{5}{11} + \frac{4}{10} > 1$ to v, and $c^*(v) \geq 0$.

$6 \leq d \leq 9$**:** The vertex v sends no charge to other vertices, and so $c^*(v) \geq 0$.

$d = 10$**:** The vertex v has initial charge 4, and by Rule R2, it sends to each neighbor a charge $\leq \frac{4}{10}$. Hence, $c^*(v) \geq 0$.

$11 \leq d \leq 21$**:** If v_i is a 4-neigbour of v such that $vv_{i-1}v_i$ and vv_iv_{i+1} (all indices modulo d) are 3-faces, then both v_{i-1} and v_{i+1} have degree ≥ 5, and one of them has degree ≥ 6 (otherwise, $vv_{i-1}vv_{i+1}$ would be a light 4-cycle).

Suppose that $d \geq 12$. Each v_i receives a charge $\leq \frac{1}{2}$ from v, except when $d(v_i) = 4$ and $vv_{i-1}v_i$ and vv_iv_{i+1} are 3-faces. In the exceptional case v sends 1 to u. If $d(v_{i-1}) \geq 6$ and $d(v_{i+1}) \geq 6$, then v sends no charge to them. If one of these vertices, say v_{i-1}, is a 5-vertex such that vv_{i-1} is incident with a face of size ≥ 4, then v sends no charge to v_{i-1} and v_{i+1}. In both cases, we may think of the charge 1 sent from v to v_i being sent $\frac{1}{2}$ along the edge vv_i and $\frac{1}{4}$ along vv_{i+1} and $\frac{1}{4}$ along vv_{i-1}. If $d(v_{i-1}) = 5$, $d(v_{i+1}) \geq 6$, and $vv_{i-2}v_{i-1}v$ is a 3-face, then v sends $\frac{1}{2}$ to v_{i-1}. Since $vv_{i-2}v_{i-1}v_i$ is a 4-cycle if $d(v_{i-2}) \leq 5$, we have $d(v_{i-2}) \geq 6$. Again, we may think of v sending $\frac{1}{2}$ directly to v_i, $\frac{1}{4}$ to v_i via v_{i-2}, and $\frac{1}{4}$ to v_i via v_{i+1}. Then again, along any edge vv_j $(1 \leq j \leq d)$ a charge $\leq \frac{1}{2}$ is sent from v. Consequently, v sends a charge $\leq d \cdot \frac{1}{2} \leq d - 6$ to its neighbors. This implies that $c^*(v) \geq 0$.

If $d = 11$ then multiply all charges used above with $\frac{10}{11}$. Again, $d \cdot \frac{5}{11} \leq d - 6$, and so $c^*(v) \geq 0$.

$d = 22$**:** Then all faces containing v are of size 3. Let ϕ_i be the charge sent from v to v_i $(i = 1, \dots, d)$. Let v_i be a neighbor of v. We claim that either $\phi_i \leq \frac{1}{2}$ or $\phi_i + \phi_{i+1} \leq 1$ or $\phi_i + \phi_{i+1} + \phi_{i+2} \leq 2$. If none of the first two inequalities hold, then $d(v_i) = 4$ and $d(v_{i+1}) = 5$. If $d(v_{i+2}) = 4$, then $vv_iv_{i+1}v_{i+2}$ is a light 4-cycle. This implies that $d(v_{i+2}) \geq 5$, and the last inequality holds. Now, it is easy to see that the total charge sent from v to its neighbors is $\leq 7 \cdot 2 + \frac{1}{2}$, so $c^*(v) > 0$.

$d \geq 23$**:** Similarly as above, we have $\phi_i + \phi_{i+1} \leq \frac{3}{2}$. This implies that the total change sent from v to its neighbors is $\leq \frac{3}{2} \cdot \lfloor \frac{d}{2} \rfloor \leq d - 6$, and so $c^*(v) \geq 0$. □

Let K_4^- denote the 4-cycle with one diagonal. The following example shows that K_4^- is not light. Let W_s be a wheel with center x, cycle $y_1y_2 \cdots y_s$ and spokes xy_i for $1 \leq i \leq s$. Let W_s' be a copy of W_s with vertices $x', y_1', \dots, y_s'$, and let H_s be the graph obtained from $W_s \cup W_s'$ by adding new vertices $z_1, \dots, z_s$ which are joined to W_s and W_s' by the edges $z_iy_i, z_iy_{i+1}, z_iy_i'$, and z_iy_{i+1}' for all $1 \leq i \leq s$ (indices modulo s). The graph H_s is 3-connected, planar, of minimum degree 4 and without adjacent 4-vertices. On the other hand, every K_4^- in H_s contains an s-vertex.

However, the proof of Theorem 6.1 shows that

Corollary 6.2. *In the subclass of triangulations with minimum degree* 4 *and without adjacent* 4*-vertices, the graph* K_4^- *is light with* $w(K_4^-) \leq 35$.

7. The lightness of $K_{1,4}$, C_5, and C_6

In this section we shall show that $K_{1,4}$, C_5, and C_6 are light in the class $\mathcal{G}$ of all planar graphs of minimum degree ≥ 4 having no adjacent 4-vertices. Let H be a plane graph. With $\varphi(H)$ we denote the smallest integer with the property that each graph $G \in \mathcal{G}$ contains a subgraph K isomorphic to H such that all vertices of K have degree $\leq \varphi(H)$. (If such an integer does not exist, we write $\varphi(H) = \infty$.)

Theorem 7.1. $\varphi(K_{1,4}) \leq 107$, $\varphi(C_5) \leq 107$, *and* $\varphi(C_6) \leq 107$.

Proof. For brevity, let $\omega := 108$. A vertex v and a face f are said to be *big* if $d(v) \geq \omega$ and $r(f) \geq 4$, respectively.

Suppose that there is a counterexample for the stated bounds. We may assume that

(0) *G is 2-connected (and hence every facial walk is a cycle of G).*

If G has more than one block (2-connected component), let B be an endblock of G and let v be the cutvertex of G contained in B. We may assume that v is on the outer face of B. Let u be another vertex of B on the outer face of B. Take ω distinct copies B_i of B ($i = 0, 1, \ldots, \omega - 1$), and denote by v_i and u_i the copies of v and u (respectively) in B_i. Let G' be the graph obtained from $B_0 \cup B_1 \cup \cdots \cup B_{\omega-1}$ by identifying all copies of v into a single vertex, and adding edges $u_0 u_i$ for $i = 1, \ldots, \omega - 1$. Then G' is a 2-connected planar graph with minimum degree ≥ 4 and no two adjacent 4-vertices. If H is a connected subgraph of G' containing no vertices of degree $\geq \omega$, then H determines an isomorphic subgraph in $B - v$ (hence in G), and its degrees in G are also $< \omega$. This proves (0).

Suppose now that a big vertex v is incident with a big face. By adding edges incident with v we can triangulate the big face, and the resulting graph is still a counterexample to our theorem. Hence, we can achieve the following property:

(1) *Every vertex $v \in V(G)$ with $d(v) \geq \omega$ is incident only with 3-faces.*

Observe that in order to achieve (1), we may have introduced parallel edges. So, we shall work in a slightly bigger class $\mathcal{G}' \supseteq \mathcal{G}$ of graphs obtained from $\mathcal{G}$ by triangulating neighborhoods of large vertices. So, we may have parallel edges, but every parallel edge has at least one big endvertex. Moreover, after replacing all parallel edges by single edges, we get a graph in the class $\mathcal{G}$. We make some further assumptions:

(2) *Among all counterexamples in* $\mathcal{G}'$ *satisfying* (0) *and* (1), *we select one with minimum number of vertices. Subject to this assumption, we choose one with maximum number of edges. Furthermore, among all embeddings of* G *in the plane we select one with minimum number of pairs* (v, f), *where* $v \in V(G)$ *is a* 4*-vertex and* f *is a big face incident with* v.

Now we consider the following discharging rules:

Rule R1. Suppose that f is a face with $r(f) \geq 6$ and e is an edge on f incident with a 3-face Δ. Let v be the vertex of Δ which is not an endvertex of e. Suppose first that the neighbor faces of Δ different from f are triangles.

(a) If $d(v) = 4$ and $r(f) \geq 6$ then f sends $\frac{1}{2}$ to v.
(b) If $d(v) = 5$ and $r(f) \geq 7$ then f sends $\frac{1}{6}$ to v.

Suppose now that precisely one neighbor face Δ' of Δ is a 3-face and Δ' is adjacent to two triangles incident with v.

(c) If $d(v) = 4$ and $r(f) \geq 7$ then f sends $\frac{1}{4}$ to v.

Rule R2. Suppose that f is a face with $r(f) \geq 7$ and e is an edge on f incident with a 3-face Δ. Suppose that a neighbor face Δ' of Δ is a triangle. Let u denote the common vertex of f, Δ, and Δ', and let v be the vertex of Δ' not in Δ. Suppose that $d(v) = 4$, that v is contained in at least three 3-faces, and if it is contained in three 3-faces, then it has no big neighbors.

(a) If $d(u) = 7$, then f sends $\frac{1}{6}$ to v.
(b) If $d(u) = 6$, then f sends $\frac{1}{2}$ to v.
(c) If $d(u) = 5$ and four faces incident with u are triangles, then f sends $\frac{1}{4}$ to v via e (i.e., f sends $\frac{1}{2}$ to v via the two edges of f incident with u).

Rule R3. Suppose that f is a face and u is a vertex incident with f.

(a) If $d(u) = 4$ and $r(f) \geq 4$ then f sends 1 to u.
(b) If $d(u) = 5$ and $r(f) = 4$ then let c denote the charge which remains at f after the application of Rule R3(a). Then f sends the remaining charge c equally distributed to all 5-vertices on f.
(c) If $d(u) = 5$ and $r(f) = 5$ then f sends $\frac{2}{3}$ to u.
(d) If $d(u) = 5$, $r(f) \geq 6$ and none of the Rules R1 and R2 applies in f for the two edges which are f-incident with u, then f sends 1 to u.

(e) If $d(u) = 5$, $r(f) \geq 6$ and at least one of the Rules R1 and R2 applies at an edge which is f-incident with u, then f sends $\frac{1}{2}$ to u.

Rule R4. Suppose that u is a 4- or 5-vertex adjacent to a vertex v of degree $\geq \omega$.

(a) If $d(u) = 4$ then v sends 1 to u.
(b) If $d(u) = 5$, let u_1 and u_2 (respectively u_1' and u_2') be the first and the second successor (respectively predecessor) of u with respect to the local clockwise rotation around v. If the following three conditions are satisfied
  (b_1) $d(u_1) \geq 6$ or $d(u_1') \geq 6$;
  (b_2) if $d(u_1) = 4$ then $d(u_2) \geq 6$, and if $d(u_1') = 4$ then $d(u_2') \geq 6$;
  (b_3) $u_1 u_1' \in E(G)$;
  then v sends $\frac{2}{3}$ to u. In all other cases, v sends $\frac{1}{2}$ to u.

Rule R5. Suppose that $f = vxy$ and $\Delta = xyu$ are distinct 3-faces with the common edge xy. Suppose that $d(v) \geq \omega$.

(a) If $d(u) = 4$ and Δ is adjacent to three 3-faces, then v sends $\frac{1}{2}$ to u.
(b) If $d(u) = 5$ and Δ is adjacent to three 3-faces, then v sends $\frac{1}{6}$ to u.
(c) If $d(u) = 4$ and Δ is a adjacent to precisely two 3-faces, then v sends $\frac{1}{4}$ to u.

Rule R6. Suppose that w is a 6- or 7-vertex adjacent with at least six 3-faces. Let v_i, $i = 1, \ldots, d(w)$, be the neighbors of w in the clockwise order around w such that the possible face f' of size ≥ 4 lies between v_6 and v_7. Suppose that v_1 is big, v_2 and v_3 are not big, $d(v_4) = 4$, v_5 is again big, and v_6 and the possible vertex v_7 have arbitrary degrees.

(a) If $d(w) = 6$ then v_1 sends $\frac{1}{2}$ to v_4 via w.
(b) If $d(w) = 7$ then v_1 sends $\frac{1}{6}$ to v_4 via w.

Rule R7. Suppose that the edge uv belongs to two 3-faces.

(a) Suppose that $d(v) = 7$ and $d(u) = 4$. If v has at most two neighbors of degree 4, then v sends $\frac{1}{2}$ to u. If v has more than two neighbors of degree 4, then v sends $\frac{1}{3}$ to u.
(b) If $8 \leq d(v) \leq \omega - 1$ and $d(u) = 4$ then v sends $\frac{1}{2}$ to u.

Rule R8. Suppose that u is a 5-vertex adjacent to a 4-vertex v. If the edge uv belongs to two 4-faces, then v sends $\frac{1}{2}$ to u.

Rule R9. Suppose that v is a 4-vertex which bears a positive charge $c > 0$ after applying all previous rules and has $r > 0$ neighbors of degree 5. Then v sends $\frac{c}{r}$ to every neighbor of degree 5.

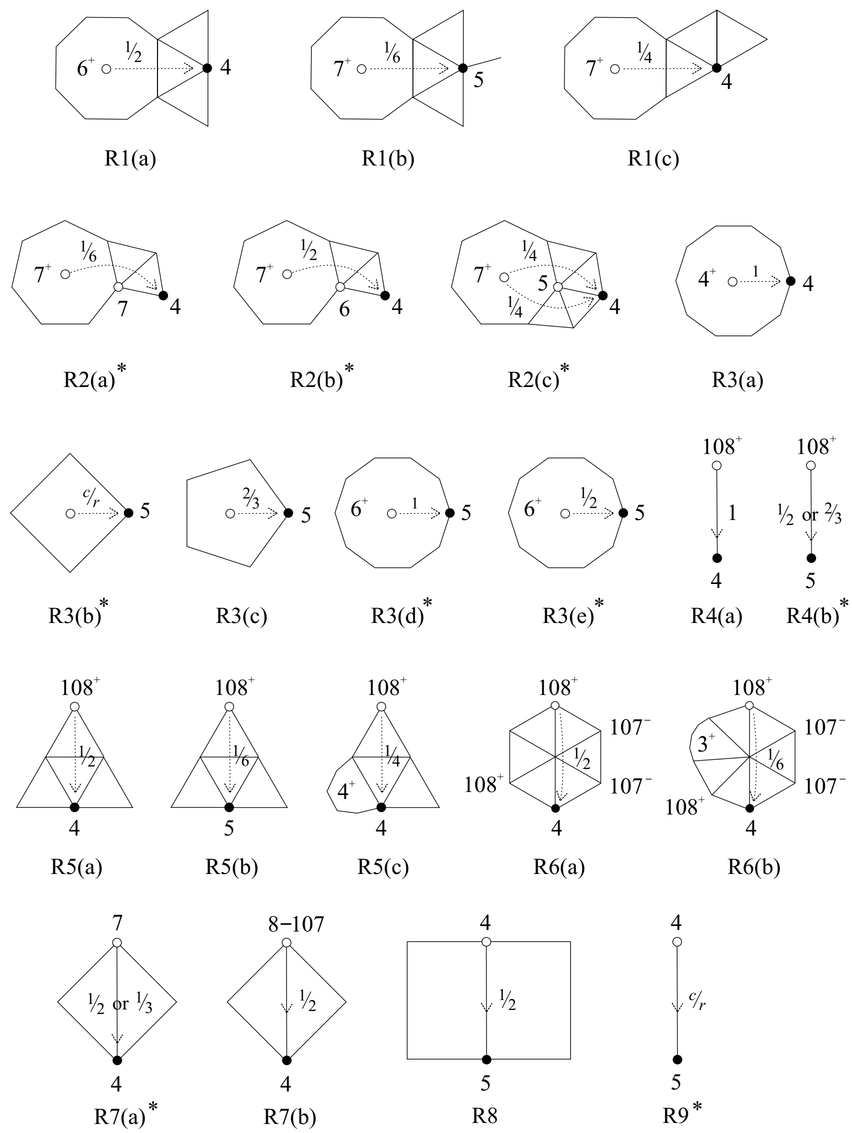

FIGURE 3. Discharging rules for the proof of Theorem 7.1. *Additional details or requirements are provided in the text.

For the reader's convenience, the rules R1–R9 are represented in Figure 3. The numbers at vertices or faces represent their degree or size, respectively, and the sign "+" (or "–") at the number means that the degree or size is $\geq$ (or $\leq$, respectively) to the given number.

We shall prove that after applying Rules R1–R9, $c^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$. Let us first consider vertices. Let v be a vertex of G and let $d = d(v)$. We introduce the following notation. Let $v_1, \ldots, v_d$ be the neighbors of v in the clockwise order around v. For $i = 1, \ldots, d$, let f_i be the face containing v_i, v, and v_{i+1} (indices modulo d). If f_i is a 3-face, let f'_i be the other face containing the edge v_iv_{i+1}. If f'_i is a 3-face as well, let u_i denote its vertex distinct from v_i and v_{i+1}. If f_i is not a 3-face, let v'_i be the vertex distinct from v which is f_i-adjacent to v_i. In the sequel, we will consider subcases depending on d.

$d = 4$**:** If v is incident with four or three big faces or with two opposite big faces, then by Rule R3(a), v receives a charge at least 4, 3, or 2, and by R8, v sends at most 2, 1, or 0, respectively. Hence, $c^*(v) \geq 0$. If v is only incident with two adjacent big faces then by Rule R3(a) the vertex v receives the charge 2 and by Rule R8 the vertex v sends a positive charge, namely $\frac{1}{2}$, if and only if the two big faces are 4-faces. The vertices on big faces have degree $< \omega$. If the fourth neighbor of v has degree $< \omega$, then the neighborhood of v contains a light C_5 and a light $K_{1,4}$. We may assume that the two 4-faces are $f_2 = vv_2v'_2v_3v$ and $f_3 = vv_3v'_3v_4v$. Then $C = vv_3v'_3v_4v_1v_2v$ is a light 6-cycle unless $v'_3 = v_2$. In the latter case, $v'_2 \neq v_4$ (by planarity) and, consequently, $C = vv_3v'_2v_2v_1v_4v$ is a light 6-cycle. The remaining case is when v has a big neighbor sending 1 to v, in which case $c^*(v) \geq 0$.

Next, we consider the case when v is incident with precisely one big face, say f_4. If a neighbor of v is big, it sends 1 to v; the second 1 is sent by the big face. Hence we may assume that v has no big neighbors. In the subcases C_5 and $K_{1,4}$ the neighborhood $N(v)$ of v contains both a C_5 and a $K_{1,4}$, a contradiction. It remains to consider the subcase C_6. For $i = 1, 2, 3$, consider the face f'_i. Suppose that f'_i is a 3-face. Suppose that $d(u_i) < \omega$. If $i = 2$, then $vv_1v_2u_2v_3v_4v$ is a light 6-cycle. If $i = 1$, then $vv_1u_1v_2v_3v_4v$ is a light 6-cycle except when $u_1 = v_4$. However, in that case we may reembed the edge v_1u_1 into the face f_4. Since we lost the pair (v, f_4) counted in the last minimality condition assumed in (2), there must be a new pair (v', f'), where $d(v') = 4$ and $r(f') \geq 4$. In that case, v_1u_1v' would originally be a 3-face, and $C = v_1v'u_1v_3v_2vv_1$ would be a light C_6. Similarly if $i = 3$; then $u_3 = v_1$ would be an exception which could be dismissed in the same way as above. The conclusion is that u_i is a big vertex and sends $\frac{1}{4}$ or $\frac{1}{2}$ to v regarding to the Rules R5(c) and R5(a) (if $i = 1, 3$ or $i = 2$, respectively).

Suppose now that f'_i is a big face. If $r(f'_i) = 6$, then f'_i determines a light C_6 by (0) and (1). If $r(f'_i) = 5$, let $f'_i = v_i u'_i x u''_i v_{i+1} v_i$. Then $v_i u'_i x u''_i v_{i+1} v v_i$ is a light C_6 if $x \neq v$. If $x = v$, then $f'_i = f_4$ and hence $u'_i = v_1$ and $u''_i = v_4$. In each case we get a loop or parallel edges joining two vertices of degree $< \omega$, a contradiction. Suppose now that $r(f'_i) = 4$, $f'_i = v_i u'_i u''_i v_{i+1} v_i$. If $i = 2$ then either $Q_1 = v_i u'_i u''_i v_{i+1} v_{i+2} v v_i$ or $Q_2 = v_i u'_i u''_i v_{i+1} v v_{i-1} v_i$ is a light C_6. If $i = 1$ then Q_1 is a light C_6 unless $u'_1 = v_3$. However, in that case $v_3 u''_1 v_2 v_1 v v_4 v_3$ is a light C_6. Similarly if $i = 3$. We may thus assume that $r(f'_i) \geq 7$. The big face f'_i sends $\frac{1}{4}$ or $\frac{1}{2}$ to v according to the Rules R1(c) and R1(a). This shows that the three faces neighboring the three triangles incident with v and not containing v send the total charge ≥ 1 to v. The second 1 is sent to v from f_4. Hence, $c^*(v) \geq 0$.

It remains to consider the case when v is incident with four 3-faces. If v has two big neighbors, then v receives the charge 2 from them by R4(a). Suppose firstly that v has no big neighbors. (Then we may restrict ourselves to the case C_6 since the neighborhood of v contains light C_5 and $K_{1,4}$.) As above we see that each face f'_i (or its big vertex u_i) sends $\frac{1}{2}$ to v by R1(a) (or R5(a)). Consequently, v receives the total charge 2 from f'_1, f'_2, f'_3, and f'_4.

Suppose secondly, v has precisely one big neighbor, say v_2. For $i = 1, \ldots, 4$, denote by u'_i the vertex which is f'_i-adjacent to v_i and distinct from v_{i+1}. Since $f_3 \cup f_4 \cup f'_3 \cup f'_4$ contains no light $K_{1,4}$, C_5 or C_6 (respectively), at least one of f'_3 and f'_4, say f'_4, is a face of size ≥ 4 or a triangle with the big vertex u'_4. In the subcase of $K_{1,4}$, the faces f'_3 and f'_4 are both triangles (otherwise we would get a light $K_{1,4}$ centered at v_4), and u'_3, u'_4 are both big vertices. Then each of them sends $\frac{1}{2}$ to v by R5(a). Hence $c^*(v) \geq 0$. In the subcase of C_5, faces f'_3 and f'_4 cannot be of size 4 or 5. If f'_3 is a triangle and u'_3 is not big, then there is a light C_5. Similarly if f'_4 is a triangle and u'_4 is not big. Hence, v receives 1 from v_2, $\frac{1}{2}$ from u'_3 (by R5(a)) or from f'_3 (by R1(a)), and $\frac{1}{2}$ from u'_4 or f'_4. This completes the C_5 case.

We are left with the subcase C_6. By the above, v receives 1 from v_2 and $\frac{1}{2}$ from f'_4 or u'_4. Since it does not receive another $\frac{1}{2}$ from f'_3 or u'_3, f'_3 is a 3-face and u'_3 is of degree $< \omega$. If v_3 has degree ≥ 8 then by Rule R7(b), v receives $\frac{1}{2}$ from v_3. Hence, $d(v_3) \in \{5, 6, 7\}$. Let f be the neighbor face of f'_3 containing the edge $v_3 u'_3$.

Suppose first that f has size ≥ 4. Then it has size ≥ 7. (Otherwise G would contain a light C_6; observe that f may contain v_4 if $r(f) = 5$. In that case, the light C_6 goes through v as well.) If $d(v_3) = 7$, then by Rules R7(a) and R2(a) the vertex v receives $\frac{1}{3}$ from v_3 and $\frac{1}{6}$ from f. So, v receives the total charge 2 from the vertices v_2, v_3, and the faces

f_4' and f, and $c^*(v) \geq 0$. If $d(v_3) = 6$ then by Rule R2(b), v receives the total charge 2 from v_2, f_4', and f. Next, suppose that $d(v_3) = 5$. Since f_2' is incident with the big vertex v_2, it is a 3-face. Therefore v receives $\frac{1}{2}$ from f by Rule R2(c). In all cases $c^*(v) \geq 0$.

Finally, let f be a triangle, say $f = v_3u_3'z$. Suppose first that z is a big vertex. If z belongs to f_2', then it sends $\frac{1}{2}$ to v by R5(a). If z is not in f_2', then we may assume that $d(v_3) \in \{6, 7\}$. If $d(v_3) = 6$ then by Rule R6(a) the vertex z sends $\frac{1}{2}$ to v. If $d(v_3) = 7$ then by Rule R6(b) the vertex z sends $\frac{1}{6}$ to v, and by Rule R7(a) the vertex v_3 sends $\frac{1}{3}$ to v. In all cases v receives the total charge 2 from v_2, f_4', f, z, and v_3. This proves that z is not a big vertex. Now, $v_1v_4u_3'zv_3vv_1$ is a light C_6 unless $z = v_1$, which we assume henceforth.

Since f_1', f_2' are incident with the big vertex v_2, they are triangles. By R5(a) we may assume that vertices u_1' and u_2' are not big. Because of R7 we may also assume that none of v_1, v_3, v_4 has degree ≥ 8, and if it is of degree 7, it has three neighbors of degree 4. Suppose that in the local clockwise ordering of edges incident with v_1, the edges v_1v_3 and v_1v are consecutive. Then $f_1' = v_1v_2v_3v_1$ where the edge v_2v_3 is not an edge of f_3. Since the two parallel edges joining v_2 and v_3 do not form a face, there is an edge between them in the clockwise ordering of edges around v_3. Since $d(v_3) \leq 7$, there is precisely one such edge v_3v'. Since all faces incident with v_2 are triangles, v' and v_2 are joined by more than one edge in parallel. Therefore, $d(v') \geq 5$ and consequently, v_3 has at most two neighbors (v and possibly u_3') of degree 4. This contradiction shows that there is an edge between v_1v_3 and v_1v. In particular, v_1 has at least 6 distinct neighbors. The same conclusion can be made for v_3.

Let G' be the graph obtained from $G - v$ by adding the edge v_2v_4. Clearly, G' is 2-connected and has no light C_6. All vertices of G' have the same degree as in G except v_1 and v_3 whose degrees have been decreased by one. The conclusions of the previous paragraph imply that $G' \in \mathcal{G}'$ is a "legal" counterexample, contrary to (2). This completes the proof in the case when $d = 4$.

$d = 5$**:** If two faces incident with v have size ≥ 5 then by R3(c)–(e) the vertex v receives a charge $\geq \frac{1}{2}$ from each of these faces, i.e., a total charge ≥ 1. If two neighbors of v are big, then v receives total charge ≥ 1 by R4(b). Hence we may assume that at most one face incident with v and at most one neighbor of v have size ≥ 5 or degree $\geq \omega$, respectively. This implies that the neighborhood of v contains a light $K_{1,4}$. So we may henceforth consider only the cases C_5 and C_6. Up to symmetries, it is sufficient to consider the following six cases.

Case 1. *The faces f_1, f_2, f_5 are of size ≥ 4.*

Case 1.1. *f_1, f_2, f_5 all have size 4.* If $d(v_1) = d(v_2) = 4$ then by Rule R8 the vertices v_1 and v_2 send charge 1 to v. If $d(v_1) = 4$ and $d(v_2) \geq 5$ then v_1 sends $\frac{1}{2}$ to v by R8, and each of f_1 and f_2 sends charge $\geq \frac{1}{3}$ to v by R3(b). If $d(v_1) \geq 5$ and $d(v_2) \geq 5$ then by Rule R3(b) each of f_1, f_2, and f_5 sends charge $\geq \frac{1}{3}$ to v. In all cases v receives charge ≥ 1.

Case 1.2. *f_1 and f_2 are 4-faces, f_5 has size ≥ 5.* If $d(v_2) = 4$ then v_2 sends $\frac{1}{2}$ to v by R8, so v receives charge ≥ 1 from f_5 and v_2. If $d(v_2) \geq 5$ then by Rule R3(b) both faces f_1 and f_2 send $\geq \frac{1}{3}$ to v, so v receives total charge ≥ 1 from f_1, f_2, and f_5.

Case 1.3. *f_1 has size ≥ 5, f_2 and f_5 are 4-faces.* If f_1 has size ≥ 6 then by R3(d) f_1 sends 1 to v. Hence we may assume that f_1 is of size 5. If f_3 or f_4 has size ≥ 4 then we repeat the proof of Case 1.2 with f_1, f_2, f_3 or f_4, f_5, f_1 instead of f_5, f_1, f_2, respectively. So, we may assume that f_3 and f_4 have size 3. Then $C = vv_2v_2'v_3v_4v$ is a 5-cycle, and $C' = vv_2v_2'v_3v_4v_5v$ is a 6-cycle (if $v_2' \neq v_5$) or $C' = vv_3v_4v_5v_5'v_1v$ is a 6-cycle (if $v_2' = v_5$ since then $v_3 \neq v_5'$ by planarity). The cycles C and C' are light unless v_4 is a big vertex. So, v receives charge ≥ 1 from f_1 and v_4.

Case 2. *f_2, f_3, f_5 are big faces, and f_1 and f_4 are 3-faces.* Since all neighbors of v are on big faces, they have degree $< \omega$ by (1). At least two of f_2, f_3, f_5 are 4-cycles, and therefore G contains a light C_5. If $r(f_5) = 4$, then $f_1 \cup f_4 \cup f_5$ contains a light C_6. If $r(f_5) > 4$, then $r(f_2) = r(f_3) = 4$. Now, $f_2 \cup f_3$ contains a C_6 unless $v_2' = v_4$ or $v_3' = v_2$. By symmetry we may assume that $v_2' = v_4$. Observe that in this case, $v_3' \neq v_2$ and $v_1 \neq v_3'$ by planarity. Hence $C = vv_1v_2v_2'v_3'v_3v$ is a light C_6. This completes the proof.

Case 3. *f_1 and f_4 are big faces, f_1 is a 4-face, and f_2, f_3, f_5 are 3-faces.* All neighbors of v distinct from v_3 are light because they lie in big faces. Faces f_1 and f_5 induce a subgraph with a light C_5. So, only in the C_6-case the investigations have to be continued. Faces f_1, f_2, f_5 induce a subgraph with a C_6. Hence, v_3 is a big vertex.

If the size of f_4 is ≥ 5 then by Rules R3(c)–(e) and R4(b), the face f_4 and the vertex v_3 send charge ≥ 1 to v. Consequently, f_1 and f_4 are 4-faces. Since f_5 is a triangle, one of v_1, v_5 has degree ≥ 5. Hence we may assume that f_1 contains at most one vertex of degree 4. Then f_1 sends charge $\geq \frac{1}{3}$ to v by R3(b). If f_4 contains at most one vertex of degree ≥ 4, then f_4 sends $\geq \frac{1}{3}$ to v as well. So, v receives $\geq \frac{1}{2} + \frac{1}{3} + \frac{1}{3} > 1$ from v_3, f_1, and f_4. Next, suppose that f_4 contains two 4-vertices, v_4 and v_5. Let $f^*, f^* \neq f_3$, denote the neighbor face of f_4 containing v_4.

If $f^* = v_4w'v_4'$ is a 3-face then $v_1vv_4w'v_4'v_5v_1$ is a 6-cycle (if $w' \neq v_1$) or $v_1v_1'v_2vv_5v_4'w'$ is a 6-cycle (if $w' = v_1$). Therefore the vertex w' is big and sends 1 to v_4 by Rule R4(a). If f^* is a big face, then f^* sends 1 to v_4 by R3(a). By Rules R4(a) and R3(a), each of v_3 and f_4 sends 1 to v_4. So, in all cases v_4 receives a total charge ≥ 3. The vertex v_4 may send $\frac{1}{2}$ to v_4' by R8. In any case, it sends a charge $\geq \frac{1}{6}$ to v (by Rule R9) since v_4 has at most three non-big neighbors. Thus, v receives a total charge $\geq \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ from v_3, f_1, and v_4, respectively.

Case 4. *f_2 and f_3 are the only big faces, and f_2 is a 4-face.* Suppose that a neighbor u of v is a big vertex. If the size of f_3 is at least 5 then by Rule R3(c)–(e), the face f_3 sends $\geq \frac{1}{2}$ to v. So, v receives ≥ 1 from the unique big vertex and f_3. Now, let f_3 be a 4-face. If $d(v_3) = 4$, then v receives $\frac{1}{2}$ from the big neighbor and $\frac{1}{2}$ from v_3 by R8. If $d(v_3) \geq 5$, then v receives $\frac{1}{2}$ from the big neighbor and $\frac{1}{3}$ from each of f_2 and f_3 by R3(b).

The other possibility is that none of $v_1, \dots, v_5$ is big. Then the neighborhood of v contains a light C_5. It also contains a light C_6 unless $v_2' = v_5$. In that case, v_5 is adjacent to v_4, v, v_1, v_2, and v_3. In particular, $d(v_5) \geq 5$. If $r(f_3) \leq 6$, then G contains a light C_6. Therefore $r(f_3) \geq 7$ and v receives $\geq \frac{1}{2}$from f_3 by R3(d) or R3(e). We are done if Rule R3(d) is applied. Otherwise, Rule R1 or R2 has been applied from f_3 via edges incident with v. Since $r(f_2) = 4$ and $d(v_5) \geq 5$, the only possibility is that the charge $\frac{1}{6}$ was sent from f_3 via the edge vv_4 to the 5-vertex v_5. For R1(b) to be applied, the face f_4' must be a triangle. Since $d(v_5) = 5$, $f_4' = v_4v_5v_3v_4$. Having the edge v_3v_4, there is a light C_6, a contradiction.

Case 5. *f_5 is the only big face incident with v.* The faces f_1, f_2, f_3, f_4 induce a subgraph containing both a C_5 and a C_6. Hence, v has a big neighbor u not belonging to f_5. If f_5 has size ≥ 5 then by Rules R3(c)–(e) the face f_5 sends $\geq \frac{1}{2}$ to v, and v receives ≥ 1 from f_5 and u. If f_5 is a quadrangle then the neighborhood of v contains a light C_5. It also contains a light C_6 unless v_2 or v_4 is a big vertex and $v_3 = v_5'$. In that case we may assume that v_2 is big. Observe that G has another embedding in the plane in which the local clockwise order around v is v_1, v_5, v_4, v_2, v_3. In that embedding we have the 4-face $vv_4v_1v_2v$ in which we can add an additional edge v_4v_2 without creating a light C_6. This contradicts (2).

Case 6. *v is incident only with 3-faces.* Then precisely one neighbor of v is a big vertex, say v_3. In the C_5 case the neighborhood of v contains a light C_5, a contradiction. Consider the C_6-case. If f_i', $i \in \{1, 4, 5\}$ has size 4, 5, or 6, then G contains a light C_6, a contradiction. Hence,

f_i' is a triangle or a face of size ≥ 7. If f_i' is a triangle and u_i is a big vertex, then u_i sends $\frac{1}{6}$ to v by Rule R5(b). If f_i' is a face of size ≥ 7 then f_i' sends $\frac{1}{6}$ to v by Rule R1(b). So, the vertex v receives the total charge $\frac{1}{2}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=1$ from v_3, f_1', f_4', and f_5', unless one of f_i' is a 3-face and u_i is not big. In that case we get a light C_6 unless u_i is a neighbor of v. This is not possible for $i=5$. By symmetry we may assume that $i=1$ and $u_1=v_4$. Now, v receives $\frac{1}{6}+\frac{1}{6}$ from f_4' (or u_4) and f_5' (or u_5). We claim that v receives the additional charge $\frac{2}{3}$ from v_3 by Rule R4(b). The condition (b_3) of that rule is satisfied because of the edge v_2v_4. If $d(v_4)\geq 6$, then (b_1) is satisfied as well. Now, if $d(v_2)=4$, then the successor of v_2 around v_3 is v_4, whose degree is ≥ 4 by assumption. Hence, also (b_2) is satisfied. It remains to consider the case when $d(v_4)<6$. Since v_4 is adjacent to v, v_1, v_2, v_3, and v_5, it must be $d(v_4)=5$. Consider the face F containing consecutive edges v_3v_4 and v_4v_2. By (1), F is a 3-face. Since v_3 has neighbors distinct from v_2, v, v_4, the third edge of F is a parallel edge joining v_2 and v_3. Since these two parallel edges do not form a 2-face and since G is 2-connected, there is an edge incident with v_2, which lies between the two parallel edges in the local clockwise order around v_2. This implies that $d(v_2)\geq 6$, and hence (b_1) and (b_2) hold. The proof of this case is complete.

$d=6$**:** In this case, v neither sends nor receives charge. So, $c^*(v)=0$.

$d=7$**:** By Rule R7(a), v sends $\frac{1}{3}$ or $\frac{1}{2}$ to a 4-neighbor u if the edge vu is incident with two triangles. Since v can have only three 4-neighbors with this property, it sends a charge ≤ 1 to its neighbors.

$8\leq d\leq \omega-1$**:** By Rule R7(b), the vertex v sends $\frac{1}{2}$ to a 4-neighbor v if the edge vu is incident with two triangles. Hence, v sends a charge, namely $\frac{1}{2}$, to at most every second neighbor. So, v sends a total charge $\leq \frac{d}{2}\cdot\frac{1}{2}\leq d-6$ to its neighbors.

$d\geq\omega$**:** By (1), v is incident only with 3-faces. The initial charge of v is $d-6$. In order to count the total charge which receives the neighborhood of the vertex v from v, we *assign* to each neighbor v_i of v the sum of charges, denoted by ϕ_i, consisting of the charge p directly sent from v to v_i (by R4) or through v_i (by R6), a half of the charge q^- sent from v over the edge $v_{i-1}v_i$ (by R5), and a half of the charge q^+ sent from v over the edge v_iv_{i+1} (by R5). Thus, $\phi_i=p+\frac{1}{2}(q^-+q^+)$ is assigned to v_i. Obviously, the sum of all charges assigned to the neighbors of v equals the total charge sent from v to its neighborhood.

We have to investigate the applications of the Rules R4, R5, and R6 at v. Let v_i be an arbitrary neighbor of v. Our goal is to prove that $\phi_i \leq 1 - \frac{1}{18}$ (in average). Consider the following subcases.

$d(v_i) \geq 8$: Then $p = 0$, $q^+ \leq \frac{1}{2}$, $q^- \leq \frac{1}{2}$. Thus, $\phi_i \leq \frac{1}{2}$.

$d(v_i) = 7$: If v sends 0 through v_i, then $\phi_i \leq \frac{1}{2}$. If v sends $\frac{1}{6}$ through v_i by R6(b), then u_i or u_{i-1} is adjacent to a 4-vertex, and $d(u_i) \geq 5$ or $d(u_{i-1}) \geq 5$. Hence, $q^+ + q^- \leq \frac{1}{6} + \frac{1}{2}$. So, $\phi_i \leq \frac{1}{6} + \frac{1}{2}(\frac{1}{6} + \frac{1}{2}) = \frac{1}{2}$.

$d(v_i) = 6$: If v sends 0 through v_i, then $\phi_i \leq \frac{1}{2}$. If v sends $\frac{1}{2}$ through v_i by Rule R6(a), the vertex v_i is only incident with 3-faces, and the neighbor of v_i opposite to v has degree 4. Hence, $d(u_{i-1}) \geq 5$, $d(u_i) \geq 5$, and by Rule R6(a) one of these vertices has degree $\geq \omega$. Thus, $\phi_i \leq \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{6} = \frac{7}{12}$.

$d(v_i) = 5$: The vertex v sends $\frac{1}{2}$ (*poor case*) or $\frac{2}{3}$ (*rich case*) to v_i by Rule R4(b). Let us first assume that v sends $\frac{1}{2}$ through v_iv_{i+1} (say) to the vertex u_i by Rule R5(a). Then $d(u_i) = 4$ and, if it sends > 0 through $v_{i-1}v_i$ to u_{i-1}, then u_{i-1} is a neighbor of u_i. So, $d(u_{i-1}) \geq 5$, and v sends $\leq \frac{1}{6}$ to u_{i-1}. In this case and in all other cases, $\phi_i \leq \frac{1}{2} + \frac{1}{2} \cdot (\frac{1}{2} + \frac{1}{6}) = \frac{5}{6} < 1 - \frac{1}{18}$ (in the poor case), and $\phi_i \leq 1$ (in the rich case). Assuming the rich case, suppose first that $d(v_{i+1}) \geq 6$. Then $\phi_i + \phi_{i+1} \leq 1 + \frac{7}{12} < 2 - \frac{2}{18}$. If $d(v_{i+1}) = 5$, then the rich case cannot occur at v_{i+1} since the condition (b_3) cannot be satisfied at v_i and at v_{i+1} simultaneously. Therefore, $\phi_i + \phi_{i+1} \leq 1 + \frac{5}{6}$. If $d(v_{i+1}) = 4$, then $d(v_{i+2}) \geq 6$ by (b_2). The estimate in the next case below shows that $\phi_{i+1} \leq 1 + \frac{1}{6}$. Consequently, $\phi_i + \phi_{i+1} + \phi_{i+2} \leq 1 + (1 + \frac{1}{6}) + \frac{7}{12} < 3 - \frac{3}{18}$.

$d(v_i) = 4$: The vertex v sends 1 to v_i, and $q^+ \leq \frac{1}{6}$, $q^- \leq \frac{1}{6}$. Thus, $\phi_i \leq 1 + \frac{1}{6}$. If $d(v_{i+1}) \geq 6$ then $\phi_{i+1} \leq \frac{7}{12}$, and $\phi_i + \phi_{i+1} \leq (1 + \frac{1}{6}) + \frac{7}{12} = 2 - \frac{1}{4}$. Next, assume that $d(v_{i+1}) = 5$. Suppose first that we have the rich case at v_{i+1}. Then, by (b_1) in R4(b), $d(v_{i+2}) \geq 6$, and hence $\phi_i + \phi_{i+1} + \phi_{i+2} \leq (1 + \frac{1}{6}) + 1 + \frac{7}{12} < 3 - \frac{3}{18}$. Otherwise, v sends $\frac{1}{2}$ to v_{i+1}. If v sends $\leq \frac{1}{6}$ through $v_{i+1}v_{i+2}$, then $\phi_{i+1} \leq \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, and $\phi_i + \phi_{i+1} \leq (1 + \frac{1}{6}) + \frac{2}{3} = 2 - \frac{1}{6}$. If v sends $\frac{1}{2}$ through $v_{i+1}v_{i+2}$, this charge is sent to a neighbor of v_{i+2} of degree 4, so $d(v_{i+2}) \geq 5$. Hence, $\phi_{i+1} \leq \frac{5}{6}$ and $\phi_{i+2} \leq \frac{5}{6}$, so $\phi_i + \phi_{i+1} + \phi_{i+2} \leq 1 + \frac{1}{6} + \frac{5}{6} + \frac{5}{6} = 3 - \frac{1}{6}$.

We conclude that the average charge assigned to neighbours of v is $\leq 1 - \frac{1}{18}$. Hence, the total charge sent from v to its neighbors is $\leq (1 - \frac{1}{18}) \cdot d \leq d - 6$. Therefore, the resulting charge $c^*(v)$ is nonnegative.

Suppose now that f is a face of G. Consider the following cases.

$r(f) = 4$**:** The 4-face f with initial charge $c(f) = 2$ has at most two 4-vertices and by Rules R3(a) and (b), $c^*(f) \geq 0$.

$r(f) = 5$**:** The 5-face f with initial charge $c(f) = 4$ has at most two 4-vertices and by Rules R3(a) and (c), $c^*(f) \geq 0$.

$r(f) = 6$**:** The 6-face f has initial charge $c(f) = 6$. If by Rule R1(a) the face f sends the charge $\frac{1}{2}$ across an edge $e = uv$, then $d(u) \geq 5$ and $d(v) \geq 5$, and by Rule R3(e), u and v each receive $\leq \frac{1}{2}$ from f. Otherwise, by Rules R3(a) and R3(d), a 4- or 5-vertex receives 1 from f. This implies that f sends a charge ≤ 6 to its neighboring vertices and faces, and $c^*(f) \geq 0$.

$r(f) \geq 7$**:** Let the bounding cycle of the face f be oriented, and u^+ and u^- be the successor and the predecessor of the vertex u, respectively. Let $f(u)$, $f(u) \neq f$, denote the face incident with the edge uu^+. If $f(u)$ is a 3-face, let w denote the vertex of $f(u)$ with $w \notin \{u, u^+\}$. Further, in this case let $f^+(u)$ and $f^-(u^+)$ denote the neighboring faces of $f(u)$ incident with uw and wu^+, respectively. The notation $f^+(u)$ or $f^-(u^+)$ is only of importance if these faces are triangles. If such a face is not triangle then it receives charge 0 from f.

The initial charge of f is $2r(f) - 6$. In order to count the total charge which is sent to the neighborhood of f from f, we assign to each vertex u on f the sum ϕ_u of charges consisting of the charges sent to $u, w, f^-(u)$, and $f^+(u)$. Obviously, the sum of all charges assigned to the vertices incident with f equals the total charge sent from f to its neighborhood. We have to investigate the applications of the Rules R1, R2, and R3 to f. Let u be an arbitrary vertex incident with f. Consider the following subcases.

$d(u) \geq 8$**:** Then f sends 0 to u, 0 to $f^-(u)$, 0 to $f^+(u)$, and $\leq \frac{1}{2}$ to w. So, $\phi_u \leq \frac{1}{2}$.

$d(u) = 7$**:** Then f sends 0 to u, $\leq \frac{1}{6}$ to $f^-(u)$, $\leq \frac{1}{6}$ to $f^+(u)$, and $\leq \frac{1}{2}$ to w. So, $\phi_u \leq 1 - \frac{1}{6}$.

$d(u) = 6$**:** Then f sends 0 to u, $\leq \frac{1}{2}$ to each of $f^-(u)$ and $f^+(u)$ (by R2(b)) and $\leq \frac{1}{2}$ to w. However, if $\frac{1}{2}$ or $\frac{1}{4}$ is sent to w, then $d(w) = 4$, so 0 is sent to $f^+(u)$. Consequently, $\phi_u \leq 1$. Otherwise, $\phi_u \leq 1$ as well, except in the following case: $\frac{1}{2}$ is sent to $f^-(u)$ and to $f^+(u)$ and $\frac{1}{6}$ is sent to w. In that case, $\phi_u = 1 + \frac{1}{6}$.

$d(u) = 5$**:** If f sends 1 to u then by Rule R3(d) the face f sends 1 to u and 0 to $\{w, f^-(u), f^+(u)\}$. So, $\phi_u = 1$. Next, suppose that f sends $\frac{1}{2}$ to u. If f sends 0 to $f^+(u)$ then f sends $\frac{1}{2}$ to u, 0 to $f^-(u)$, 0 to $f^+(u)$ and $\leq \frac{1}{2}$ to w. So, $\phi_u \leq 1$. If f sends $\frac{1}{4}$ to $f^+(u)$ then by Rule R2(c) a neighbor of w has degree 4. Hence, $d(w) \geq 5$, and f sends $\frac{1}{2}$ to u, $\frac{1}{4}$ to $f^-(u)$, $\frac{1}{4}$ to $f^+(u)$, and 0 or $\frac{1}{6}$ to w. So, $\phi_u \leq 1 + \frac{1}{6}$.

We remark that $\phi_u = 1 + \frac{1}{6}$ if and only if f sends $\frac{1}{2}$ to u, $\frac{1}{4}$ to $f^-(u)$, $\frac{1}{4}$ to $f^+(u)$, and $\frac{1}{6}$ to w. Then, by Rule R2(c), u is incident with four 3-faces. If none of the neighbors of u is big, we have light $K_{1,4}$, C_5, and C_6 in the neighborhood of u. Hence, w^- (as the only possibility) has degree $\geq \omega$.

$d(u) = 4$: Since $d(u) = 4$ the degree $d(w) \geq 5$. Hence, f sends 1 to u, 0 to $f^-(u)$, 0 to $f^+(u)$, and $\leq \frac{1}{6}$ to w. So $\phi_u \leq 1 + \frac{1}{6}$. We remark that $\phi_u = 1 + \frac{1}{6}$ only when $d(w) = 5$ and f sends $\frac{1}{6}$ to w. Otherwise $\phi_u = 1$.

Since to each vertex u on the boundary of f a charge $\leq 1 + \frac{1}{6}$ is assigned, the total charge sent by f is $\leq (1 + \frac{1}{6})d(v)$. This is not larger than the initial charge $2(r(f) - 3)$ if $r(f) \geq 8$. Hence, the final charge $c^*(f) \geq 0$. Finally, let $r(f) = 7$. If to one vertex u on the boundary of f a charge ≤ 1 is assigned then the total charge sent by f is $\leq 1 + (1 + \frac{1}{6})(r(f) - 1)$. This is again not larger than the initial charge, and $c^*(f) \geq 0$. Suppose that to each vertex on the boundary of f the charge $1 + \frac{1}{6}$ is assigned. Then all these vertices have degrees 4, 5, or 6. Then there is a 5- or 6-vertex u such that u^+ is also a 5- or 6-vertex. Suppose first that $d(u) = 5$. By our remark in the proof of case "$d(u) = 5$" the vertex w^- has degree $\geq \omega$. Hence, a charge ≤ 1 is assigned to u^-, a contradiction.

Suppose now that $d(u) = 6$. As remarked in the "$d(u) = 6$" case, $\frac{1}{2}$ is sent to $f^-(u)$ and to $f^+(u)$, and $\frac{1}{6}$ is sent to w. Let w^- and w^+ be the vertices of degree 4 in $f^-(u)$ and $f^+(u)$ to which $\frac{1}{2}$ is sent. By Rule R2(b), w^+ is contained in at least three 3-faces. Since w^- is of degree 4, it is not a neighbor of w^+. Therefore, w^+ is contained in precisely three 3-faces, say xyw^+, yww^+, and wuw^+. By the requirement in Rule R2(b), the vertices x, y are not big. Since $\phi_{u^+} = 1 + \frac{1}{6}$, f sends $\frac{1}{2}$ also to $f^-(u^+)$. Hence, the fifth neighbor of w, denote it by x', has degree 4. (The other neighbors of w are u^+, u, w^+, and y.) In particular, $x' \neq x$ since x is adjacent to the 4-vertex w^+. Now, the neighborhood of w contains light $K_{1,4}$, and C_5. Moreover, $uw^+xywx'u^+u$ is a light 6-cycle. This contradiction shows that $c^*(f) \geq 0$ and completes the proof of the theorem. ☐

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