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European Journal of Combinatorics 25 (2004) 55–64

European Journal of Combinatorics

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Cubic inflation, mirror graphs, regular maps, and partial cubes

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Received 23 July 2003; received in revised form 9 September 2003; accepted 9 September 2003

Abstract

Partial cubes are, by definition, isometric subgraphs of hypercubes. Cubic inflation is an operation that transforms a 2-cell embedded graph G into a cubic graph embedded in the same surface; its result can be described as the dual of the barycentric subdivision of G. New concepts of mirror and pre-mirror graphs are also introduced. They give rise to a characterization of Platonic graphs (i) as pre-mirror graphs and (ii) as planar graphs of minimum degree at least three whose cubic inflation is a mirror graph. Using cubic inflation we find five new prime cubic partial cubes.

MSC (2000): 05C10; 05C75; 05C12

Keywords: Graph embeddings; Barycentric subdivision; Platonic graphs; Isometric subgraphs; Hypercubes; Graph automorphisms

1. Introduction

Graphs that can be isometrically embedded into hypercubes are called *partial cubes*. They were introduced by Graham and Pollak [14] and intensively studied afterwards. Djoković [10] gave the first characterization of partial cubes, several more followed in [2, 4, 26, 31], cf. the book [8] for more information on these characterizations. Partial cubes

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0195-6698/$ - see front matter
doi:10.1016/j.ejc.2003.09.004

were applied in different situations, see, for instance, [5, 6, 11, 21]. Distance regular partial cubes are characterized in [29], in [20] this result is extended to a certain broader metrical hierarchy. For the complexity issues on partial cubes we refer to [1, 16, 17] and for yet more information on these graphs see also the books [8, 17], recent studies in [19], and references therein.

One of the most challenging open problems in the area is to classify regular partial cubes, in particular the cubic ones. For one of the most important subclasses of partial cubes—median graphs—Mulder [24] proved that hypercubes are the only regular examples. Besides hypercubes, the even cycles are also regular partial cubes. Observe that the Cartesian product of two (regular) partial cubes is a (regular) partial cube. We say that a regular partial cube is *prime* if it cannot be written as a Cartesian product of two (necessarily regular) partial cubes, each containing at least two vertices.

Restricting to the cubic case, it was verified in [3] by a computer search that up to 30 vertices, there are only three prime cubic partial cubes: the generalized Petersen graph $P(10, 3)$ on 20 vertices, the permutahedron Π_3 from Fig. 2 on 24 vertices, and a sporadic example on 30 vertices. Some prime cubic partial cubes on more vertices are also known, for instance the truncated cuboctahedron on 48 vertices and the truncated icosidodecahedron on 120 vertices [7].

Motivated by the search for regular/cubic partial cubes, mirror graphs are introduced in the next section. It is then proved that they are partial cubes. In Section 3 the concept of cubic inflation is described. It is observed that the cubic inflation of an arbitrary graph embedded in some surface contains a Hamilton cycle, which leads us to conjecture that every cubic partial cube is Hamilton. In the following section the concept of pre-mirror graphs is introduced in order to characterize Platonic graphs as pre-mirror graphs and as planar graphs of minimum degree at least three whose cubic inflation is a mirror graph. In the final section our efforts give us five new prime cubic partial cubes.

The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of k copies of K_2 is a (k-dimensional) *hypercube* or *k-cube* Q_k. The 3-cube is also known as the *cube*. A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$, where $d_G(u, v)$ denotes the usual shortest path distance.

2. Mirror graphs

Let $G = (V, E)$ be a connected graph. Call a partition $\mathcal{P} = \{E_1, E_2, \ldots, E_k\}$ of E a *mirror partition* if for every $i \in \{1, \ldots, k\}$, there is an automorphism α_i of G such that

(M1) for every edge $uv \in E_i$, $\alpha_i(u) = v$ and $\alpha_i(v) = u$, and
(M2) $G - E_i$ consists of two connected components G_1^i and G_2^i, and α_i maps G_1^i isomorphically onto G_2^i.

Since α_i is an automorphism of G, E_i is a matching in G joining G_1^i and G_2^i.

A connected graph is a *mirror graph* if it admits a mirror partition. Note that hypercubes and even cycles are mirror graphs. Also, if G_1 and G_2 are mirror graphs, then their

Cartesian product $G_1 \square G_2$ is also a mirror graph. Furthermore, as the mirror partition condition is quite strong, mirror graphs that cannot be written as Cartesian products of other graphs are rather specific.

To show that mirror graphs are partial cubes, we need the following notion. Two edges $e = xy$ and $f = uv$ of a graph G are in the Djoković–Winkler [10, 31] relation Θ if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. Winkler [31] proved that a connected graph is a partial cube if and only if it is bipartite and Θ is transitive (and hence an equivalence relation).

Proposition 1. *Every mirror graph is a partial cube. Moreover, its mirror partition coincides with its Θ-equivalence classes.*

Proof. Let G be a mirror graph with a mirror partition $\mathcal{P}$. We first show that a mirror graph G is bipartite. If not, let $C = u_1u_2 \ldots u_{2s+1}u_1$ be a shortest odd cycle, and let $u_1u_2 \in E_i$, where E_i is a part of a mirror partition of G. Let $u_1 \in G_1^i$ and $u_2 \in G_2^i$. By (M2), there is another edge $u_r u_{r+1}$ of C that belongs to E_i. Let us assume that vertices of C have been enumerated so that r is the minimum possible. Then, clearly, $r \leq s+1$. Since C is a shortest odd cycle, it is isometric in G. Therefore, $d_G(u_1, u_{r+1}) \geq r - 1$ and $d_G(u_2, u_r) = r - 2$. But this contradicts the fact that $\alpha_i(u_1) = u_2$ and $\alpha_i(u_{r+1}) = u_r$.

Let uv be an edge of $E_i \in \mathcal{P}$, where $u \in G_1^i$ and $v \in G_2^i$. Let $z \in G_1^i$. We claim that $d(v, z) = d(u, z) + 1$. Let P be a (v, z)-geodesic path and let ww' be the first edge of P with $w \in G_2^i$ and $w' \in G_1^i$. Then $d_G(u, w') = d_G(v, w)$ which implies that $d(v, z) > d(u, z)$. Clearly, $d(v, z) \leq d(u, z) + 1$.

Let $uv, xy \in E_i$. We may assume that $u, x \in G_1^i, v, y \in G_2^i$. By the above, $d(u, y) = d(u, x) + 1$ and $d(v, x) = d(v, y) + 1$. Thus $uv \, \Theta \, xy$.

Assume that $uv \, \Theta \, xy$ where uv is an edge of E_i and $u \in G_1^i$, $v \in G_2^i$. We need to show that $xy \in E_i$ as well. Suppose not, and assume without loss of generality that $x, y \in G_1^i$. Then $d(v, x) = d(u, x)+1$ and $d(v, y) = d(u, y)+1$ thus $d(v, x)+d(u, y) = d(v, y) + d(u, x)$, a contradiction. $\square$

We next wish to find examples of mirror graphs. For this sake, the concept of the cubic inflation is introduced first.

3. Cubic inflation

An *embedded graph* or a *map* is a connected graph together with a 2-cell embedding in some closed surface. Let G be a map without vertices of degree one. Then we define the map $\mathcal{CI}(G)$ as follows. First, we replace each vertex $v \in V(G)$ by a cycle Q_v of length $2 \deg_G(v)$, and then replace every edge uv of G by two edges joining Q_u and Q_v in such a way that a cubic map on the same surface is obtained in which all cycles Q_v are facial and all edges of G give rise to 4-faces in that map. The result of such a change is shown locally in Fig. 1. The resulting map $\mathcal{CI}(G)$ is called the *cubic inflation* of G. The map $\mathcal{CI}(K_4)$ is illustrated on Fig. 2; it is interesting to note that $\mathcal{CI}(K_4)$ is isomorphic to the permutahedron Π_3, cf. [32, p. 16].

There is an alternative way to describe the cubic inflation. Let G be an embedded graph. Recall that the *barycentric subdivision* $B(G)$ of G is a triangulation obtained as

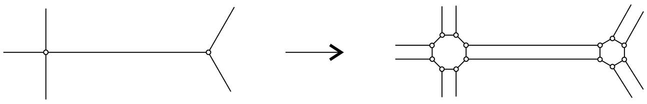

Fig. 1. Cubic inflation locally.

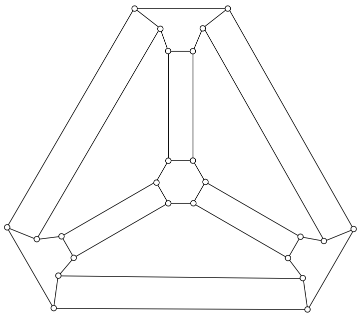

Fig. 2. Inflated tetrahedron.

follows [23]. Subdivide each edge of G by one vertex, and in the interior of each face add a vertex which is joined to all vertices (including the new subdivision vertices) on the corresponding face boundary. Denote by G^* the dual map of the map G. The following result follows easily from the fact that $B(G) = B(G^*)$ for every embedded graph G.

Proposition 2. *For every embedded graph G without vertices of degree one, we have*

$$\mathcal{CI}(G) = B(G)^* = \mathcal{CI}(G^*).$$

Yet another way to describe the cubic inflation $\mathcal{CI}(G)$ of G is that $\mathcal{CI}(G)$ is just the truncation of the medial graph of G: $\mathcal{CI}(G) = \mathrm{Tr}(\mathrm{Med}(G))$. Let G be an embedded graph. Then the vertices of the *medial* $\mathrm{Med}(G)$ of G are the edges of G. Each face $F = e_1 e_2 \dots e_k$ determines the edges $e_1e_2, \dots, e_{k-1}e_k, e_ke_1$ of $\mathrm{Med}(G)$. See [13, Section 17.2] for a more detailed explanation of this concept. The *truncation* $\mathrm{Tr}(G)$ of G is a graph obtained from G by replacing each vertex u of degree k with k new vertices that form a cycle and are each adjacent to the corresponding vertices of the neighbors of u. Since G is a graph embedded in a surface, there is a natural order for the new adjacencies. For a more exact definition we again refer to [13, p. 126].

The notion of cubic inflation is also related to Delaney symbols used in tiling theory (see, for example, [15]).

Cubic inflations on more general surfaces may also yield partial cubes. Such examples are shown by an example in Fig. 3. Here we start with an n-cycle embedded as a horizontal "meridian" in the torus, and then add $k \geq 1$ loops embedded as shown in the figure. Each vertex becomes incident with zero or more loops. The graph of the cubic inflation is isomorphic to $C_{2n+2k} \,\square\, K_2$, hence it is a (nonprime) partial cube and also a mirror graph. By Proposition 2, the dual map has the same cubic inflation. Let us observe that the dual

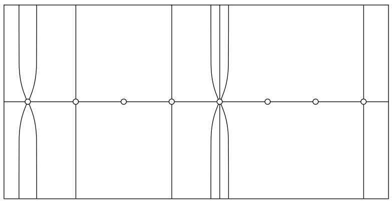

Fig. 3. Toroidal examples.

map admits the same structure as exhibited in Fig. 3. It would be of certain interest to find other examples of this kind.

The following simple result shows that cubic inflations of arbitrary maps are Hamilton.

Proposition 3. *Let H be the cubic inflation of a graph G embedded in some surface. Then H contains a Hamilton cycle.*

Proof. Let $\mathcal{C}_1$ be the collection of all cycles of H that correspond to vertices of G, and let T be a spanning tree of G. Let $\mathcal{C}_2$ be the set of all 4-cycles of H that correspond to the edges of T. Then the symmetric difference $\mathcal{C}_1 + \mathcal{C}_2$ is a Hamilton cycle of H. □

In the last section, further examples of (cubic) partial cubes obtained by cubic inflation are presented. Since this is a rare phenomenon, Proposition 3 led us to the following

Conjecture 4. *Every cubic partial cube is Hamilton.*

It is possible that every regular partial cube is Hamilton. We do not dare to conjecture this since a much weaker well-known conjecture is far from being understood. Namely, the middle level graphs (which are regular partial cubes) are conjectured to be Hamilton, and no real progress has been made towards a proof of this conjecture. See [27] for more details.

4. Inflated graphs with mirror partitions

In this section we characterize mirror graphs that can be obtained by the cubic inflation from some plane map.

Let B be a Eulerian graph embedded in some surface. A *straight-ahead walk* in B is a closed walk such that every pair of consecutive edges (including the transition from the last edge back to the initial edge of the walk) passes through the corresponding vertex straight-ahead with respect to the local rotation at that vertex. Two straight-ahead walks are considered the same if one is a cyclic shift or the inverse of a cyclic shift of the other. Then every edge of B determines precisely one straight-ahead walk containing that edge.

Let $B = B(G)$ be the barycentric subdivision of a map G, and let $W = v_1 v_2 \ldots v_r v_1$ be a straight-ahead walk in B. The vertices $v_i \in V(B)$ appearing in W correspond to vertices, edges, and faces of G. We say that v_i *appears essentially* in W if v_i is either a vertex of G, or v_i is an edge of G and v_{i-1} and v_{i+1} (indices considered modulo r) are faces of G. Then W determines a cyclic sequence of vertices and edges of G that is obtained by taking all essential appearances in W. Every such sequence of vertices and edges of G is said to be an *SA-walk* in G. Note that the collection of those SA-walks in G that contain at least one edge of G induces a partition of $E(G)$. The notion of SA-walks appears in other contexts as traverse [12], straight ahead [25], straight Eulerian [13], cut-through [18], and central-circuit [9].

We are interested in graphs with special SA-walks that are somehow similar to the mirror partition condition. If S is an SA-walk in G (so, S is a sequence of consecutively incident vertices, edges, and faces of G), let $G - S$ be the subgraph of G obtained by removing all edges and vertices that occur in S. Let us call a plane graph G a *pre-mirror graph* if for every SA-walk S of G:

(PM1) $G - S$ consists of two connected components G_1^S, G_2^S, and
(PM2) there is an automorphism α_S of G that maps G_1^S isomorphically onto G_2^S, where any element of S is invariant under α_S.

The main question in our investigations is which mirror graphs are cubic inflations. Recall that the Platonic graphs are tetrahedron, cube, octahedron, icosahedron, and dodecahedron.

A map G is *regular* (or *flag-transitive*) if its automorphism group acts transitively on the triples $(v, e, F) \in V(G) \times E(G) \times F(G)$ (also called *flags*) whose vertex v is incident with the edge e, and e is incident with the face F. It is known that regular maps in the sphere are precisely the Platonic maps and all cycles.

Theorem 5. *Let G be a map in the plane with minimum vertex degree at least three. Then the following assertions are equivalent.*

(i) *$\mathcal{CI}(G)$ is a mirror graph.*
(ii) *G is a pre-mirror graph.*
(iii) *G is a Platonic graph.*

Proof. (i) $\Rightarrow$ (iii). Let G be a map in the plane with minimum vertex degree at least three such that $\mathcal{CI}(G)$ is a mirror graph.

We first observe that mirror graphs are vertex-transitive. First of all, it is clear by (M2) that every mirror graph H is connected. Let x and y be vertices of H, and let P be a path of length r from x to y. Let $i_1, \ldots, i_r$ be integers in $\{1, \ldots, k\}$ such that the jth edge on P belongs to the part E_{i_j} of the mirror partition, $j = 1, \ldots, r$. Then $\alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_r}$ is an automorphism of H that maps x to y.

Next we show that every automorphism of the mirror graph $\mathcal{CI}(G)$, which appears in the mirror partition condition, fixes the set of faces of $\mathcal{CI}(G)$ that correspond to vertices of G (hence it also fixes the set of faces that correspond to faces of G). Recall first that Mader [22] and Watkins [28] (cf. also [13]) proved that vertex connectivity of a vertex-transitive graph of degree k is at least $2(k + 1)/3$. Since $\mathcal{CI}(G)$ is a mirror graph, it is vertex-transitive. Moreover, it is cubic and hence 3-connected. By a theorem of Whitney [30] (cf. also [23]), every automorphism of a 3-connected planar graph maps facial cycles onto facial cycles.

Now, let α be an automorphism of $\mathcal{CI}(G)$ which appears in the mirror partition condition. It is obvious that α fixes any facial cycle C which contains an edge whose ends are interchanged by α (applying also Whitney's theorem). Since every edge is in precisely two faces, we infer that faces incident to C are also mapped properly (that is, faces corresponding to vertices of G are mapped to faces corresponding to vertices of G, and likewise for the faces corresponding to faces of G). By connectivity of $\mathcal{CI}(G)$ (we apply successively the former observation for incident faces) we infer that α fixes the set of faces which correspond to vertices of G. Obviously, since compositum preserves this property, we infer that for each pair of vertices $u, v \in \mathcal{CI}(G)$ there exists an automorphism that maps u to v and fixes the set of faces that correspond to vertices of G.

Note that every vertex of $\mathcal{CI}(G)$ is incident with precisely three faces which correspond to a vertex of G, its incident edge in G, and one of the faces in which this edge lies in G, respectively. In other words, there is a bijection between vertices of $\mathcal{CI}(G)$ and flags (v, e, F) of G. Moreover, by observations of the previous paragraph, for every given pair $u, v \in \mathcal{CI}(G)$, there exists an automorphism of $\mathcal{CI}(G)$ which maps u to v, and at the same time fixes the set of faces which correspond to vertices of G. From this we infer that for each given pair of flags in G, there is an automorphism of G, which maps one flag to the other. Hence G is a regular map, and since it is of degree at least 3, we derive that it is a Platonic graph.

(iii) $\Rightarrow$ (ii). It is a straightforward check that all five Platonic graphs are pre-mirror.

(ii) $\Rightarrow$ (i). It follows directly from definitions of both classes and cubic inflation that if G is pre-mirror then $\mathcal{CI}(G)$ is mirror. $\square$

Theorem 5 characterizes plane graphs of minimum degree ≥ 3 whose cubic inflations are mirror graphs. They are precisely the Platonic graphs. If G is a plane graph with minimum degree 2 and its cubic inflation is a mirror graph, then it is easy to see that G is a cycle C_n, $n \geq 3$. Conversely, $\mathcal{CI}(C_n)$ is isomorphic to the Cartesian product of C_{2n} and K_2, and hence it is a mirror graph. However, C_n is not pre-mirror (since for one of its SA-walks S, $C_n - S$ is the empty graph).

If we allow graphs with multiple edges and loops, then the set of all regular spherical maps extends with cycles of length 1 and 2 and with *bonds*—dual maps of the cycles.

Corollary 6. *The cubic inflation of a spherical map G with minimum degree at least 2 is a mirror graph if and only if G is a regular spherical map.*

Planar pre-mirror graphs (and all cycles) correspond bijectively to mirror graphs which are cubic inflations. The natural question is, are there any nontrivial prime mirror graphs that are not cubic inflations of regular maps? Secondly, are there any prime mirror graphs that are not planar? Is there a similar characterization of those maps on some other surface whose cubic inflation is a mirror graph? Perhaps this could be done by using some kind of SA-walks or their unions.

We have only partly solved the question for which plane graphs their cubic inflation is a partial cube. Perhaps the following related question could be easier to attack: For which plane graphs their SA-walks are not self-crossing? An SA-walk is called self-crossing if there exist two elements (two vertices, two edges, or a vertex and an edge) of this walk that share a common face, but are not opposite on that face. Note that if G is embedded in the plane and $\mathcal{CI}(G)$ is a partial cube, then the SA-walks of G are not self-crossing. On the other hand, even if no SA-walk in G is self-crossing, $\mathcal{CI}(G)$ is not necessarily

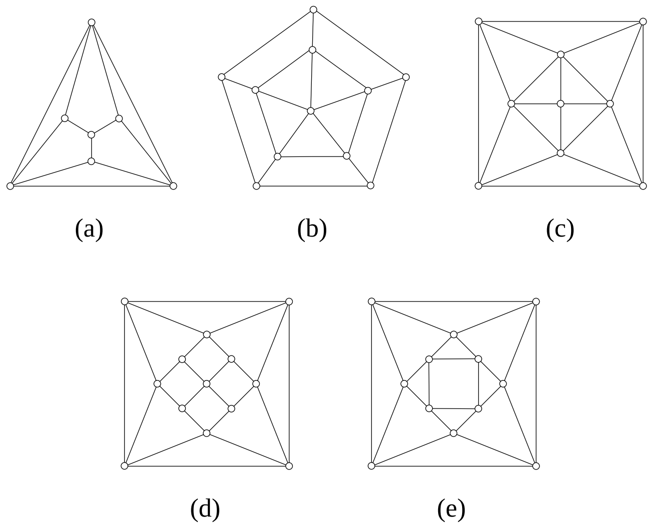

Fig. 4. Planar maps yielding cubic partial cubes.

a partial cube. However, answering this question would considerably reduce the class of graphs, for which the first question is relevant.

5. Hunting for cubic partial cubes

We now return to our starting point—searching for more cubic partial cubes. Besides the cubic partial cubes mentioned in the introduction, three more sporadic examples on 36, 42, and 48 vertices are presented in [3]. In this section we obtain five new such examples using the concept of the cubic inflation.

By Theorem 5, $\mathcal{CI}(G)$ is a mirror graph if G is a Platonic graph, hence Proposition 1 implies:

Corollary 7. *Let G be any of the five Platonic graphs. Then $\mathcal{CI}(G)$ is a prime cubic partial cube.*

As we already know, $\mathcal{CI}(K_4)$ is the permutahedron Π_3. Since octahedron O is the dual of the cube, Proposition 2 implies that $\mathcal{CI}(Q_3)$ and $\mathcal{CI}(O)$ are isomorphic graphs on 48 vertices—the truncated cuboctahedron. It embeds isometrically into Q_9 [7]. Note that this graph is not isomorphic to the graph B_2 on 48 vertices from [3] since both are 3-connected but B_2 has a facial cycle of length 12. As the icosahedron is the dual of the dodecahedron, their cubic inflations are isomorphic graphs on 120 vertices—the truncated icosidodecahedron that embeds isometrically into Q_{15} [7].

Proposition 8. *Cubic inflations of plane maps shown in Fig. 4(a)–(e) are partial cubes.*

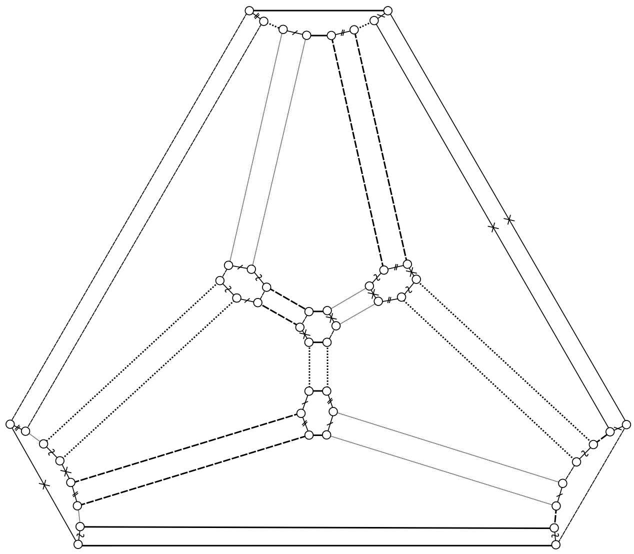

Fig. 5. Θ-classes of a cubic partial cube.

In order to prove Proposition 8, one has to verify that the relation Θ is transitive. This was checked by computer. As an example, the embedding of the cubic inflation of the graph in Fig. 4(a) is shown in Fig. 5. Different drawing styles and marks on the edges indicate the Θ-equivalence classes of the graph.

To obtain graphs of Proposition 8, we have not used any particular method. Yet, the intuitive reason for finding them lies in a fact that they are close to pre-mirror graphs in the sense, that they possess some SA-walks with properties (PM1) and (PM2). In the graphs of Fig. 4(a)–(e), there are $3, 5, 4, 4$ and 4 such SA-walks, respectively. Of course, since these graphs are not pre-mirror, they also have other SA-walks. Observe that we only need to check for these other SA-walks, if the corresponding edges in the cubic inflation form whole Θ-classes. If they do, then the graph is a partial cube.

The cubic inflation of the graph in Fig. 4(a) has 48 vertices. It is neither isomorphic to $\mathcal{CI}(Q_3)$, that is, to the truncated cuboctahedron (since it has two adjacent 8-faces) nor to the 48-vertex partial cube from [3] (which has adjacent 4-faces). The graphs (b) and (c) inflate into cubic graphs on 80 vertices while the graphs (d) and (e) inflate to 96 vertices. Since the face lengths of these pairs of graphs are pairwise different, Proposition 8 gives rise to five new examples of prime cubic partial cubes.

Note that the cubic inflation of every cycle C_n ($n \geq 2$) is also a cubic partial cube. However, $\mathcal{CI}(C_n) = C_{2n} \,\Box\, K_2$ is not prime.

All examples of cubic partial cubes that we have obtained so far, have the property that by removing any edge, graphs are no longer partial cubes. Partial cubes with this property are said to be *edge-critical* [3]. Therefore, we finish the paper with the following question: Is every cubic partial cube edge-critical?

Acknowledgements

We thank one of the referees for invaluable suggestions for many improvements of the presentation. The authors are supported by the Ministry of Education, Science and Sport of Slovenia under the grants Z1-3073-0101-01, J1-0504-0101-01, J1-0504-0101-01, and J1–0502–0101–01, respectively.

References

[1] F. Aurenhammer, J. Hagauer, Recognizing binary Hamming graphs in $O(n^2 \log n)$ time, Math. Systems Theory 28 (1995) 387–395.
[2] D. Avis, Hypermetric spaces and the Hamming cone, Canad. J. Math. 33 (1981) 795–802.
[3] C.P. Bonnington, S. Klavžar, A. Lipovec, On cubic and edge-critical isometric subgraphs of hypercubes, Australas. J. Combin. 28 (2003) 217–224.
[4] V. Chepoi, d-Convexity and isometric subgraphs of Hamming graphs, Cybernetics 1 (1988) 6–9.
[5] V. Chepoi, On distances in benzenoid graphs, J. Chem. Inform. Comput. Sci. 36 (1996) 1169–1172.
[6] V. Chepoi, S. Klavžar, The Wiener index and the Szeged index of benzenoid systems in linear time, J. Chem. Inform. Comput. Sci. 37 (1997) 752–755.
[7] M. Deza, V.P. Grishukhin, A zoo of l_1-embeddable polytopal graphs, Bull. Inst. Math. Acad. Sinica 25 (1997) 181–231.
[8] M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer, Berlin, 1997.
[9] M. Deza, M. Shtogrin, Octahedrites, Symmetry, Polyhedra and Science and Art (2002) (special issue).
[10] D. Djoković, Distance preserving subgraphs of hypercubes, J. Combin. Theory Ser. B 14 (1973) 263–267.
[11] K. Fukuda, K. Handa, Antipodal graphs and oriented matroids, Discrete Math. 111 (1993) 245–256.
[12] M.L. Gargano, J.W. Kennedy, Gaussian graphs and digraphs, Congr. Numer. 101 (1994) 161–170.
[13] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, New York, 2001.
[14] R.L. Graham, H. Pollak, On the addressing problem for loop switching, Bell System Tech. J. 50 (1971) 2495–2519.
[15] D.H. Huson, The generation and classification of tile-k-transitive tilings of the Euclidean plane, the sphere and the hyperbolic plane, Geom. Dedicata 47 (1993) 269–296.
[16] W. Imrich, S. Klavžar, On the complexity of recognizing Hamming graphs and related classes of graphs, European J. Combin. 17 (1996) 209–221.
[17] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, Wiley, New York, 2000.
[18] D.-Y. Jeong, Realizations with a cut-through Eulerian circuit, Discrete Math. 137 (1995) 265–275.
[19] S. Klavžar, H.M. Mulder, Partial cubes and crossing graphs, SIAM J. Discrete Math. 15 (2002) 235–251.
[20] J.H. Koolen, S.V. Shpectorov, Distance-regular graphs the distance matrix of which has only one positive eigenvalue, European J. Combin. 15 (1994) 269–275.
[21] J. Lawrence, Lopsided sets and orthant-intersection by convex sets, Pacific J. Math. 104 (1983) 155–173.
[22] W. Mader, Über den Zusammenhang symmetrischer Graphen, Arch. Math. (Basel) 21 (1970) 331–336.
[23] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, 2001.
[24] H.M. Mulder, n-cubes and median graphs, J. Graph Theory 4 (1980) 107–110.
[25] T. Pisanski, T. Tucker, A. Žitnik, Eulerian embeddings of graphs, IMFM Preprint Series, University of Ljubljana, vol. 34, 1996, 531.
[26] R.L. Roth, P.M. Winkler, Collapse of the metric hierarchy for bipartite graphs, European J. Combin. 7 (1986) 371–375.
[27] C.D. Savage, Long cycles in the middle two levels of the Boolean lattice, Ars Combin. 35A (1993) 97–108.
[28] M.E. Watkins, Connectivity of transitive graphs, J. Combin. Theory 8 (1970) 23–29.
[29] P.M. Weichsel, Distance regular subgraphs of a cube, Discrete Math. 109 (1992) 297–306.
[30] H. Whitney, 2-Isomorphic graphs, Amer. J. Math. 55 (1933) 245–254.
[31] P. Winkler, Isometric embeddings in products of complete graphs, Discrete Appl. Math. 7 (1984) 221–225.
[32] G.M. Ziegler, Lectures on Polytopes, Springer, New York, 1998.