Subdivisions of Large Complete Bipartite Graphs and Long Induced Paths in k-Connected Graphs

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Abstract

It is proved that for every positive integers k, r and s there exists an integer $n = n(k, r, s)$ such that every k-connected graph of order at least n contains either an induced path of length s or a subdivision of the complete bipartite graph $K_{k,r}$.

1 Introduction

According to Ramsey's theorem, for every positive integer r there is an integer $n = n(r)$ such that every graph of order at least n contains either a complete graph K_r or an edgeless graph $\bar{K}_r$ as an induced subgraph. For connected graphs this implies the following slightly stronger result, see Proposition 9.4.1 in [2].

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Proposition 1.1 *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every connected graph of order at least n contains K_r, $K_{1,r}$ or a path of length r as an induced subgraph.* $\square$

A similar result holds for 2-connected graphs, see Proposition 9.4.2 in [2].

Proposition 1.2 *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every 2-connected graph of order at least n contains a subdivision of $K_{2,r}$ or a cycle of length at least r as a subgraph.* $\square$

In 1993, Oporowski, Oxley and Thomas [5] proved the following two results for 3- and 4-connected graphs, respectively.

Theorem 1.3 (Oporowski, Oxley and Thomas [5]) *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every 3-connected graph of order at least n contains a minor of order at least r that is either a wheel or a $K_{3,r}$.* $\square$

Theorem 1.4 (Oporowski, Oxley and Thomas [5]) *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every 4-connected graph of order at least n contains a minor of order at least r that is either a double wheel, a crown, a Möbius crown or a $K_{4,r}$.* $\square$

In the light of the above results it seems sensible to conjecture the following.

Conjecture 1.5 *For every $k, r \in \mathbb{N}$ there is a finite set $\mathcal{G}_{k,r}$ of k-connected graphs each of order at least r and an $n \in \mathbb{N}$ such that every k-connected graph of order at least n contains a minor that is either a member of $\mathcal{G}_{k,r}$ or a $K_{k,r}$.* $\square$

The main result of the present note (Theorem 1.6 below) supports this conjecture.

Theorem 1.6 *For every $k, r, s \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that every k-connected graph of order at least n contains either an induced path of length s or a subdivision of $K_{k,r}$.*

In 1981, Bondy and Locke [1] proved that if a 3-connected graph contains a path of length r, then it contains a cycle of length at least $\frac{2}{3}r + 2$. This together with Theorem 1.6 implies that, for every $r \in \mathbb{N}$, every large enough k-connected graph that does not contain a subdivision of $K_{k,r}$ contains a cycle of length at least $\frac{2}{3}r + 2$. Since every 3-connected non-planar graph which is not isomorphic to K_5 contains a subdivision of $K_{3,3}$, the above observation relates Theorem 1.6 to the following result, due to Jackson and Wormald [4].

Theorem 1.7 (Jackson and Wormald [4]) *There are real numbers $\alpha, \beta > 0$ such that every 3-connected planar graph of order at least n contains a cycle of length at least βn^{α}.* □

This result leads to our second conjecture.

Conjecture 1.8 *For every $k \in \mathbb{N}$ there are real numbers $\alpha_k, \beta_k > 0$ such that every k-connected graph of order at least n not containing $K_{k,k}$ as a minor contains a cycle of length at least $\beta_k n^{\alpha_k}$.* □

2 Proof of Theorem 1.6

First, we introduce some notation. Let $X = \{x_1, \ldots, x_k\}$ be a set of $k \geq 1$ vertices and let y be a vertex not contained in X. By a (y, X)-*fan* we mean a graph F that is the union of k paths $P_1, \ldots, P_k$ such that P_i is a (y, x_i)-path, that is a path between y and x_i, where $i = 1, \ldots, k$, and $V(P_i) \cap V(P_j) = \{y\}$ where $1 \leq i < j \leq k$.

For the proof of Theorem 1.6 we need the following well known consequence of Menger's theorem.

Lemma 2.1 *Let G be a k-connected graph where $k \geq 1$, let X be a set of k vertices of G and let y be a vertex of G not contained in X. Then G contains a (y, X)-fan.* □

By a (k, ℓ, t)-system we mean a triple $(X, Y, (F_y)_{y \in Y})$ such that the following conditions hold.

(a) X and Y are disjoint vertex sets with $|X| = k$ and $|Y| \geq \ell$.

(b) For every $y \in Y$, F_y is a (y, X)-fan with $|E(F_y)| \leq t$.

Clearly, a (k, ℓ, t)-system only exists for $t \geq k$. The proof of Theorem 1.6 is mainly based on the following result.

Lemma 2.2 *Let k, r be positive integers. Then for every integer $t \geq k$ there is an integer $\ell = \ell(k, r, t)$ such that for every (k, ℓ, t)-system $(X, Y, (F_y)_{y \in Y})$ the graph $H = \bigcup_{y \in Y} F_y$ contains a subdivision of $K_{k,r}$.*

Proof. We prove the existence of $\ell(k, r, t)$ by induction on $t \geq k$. For $t = k$, we claim that $\ell(k, r, k) = r$ has the desired property. To see this, let $(X, Y, (F_y)_{y \in Y})$ be a (k, r, k)-system. Then $|E(F_y)| = k$ and, therefore, F_y is a star. Consequently, $H = \bigcup_{y \in Y} F_y$ is a $K_{k,r}$. This proves the claim.

Now, let $t > k$ and suppose that $\ell(k, r, t-1)$ exists. Define $\ell(k, r, t) = Lr$, where

$$L = (\ell(k, r, t-1) \cdot k + 1)(t - k + 1)k.$$

To show that $\ell = \ell(k, r, t)$ has the desired property, consider a (k, ℓ, t)-system $(X, Y, (F_y)_{y \in Y})$ and the coresponding graph $H = \bigcup_{y \in Y} F_y$. We have to show that H contains a subdivision of $K_{k,r}$. Let G be the auxiliary graph with vertex set Y where two distinct vertices y and y' of G are adjacent if and only if

$$(V(F_y) \cap V(F_{y'})) \setminus X \neq \emptyset.$$

If G contains an independent set $Z \subseteq Y$ with r vertices, then, clearly, the graph $H' = \bigcup_{y \in Z} F_y$ is a subdivision of $K_{k,r}$ that is contained in H. If the indpendence number of G is smaller than r, then, because of $|V(G)| \geq Lr$, the graph G contains a vertex y_0 of degree at least L and we argue as follows. For $x \in X$, let P_x denote the (y_0, x)-path of the (y_0, X)-fan F_{y_0} and let $\tilde{P}_x = P_x - x$. Since F_{y_0} has at most t edges and $|X| = k$, we infer that $|V(\tilde{P}_x)| \leq t - k + 1$ for every $x \in X$. Furthermore, since y_0 has degree at least L in G, we conclude that there is a vertex $x_0 \in X$ such that

$$V(F_y) \cap V(\tilde{P}_{x_0}) \neq \emptyset$$

holds for at least L/k vertices $y \in Y \setminus \{y_0\}$. Let N denote the set of all these vertices and let $\tilde{N} = N \setminus V(\tilde{P}_{x_0})$. Then

$$|\tilde{N}| \geq \frac{L}{k} - (t - k + 1) = \ell(k, r, t-1) \cdot k \cdot (t - k + 1).$$

Consequently, there exists a vertex $u \in V(\tilde{P}_{x_0})$ and a subset N' of $\tilde{N}$ with

$$|N'| \geq |\tilde{N}|/(t - k + 1) \geq \ell(k, r, t-1) \cdot k$$

such that $u \in V(F_y)$ for every $y \in N'$. Finally, we conclude that there is a a vertex x' of X and a subset Y' of N' with $|Y'| \geq |N'|/k \geq \ell(k, r, t-1)$

such that, for every $y \in Y'$, the vertex u belongs to the (y, x')-path of the (y, X)-fan F_y. Now, let $X' = X - \{x'\} \cup \{u\}$ and, for $y \in Y'$, let F'_y denote the (y, X')-fan obtained from F_y by deleting all vertices of the (u, x')-path of F_y beside the vertex u. Then, since u is not contained in $X \cup Y'$, we have $|E(F'_y)| \le |E(F_y)| - 1 \le t - 1$ and, therefore, $(X', Y', (F'_y)_{y \in Y'})$ is a $(k, \ell(k, r, t-1), t-1)$-system. Hence the induction hypothesis implies that $H' = \bigcup_{y \in Y'} F'_y$ contains a subdivision of $K_{k,r}$. Clearly, H' is a subgraph of H. This completes the proof of Lemma 2.2. □

Proof of Theorem 1.6. Let $k, r, s \in \mathbb{N}$. We have to show that there is an integer $n = n(k, r, s)$ such that every k-conected graph of order at least n contains either an induced path of length s or a subdivision of $K_{k,r}$.

Since every k-connected graph of order at least $k+1$ contains an induced path of legth 1, we have $n(k, r, 1) = k + 1$. Now suppose $s \ge 2$. Define $t = k(s-1)$ and

$$n(k, r, s) = k + \ell(k, r, t)[k(s-2) + 1]$$

where $\ell(k, r, t)$ is the function from Lemma 2.2. Let G be a k-connected graph with $|V(G)| \ge n(k, r, s)$. Suppose that G does not contain an induced path of lenght s. Then we apply Lemma 2.2 to show that G contains a subdivision of $K_{k,r}$. First, choose a set X of k vertices in G. Now, consider an arbitrary vertex $y \in V(G) \setminus X$. By Lemma 2.1, G contains a (y, X)-fan. Consequently, there is a (y, X)-fan F_y in G such that, for every $x \in X$, the (y, x)-path of F_y is an induced path in G. We call such a (y, X)-fan *strong*. Clearly, if F_y is a strong (y, X)-fan, then $|E(F_y)| \le k(s-1) = t$ and $|V(F_y) \setminus X| \le t + 1 - k = k(s-2) + 1$. Since $|V(G) \setminus X| \ge n(k, r, s) - k \ge \ell(k, r, t)[k(s-2) + 1]$, we then conclude that there exists a vertex set $Y \subseteq V(G) \setminus X$ with $|Y| \ge \ell(k, r, t)$ such that, for every $y \in Y$, the graph G contains a strong (y, X)-fan F_y. Therefore, $(X, Y, (F_y)_{y \in Y})$ is a $(k, \ell(k, r, t), t)$-system and, by Lemma 2.2, the subgraph $H = \bigcup_{y \in Y} F_y$ of G contains a subdivision of $K_{k,r}$. This completes the proof of Theorem 1.6. □

References

[1] J.A. Bondy, S.C. Locke, Relative length of paths and cycles in 3-connected graphs, *Discrete Math.* **33** (1981) 111–122.

[2] R. Diestel, *Graph Theory*, Springer, 1997.

[3] Z. Gao, X. Yu, Convex programming and circumference of 3-connected graphs of low genus, *J. Combin. Theory Ser. B* **69** (1997) 39–51.

[4] B. Jackson, N. Wormald, Longest cycles in 3-connected planar graphs, *J. Combin. Theory Ser. B* **54** (1992) 291–321.

[5] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, *J. Combin. Theory Ser. B* **57** (1993) 239–257.