Hajós theorem for colorings of edge-weighted graphs

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Abstract

Hajós theorem states that every graph with chromatic number at least k can be obtained from the complete graph K_k by a sequence of simple operations such that every intermediate graph also has chromatic number at least k. Here, Hajós theorem is extended in three slightly different ways to colorings and circular colorings of edge-weighted graphs. These extensions shed some new light on the Hajós theorem and show that colorings of edge-weighted graphs are most natural extension of usual graph colorings.

1 Introduction

A graph is *k-critical* if its chromatic number is k but every proper subgraph has smaller chromatic number. Critical graphs play an important role in the theory of graph colorings. However, not much is known about them. On the other hand, there is a very simple inductive construction due to Hajós [2] which gives rise to all k-critical graphs. More precisely, Hajós theorem states that every k-critical graph can be obtained from copies of the complete graph K_k by applying a finite sequence of the following two operations:

(a) Identify two nonadjacent vertices.

(b) Take two graphs G_1, G_2 constructed by these operations, delete an edge $u_i v_i$ in G_i, $i = 1, 2$, identify u_1 with u_2 and add the edge $v_1 v_2$.

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In [5], it was asked what an analogue of the Hajós theorem would be for circular colorings. An answer was obtained by Zhu in [6]. However, that result needs additional, more complicated operations.

It is shown in this paper that the Hajós theorem has very natural and simple generalizations in the case of edge-weighted graphs, both for the usual chromatic number and the circular chromatic number. Among two different extensions of Hajós theorem for the circular chromatic number, the one which is a full analogue of the Hajós theorem for weighted graphs (cf. Corollary 6.5), may need an infinite sequence (a limiting process) of operations instead of a finite sequence. We do not know if this is really necessary, but if it is, this result would shed new light on the Hajós theorem. May this be a reason why no nontrivial applications of this celebrated theorem are known?

2 The channel assignment problem

The channel assignment problem is a graph coloring problem which generalizes usual colorings to edge-weighted graphs as described below [3].

A *weighted graph* is a pair $G = (V, A)$, where V is the vertex set, and $A : V \times V \to \mathbb{R}^+ \cup \{0\}$ are the edge-weights. Let E denote the set of all unordered pairs uv ($u \neq v$) of vertices for which $A(u, v) \neq 0$. For $u, v \in V$, $a_{uv} = A(u, v)$ denotes the weight of the edge uv of G. In this paper, it is also assumed that the edge-weights are symmetric, i.e., $a_{uv} = a_{vu}$, and that there are no loops, i.e., $a_{vv} = 0$ for every $v \in V$.

Let $G = (V, A)$ and $G' = (V', A')$ be weighted graphs with $V' \subseteq V$. We say that G' is a *subgraph* of G, $G' \subseteq G$, if $a'_{uv} \leq a_{uv}$ for every $u, v \in V'$. It is a *proper subgraph* if either $V' \neq V$ or there exist $u, v \in V'$ such that $a'_{uv} < a_{uv}$.

The *chromatic number* $\chi(G)$ (also known as the *span*) of the weighted graph G is equal to the minimum real number r such that there exists a mapping (called an *r-coloring*) $c : V \to [1, r]$ (where $[1, r]$ denotes the closed interval from 1 to r) such that for any two adjacent vertices $u, v \in V$, $|c(u) - c(v)| \geq a_{uv}$.

The problem of determining the span of a weighted graph with integer edge-weights is known as the *channel assignment problem* since it has applications in assigning channels and frequencies to radio or mobile telephony transmitters. The following lemma shows that we may only consider colorings for which $c(v)$ is an integer for every $v \in V$.

Lemma 2.1 *Suppose that $c : V \to [1, r]$ is an r-coloring of a weighted graph*

G with integer edge-weights. For $v \in V$, let $c'(v) = \lfloor c(v) \rfloor$. Then c' is a $\lfloor r \rfloor$-coloring of G.

Proof. It is easy to see that $|c'(u) - c'(v)| \geq \lfloor |c(u) - c(v)| \rfloor \geq \lfloor a_{uv} \rfloor = a_{uv}$. This completes the proof. □

The channel assignment problem is related to some other problems on graphs. For example:

(a) If all edge-weights are equal to 1, then $\chi(G)$ is the usual chromatic number of G.

(b) Let G be an arbitrary (unweighted) graph with vertex set V. Let K_G be the complete graph with the same vertex set as G and edge-weights 1 (for edges of G) and 2 (for nonedges of G). Then $\chi(K_G) = |V| + \mathrm{la}(G) - 1$, where $\mathrm{la}(G)$ is the *linear arboricity* of G, i.e., the minimum number of paths whose vertex sets partition $V(G)$. In particular, $\chi(K_G) = |V|$ if and only if G has a hamiltonian path. This example shows that computation of the weighted circular chromatic number is NP-hard even for complete graphs with edge-weights 1 and 2 only.

We refer to [3] for further details about the channel assignment problem.

3 Circular chromatic number of weighted graphs

The theory of circular colorings of graphs has become an important branch of chromatic graph theory with many interesting results, leading to new methods and exciting new results. We refer to the survey article by Zhu [5]. An extension of circular colorings to weighted graphs was recently introduced by the author [4].

For a positive real number p, denote by $S_p \subset \mathbb{R}^2$ the circle with radius $\frac{p}{2\pi}$ (hence with perimeter p) centered at the origin of $\mathbb{R}^2$. In the obvious way, we can identify the circle S_p with the set $\mathbb{R}/p\mathbb{Z}$ of real number modulo p. For $x, y \in S_p$, let us denote by $S_p(x, y)$ the arc on S_p from x to y in the clockwise direction, and let $\mathrm{d}(x, y)$ denote the length of this arc.

Let $G = (V, A)$ be an edge-weighted graph with at least one edge. A *circular p-coloring* of G is a function $c : V \to S_p$ such that for every edge $uv \in E$, $\mathrm{d}(c(u), c(v)) \geq a_{uv}$. Since $\mathrm{d}(c(u), c(v)) + \mathrm{d}(c(v), c(u)) = p$, a necessary condition for existence of a circular p-coloring is that

$$p \geq 2 \max\{a_{uv} \mid u, v \in V\}. \tag{1}$$

The *circular chromatic number* $\chi_c(G)$ of a weighted graph G is the infimum of all real numbers p for which there exists a circular p-coloring of G. It is known [4] that the infimum is attained, i.e., there exists a circular $\chi_c(G)$-coloring of G.

The circular chromatic number of weighted graphs generalizes some other graph invariants and can be used as a model for several well-known optimization problems:

(a) If all edge-weights are equal to 1, then $\chi_c(G)$ is the usual circular chromatic number of G (cf., e.g., [5]).

(b) If there is a function $f : V \to \mathbb{R}^+$, and weights of edges are defined as $a_{uv} = f(u)+f(v)$, then we get the notion of weighted circular colorings that were studied by Deuber and Zhu [1].

(c) Let G be an arbitrary (unweighted) graph with vertex set V. Let K_G be the complete graph with the same vertex set as G and edge-weights 1 (for edges of G) and 2 (for nonedges of G). Then $\chi_c(K_G) = |V|$ if and only if G has a hamiltonian cycle, and $\chi_c(K_G) \le |V| + 1$ if and only if G has a hamiltonian path. This example shows that computation of the weighted circular chromatic number is NP-hard even for complete graphs with edge-weights 1 and 2 only.

(d) Let $D = [d_{uv}]_{u,v\in V}$ be the cost matrix for a metric traveling salesman problem (MTSP), i.e., D is a nonnegative matrix that satisfies the triangular inequality. Then every circular p-coloring of the weighted complete graph K_V (with edge-weights D) determines a tour of the traveling salesman of cost $\le p$, and vice versa. Therefore, $\chi_c(K_V)$ is the optimum for the considered MTSP.

The notion of the circular chromatic number thus generalizes several well-known optimization problems and hence introduces the possibility to apply tools from one area into another one. As the edge-weights are not discrete integer values, one may also get use of some tools from continuous optimization.

4 Hajós theorem for the channel assignment problem

In this section it is assumed that all edge-weights are integers. Let p be a positive integer. We consider the set $\mathcal{G}(p)$ of all weighted graphs with integer edge-weights whose chromatic number is at least p. It is clear that the following two operations never decrease the chromatic number:

(a) Identify two nonadjacent vertices $u, v \in V$. Let w be the new vertex. The resulting weighted graph $G' = (V', A')$ has vertex set $V' = (V \setminus \{u, v\}) \cup \{w\}$ and the same edge-weights as G except that the weights of edges incident with w are $a'_{wz} = \max\{a_{uz}, a_{vz}\}$.

(b) Increase some edge-weight (possibly from 0 to a positive value in which case a new edge is added) or add a new vertex. By repeating this operation, every weighted graph which contains G as a subgraph can be obtained.

There is a third operation which combines two graphs from $\mathcal{G}(p)$ and gives rise to a new graph in $\mathcal{G}(p)$. For $i = 1, 2$, let $G_i = (V_i, A^{(i)})$ be a weighted graph, and let $u_i v_i$ be an edge of G_i. Let us identify u_1 and u_2 in the disjoint union of G_1 and G_2 into a new vertex u, then delete the edges uv_1 and uv_2, and finally, add the edge $v_1 v_2$ with weight $a_{v_1 v_2} = a^{(1)}_{u_1 v_1} + a^{(2)}_{u_2 v_2} - 1$. The operation is represented in Figure 1. The resulting weighted graph G is called the *Hajós sum* of G_1 and G_2.

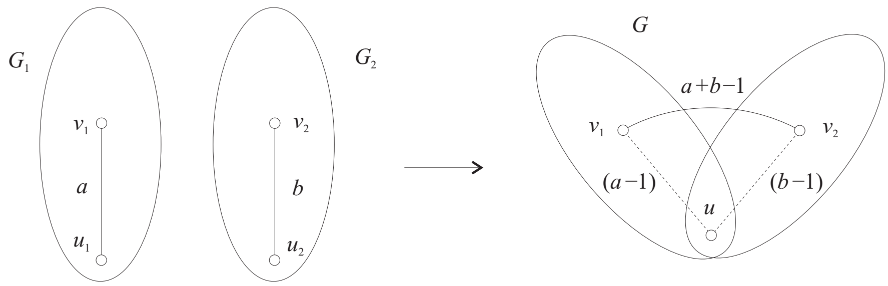

Figure 1: The (weak) Hajós sum of two graphs

If we leave the edges uv_1 and uv_2 in G having weights $a^{(1)}_{u_1 v_1} - 1$ and $a^{(2)}_{u_2 v_2} - 1$, respectively, we get the *weak Hajós sum*. In Figure 1, the edge-weights for the weak sum are given in parenthesis. All results stated in this paper carry over to the cases when the usual Hajós sum is replaced with the weak sum.

Lemma 4.1 *Let G be the Hajós sum of weighted graphs G_1 and G_2. Then $\chi(G) \geq \min\{\chi(G_1), \chi(G_2)\}$.*

Proof. If $c : V \to \{1, 2, \ldots, r\}$ is an r-coloring of G, $r = \chi(G)$, then $|c(v_1) - c(v_2)| \geq a_{v_1 v_2} = a^{(1)}_{u_1 v_1} + a^{(2)}_{u_2 v_2} - 1$. Therefore, either $|c(u) - c(v_1)| \geq$

$a^{(1)}_{u_1v_1}$ or $|c(u) - c(v_2)| \geq a^{(2)}_{u_2v_2}$. Assuming the former, the restriction of c to V_1 determines an r-coloring of G_1. This shows that $\chi(G_1) \leq \chi(G)$. □

A weighted graph $G \in \mathcal{G}(p)$ is *p-critical* if every proper subgraph of G (with integer weights) has chromatic number less than p. Observe that this implies that $\chi(G) = p$. Let $\mathcal{C}(p)$ be the set of all p-critical weighted graphs. Clearly, every graph in $\mathcal{G}(p)$ contains a subgraph which is in $\mathcal{C}(p)$.

Denote by $\mathrm{MTS_I}(p)$ the set of all integer-weighted complete graphs G with $\chi(G) = p$ whose edge-weights satisfy the triangular inequality. Examples of graphs in $\mathrm{MTS_I}(p) \cap \mathcal{C}(p)$ are complete graphs K_k with all edge-weights equal to the same integer α, where $p - 1 = \alpha(k-1)$. More complicated examples are obtained as follows. Start with a critical complete graph $G_0 \in \mathrm{MTS_I}(p_0)$ and replace the i^{th} vertex v_i of G_0 $(i = 1, \dots, n)$ by the complete graph K_{k_i} with unit edge-weights and let all edges between K_{k_i} and K_{k_j} have the same weight as the edge v_iv_j in G_0. The graph G thus obtained is in $\mathrm{MTS_I}(p) \cap \mathcal{C}(p)$ where $p = p_0 + k_1 + \cdots + k_n - n$.

Let $\mathcal{H}(p)$ be the set of weighted graphs that can be obtained from graphs in $\mathrm{MTS_I}(p)$ by a sequence of identifications of nonadjacent vertices and Hajós sums. These graphs will be called *Hajós constructible graphs*.

Theorem 4.2 *Every graph in $\mathcal{G}(p)$ contains a subgraph that is in $\mathcal{H}(p)$, i.e., $\mathcal{C}(p) \subseteq \mathcal{H}(p)$.*

Proof. Suppose that the theorem is not true. Let G be a counterexample with minimum number of vertices and, subject to this constraint, with maximum sum of edge-weights. Such a counterexample exists since every graph with an edge of weight $\geq p - 1$ contains as a subgraph the complete graph on 2 vertices with the edge of weight $p - 1$ which is a member of $\mathrm{MTS_I}(p)$.

Suppose first that G has vertices w, v_1, v_2 such that the triangular inequality does not hold: $a_{v_1v_2} > a_{wv_1} + a_{wv_2}$. For $i = 1, 2$, let G_i be the graph obtained from G by increasing the weight a_{wv_i} by 1. By the maximality of G, G_i contains a Hajós constructible subgraph G'_i. Since G does not contain such a subgraph, G'_i contains the edge wv_i, and its weight in G'_i is $a_{wv_i} + 1$.

Let G''_i be a copy of G'_i such that G''_1 and G''_2 are disjoint. Let G'' be the Hajós sum of graphs G''_1 and G''_2 with respect to the edges wv_1 and wv_2. If t_1 and t_2 are vertices of G'' that correspond to the same vertex $t \in V(G'_1) \cap V(G'_2) \setminus \{w\}$, then t_1 and t_2 are nonadjacent in G''. Therefore, we may identify all such pairs t_1, t_2 of nonadjacent vertices in G''. The resulting graph is Hajós constructible. It is easy to see that it is isomorphic to a subgraph of G. This contradiction shows that edge-weights in G satisfy the triangular inequality.

Triangular inequality in G implies that any two nonadjacent vertices have the same neighbors. Therefore, G is a complete multipartite graph. Let $V_1, \ldots, V_k$ be the partition of V into maximal independent sets. If vu and vw are edges of G incident with the same vertex v such that $uw \notin E$, then the triangular inequality shows that $a_{vu} = a_{vw}$. Therefore, all edges between two partite sets V_i, V_j have the same weight. Let G' be the induced subgraph obtained from G by taking one vertex from each partite class V_i, $i = 1, \ldots, k$. By the conclusions made above, any coloring of G' gives rise to a coloring of G with the same span. Therefore, $G' \in \mathcal{G}(p)$. Moreover, $G' \in \mathrm{MTS_I}(p')$ where $p' = \chi(G) \geq p$.

We claim that G' contains a subgraph $G'' \in \mathrm{MTS_I}(p)$. The claim is clear if G' has all edge-weights equal to 1 or if $p' = p$. Next, suppose that $p' > p$ and that the largest edge-weight is at least 2. Then the decrease of the largest edge-weight in G yields a subgraph of G which is a complete graph, satisfies the triangular inequality, and has chromatic number at least p. By repeating the decrease of a largest edge-weight one by one, we either get all edge-weights equal to 1, or decrease the chromatic number to p. In each case, the claim is established. □

Theorem 4.2 answers a question of McDiarmid stated in [3].

5 Hajós theorem for the integer circular chromatic number

Hajós theorem for circular chromatic number has been recently obtained by Zhu [6]. For that purpose, three new, rather complicated operations are introduced instead of the Hajós sum. In this paper two versions of the Hajós theorem for the circular chromatic number are obtained. The first one, presented in this section, is a weaker form, analogous to Theorem 4.2. The other one is presented in the next section.

Let us first observe that the Hajós sum of two graphs can have smaller circular chromatic number as the starting graphs. For example, the Hajós sum of two 3-cycles gives rise to the 5-cycle (all edge-weights are equal to 1). While $\chi_c(C_3) = 3$, the resulting graph C_5 has circular chromatic number equal to $\frac{5}{2}$. It will be discussed in the next section how to overcome this trouble. In this section we consider only the case when p is an integer and consider the class $\mathcal{G}'(p)$ of all weighted graphs with integer edge-weights whose circular chromatic number is strictly larger than $p - 1$. In this class, the behavior is similar as for usual colorings. This is due to the following lemma.

Lemma 5.1 *Suppose that p is an integer and that $c : V \to \mathbb{R}/p\mathbb{Z}$ determines a circular p-coloring of a weighted graph G with integer edge-weights. For $v \in V$, let $c'(v) = \lfloor c(v) + \frac{1}{2} \rfloor$. Then c' is a circular p-coloring of G.*

Proof. It is easy to see that the circular distance d' of $c'(u)$ and $c'(v)$ is strictly greater than $d - 1$, where d is the circular distance between $c(u)$ and $c(v)$. Moreover, if d is an integer, then $d' = d$. Since d' is an integer, $d' \geq \lfloor d \rfloor \geq \lfloor a_{uv} \rfloor = a_{uv}$. This completes the proof. □

Lemma 5.2 *Let G be the Hajós sum of graphs $G_1, G_2 \in \mathcal{G}'(p)$. Then $G \in \mathcal{G}'(p)$.*

Proof. Suppose that $\chi_c(G) \leq p-1$. By Lemma 5.1, G has a circular $(p-1)$-coloring whose colors are integers. This implies, in the same way as in the proof of Lemma 4.1, that either G_1 or G_2 has a circular $(p-1)$-coloring, a contradiction. □

A weighted graph $G \in \mathcal{G}'(p)$ is *critical* if every proper subgraph of G with integer edge-weights has circular chromatic number at most $p-1$. Let $\mathcal{C}'(p)$ be the set of all critical weighted graphs. Clearly, every graph in $\mathcal{G}'(p)$ contains a subgraph which is in $\mathcal{C}'(p)$. It is also easy to see that every $G \in \mathcal{C}'(p)$ has $\chi_c(G) = p$, except the graph K on two vertices and its edge of weight $\lceil p/2 \rceil$, where p is odd.

Denote by $\mathrm{MTS}'_{\mathrm{O}}(p)$ the set of all weighted complete graphs with integer edge-weights satisfying the triangular inequality and whose circular chromatic number is equal to p. These graphs correspond to the metric traveling salesman problems whose shortest hamiltonian cycle has length p. Additionally, we shall assume that $\mathrm{MTS}'_{\mathrm{O}}(p)$ contains the complete graph K of order 2 with its edge of weight $\lceil p/2 \rceil$ (even though $\chi_c(K) = p+1$ when p is odd).

Let $\mathcal{H}'(p)$ be the set of weighted graphs that can be obtained from graphs in $\mathrm{MTS}'_{\mathrm{O}}(p)$ by a sequence of identifications of nonadjacent vertices and (weak) Hajós sums. These graphs are said to be (*weakly*) *Hajós constructible*.

Having Lemma 5.1 and Lemma 5.2, the proof of Theorem 4.2 can be used also for graphs in $\mathcal{G}'(p)$. Let us observe that the addition of the graph K in $\mathrm{MTS}'_{\mathrm{O}}(p)$ (when p is odd) was necessary for the proof to work.

Theorem 5.3 *Every graph in $\mathcal{G}'(p)$ contains a subgraph that is in $\mathcal{H}'(p)$, i.e., $\mathcal{C}'(p) \subseteq \mathcal{H}'(p)$.*

Theorem 5.3 for the circular chromatic number of unweighted simple graphs is equivalent to the usual Hajós theorem. This follows from the fact that every simple graph G satisfies $\chi(G) < \chi_c(G) \leq \chi(G)$ and since the Hajós sum gives rise to a simple graph if and only if both factors are simple, hence the only graph in $\mathrm{MTS}'_{\mathrm{O}}(p)$ that is used in Hajós construction of G is the complete graph K_p.

6 Hajós theorem for general edge-weighted graphs

In this section, the circular chromatic number p and the edge-weights are arbitrary positive real numbers.

Theorem 5.3 is not the full analogy of Hajós theorem for circular colorings. As mentioned in the previous section, the (weak) Hajós sum does not preserve the property of the circular chromatic number being at least p. However, a modified operation described below behaves well in this respect.

For $i = 1, 2$, let $G_i = (V_i, A^{(i)})$ be a weighted graph, and let $u_i v_i$ be an edge of G_i. Let G be the weighted graph obtained in the same way as by taking the Hajós sum of G_1 and G_2, except that the edge $v_1 v_2$ gets weight $a_{v_1 v_2} = a^{(1)}_{u_1 v_1} + a^{(2)}_{u_2 v_2}$. The resulting weighted graph G is called the *strong Hajós sum* of G_1 and G_2.

Lemma 6.1 *Let G be the strong Hajós sum of weighted graphs G_1 and G_2. Then $\chi_c(G) \geq \min\{\chi_c(G_1), \chi_c(G_2)\}$.*

Proof. Let $c : V \to S_p$ be a circular p-coloring of G. Since $\mathrm{d}(c(v_1), c(v_2)) \geq a_{v_1 v_2}$, it follows that either for $i = 1$ or for $i = 2$, $\mathrm{d}(c(v_i), c(u)) \geq a^{(i)}_{v_i u_i}$. Therefore, the restriction of c to V_i determines a circular p-coloring of G_i. □

Let $\mathcal{G}_c(p)$ be the set of all weighted graphs whose circular chromatic number is at least p. A weighted graph $G \in \mathcal{G}_c(p)$ is *p-critical* if every proper subgraph of G has circular chromatic number less than p. Since χ_c is a continuous function of edge-weights, this implies that $\chi_c(G) = p$. Let $\mathcal{C}_c(p)$ be the set of all p-critical weighted graphs. Clearly, every graph in $\mathcal{G}_c(p)$ contains a subgraph which is in $\mathcal{C}_c(p)$.

Denote by $\mathrm{MTS}_{\mathrm{O}}(p)$ the set of all weighted complete graphs whose edge-weights satisfy the triangular inequality and whose circular chromatic number is equal to p. For $\varepsilon \geq 0$, let $\mathrm{M}_\varepsilon\mathrm{TS}_{\mathrm{O}}(p)$ be the set of all instances of weighted graphs with $\chi_c(G) = p$ whose edge-weights satisfy the ε-*triangular inequality*:

$$a_{uv} \le a_{uw} + a_{wv} + \varepsilon \qquad (u, v, w \in V).$$

Lemma 6.2 *Every weighted graph in $\mathcal{G}_c(p)$ has a subgraph that can be obtained from graphs in $\mathrm{M}_1\mathrm{TS}_{\mathrm{O}}(p)$ by identifying nonadjacent vertices and taking strong Hajós sums.*

The proof of Lemma 6.2 is similar to the proof of Theorem 4.2, and will therefore be omitted. The same proof can be used to verify the following result.

Lemma 6.3 *For every $\varepsilon > 0$, every graph in $\mathrm{M}_{2\varepsilon}\mathrm{TS}_{\mathrm{O}}(p)$ contains a subgraph that can be obtained from graphs in $\mathrm{M}_{\varepsilon}\mathrm{TS}_{\mathrm{O}}(p)$ by taking strong Hajós sums and identifying nonadjacent vertices.*

Lemma 6.2 and consecutive application of Lemma 6.3 yield:

Theorem 6.4 *For every $\varepsilon > 0$, every graph in $\mathcal{G}_c(p)$ contains a subgraph that can be obtained from graphs in $\mathrm{M}_{\varepsilon}\mathrm{TS}_{\mathrm{O}}(p)$ by taking strong Hajós sums and identifying nonadjacent vertices.*

The proof of Theorem 6.4 also shows that none of the graphs used or encountered when constructing the graph $G \in \mathcal{G}_c(p)$ of order n from graphs in $\mathrm{M}_{\varepsilon}\mathrm{TS}_{\mathrm{O}}(p)$ has more than $2n - 1$ vertices. The proof also yields a bound on the number of steps used in such a construction. As a corollary we get:

Corollary 6.5 *For every $\tau > 0$ and every $G \in \mathcal{G}_c(p)$, there exists a subgraph G' of G that can be obtained from graphs in $\mathrm{MTS}_{\mathrm{O}}(p - \tau)$ by taking strong Hajós sums and identifying nonadjacent vertices.*

Proof. Let ε be a positive real number, let n be the order of G, and let α be the minimum positive edge-weight of G. By Theorem 6.4, G has a subgraph H that is Hajós constructible from graphs $G_1, \ldots, G_m \in \mathrm{M}_{\varepsilon}\mathrm{TS}_{\mathrm{O}}(p)$. The proof of Theorem 6.4 shows that each G_i $(1 \le i \le m)$ is of order at most n, and that we may also assume that all nonzero edge-weights in G_i are greater or equal to α. If $\varepsilon < \alpha$ (which we assume), then this implies that every G_i is a complete multipartite graph.

Let $V_1, \ldots, V_r$ be the partite sets of G_i. The ε-triangular inequality applied to triples u, v, u' and u', v', v, respectively, implies that the weights of any two edges $uv, u'v'$ joining V_a and V_b ($1 \leq a < b \leq r$) do not differ too much:

$$|a_{uv} - a_{u'v'}| \leq 2\varepsilon. \tag{2}$$

Let K_i be the induced complete subgraph of G_i obtained by taking one vertex from each partite class of G_i. Then (2) easily implies that

$$\chi_c(K_i) \geq \chi_c(G_i) - 2n\varepsilon = p - 2n\varepsilon. \tag{3}$$

Let K_i' be the graph obtained from K_i by replacing each edge-weight w by $w' = \frac{\alpha(w+\varepsilon)}{\alpha+\varepsilon}$. Since K_i satisfies the ε-triangular inequality, it follows that K_i' satisfies the usual triangular inequality. It is also easy to see that $w' \leq w$, i.e., K_i' is a subgraph of K_i.

Suppose now that $\varepsilon \leq \tau/(2n + p/\alpha)$. Then the bound (3) implies that

$$\chi_c(K_i') \geq \frac{\alpha}{\alpha+\varepsilon}\,\chi_c(K_i) \geq p - 2n\varepsilon - \frac{p\varepsilon}{\alpha} \geq p - \tau.$$

By multiplying all edge-weights of K_i' by the constant $(p-\tau)/\chi_c(K_i') \leq 1$, a subgraph K_i'' of K_i' is obtained, and $K_i'' \in \mathrm{MTS}_{\mathrm{O}}(p-\tau)$.

Now, consider the constructing sequence for obtaining the subgraph H of G from $G_1, \ldots, G_m$ by using Hajós operations. In that sequence, replace each G_i by its subgraph K_i''. This gives rise to a constructing sequence for a subgraph G' of H and proves the theorem. □

Let us observe that the subgraph G' in Corollary 6.5 has $\chi_c(G') \geq p - \tau$.

Corollary 6.5 may be viewed as an "approximative" version of the Hajós theorem. The difficulty of establishing an "exact" version of this result may be related to the fact that today no nontrivial applications of the Hajós theorem are known.

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