WHAT IS...

a Graph Minor

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A minor of a graph G describes a substructure of G that is more general than a subgraph. If we take a subgraph of G and then contract some connected pieces in this subgraph to single points, the resulting graph is called a minor of G. Formally, a graph M is a *minor* of another graph G if G contains pairwise disjoint trees $T_v \subseteq G$, one for each vertex v of M, such that whenever u and v are adjacent vertices of M, there is an edge in G joining T_u and T_v. Closely related to this definition are the operations of deletion and contraction of an edge, each of which clearly produces a minor. Conversely, we can obtain any minor M of a graph G, by first deleting all edges except those in subgraphs T_v ($v \in V(M)$) and those connecting T_u, T_v for $uv \in E(M)$, and secondly contracting the edges in the trees T_v.

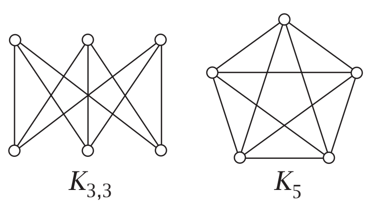

Figure 1. The Kuratowski graphs $K_{3,3}$ and K_5.

The first important appearance of graph minors is in the following version of the Kuratowski Theorem:

> *A graph G can be embedded in the plane (is planar) if and only if neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ is a minor of G.*

In topology, this theorem is usually expressed in an equivalent form saying that no subgraph of G is homeomorphic to K_5 or $K_{3,3}$.

Wagner proved in 1937 that the Four Color Theorem is equivalent to the statement that the vertices of every graph without K_5 minors can be colored with four colors so that adjacent vertices receive different colors. This led Hadwiger to conjecture that every K_n-minor-free graph is $(n-1)$-colorable. Hadwiger's Conjecture is still wide open and is one of the most challenging problems in graph theory.

At the beginning of the 1980s, Neil Robertson and Paul Seymour developed the theory of graph minors in a series of twenty long papers. It took twenty-one years [2, 3] to publish this seminal work, which had a tremendous impact not only on various branches of graph theory but also on many other areas, most notably theoretical computer science. The ultimate result of Robertson and Seymour was the proof of Wagner's Conjecture [3] that (finite) graphs are well-quasi-ordered by the graph minor relation, which is equivalent to the following statement:

> *In every infinite collection of graphs, there are two such that one is a minor of the other.*

A class $\mathcal{M}$ of graphs is *minor-closed* if for every $G \in \mathcal{M}$, all minors of G are also in $\mathcal{M}$. Examples of minor-closed classes are the collection of all planar graphs and, more generally, all graphs that can be embedded in a fixed surface. Every minor-closed class $\mathcal{M}$ can be described by specifying the set of all minor-minimal graphs that are not in $\mathcal{M}$ — these graphs are called the *forbidden minors* for $\mathcal{M}$.

The well-quasi-ordering of graphs with respect to the minor relation is equivalent to the following important result:

> *For every minor-closed family of graphs, the set of forbidden minors is finite.*

This is a far-reaching generalization of the Kuratowski Theorem with a slight disadvantage that its proof is nonconstructive—it does not yield a procedure for finding the forbidden minors for the family $\mathcal{M}$, nor does it yield a constructive bound on the number of forbidden minors.

Robertson and Seymour also proved that for every fixed graph M, there is an algorithm that has time complexity $O(n^3)$ and that for a given graph G of order n decides if M is a minor of G. This yields another powerful result:

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For every minor-closed family $\mathcal{M}$ of graphs there exists a cubic time algorithm for testing membership in $\mathcal{M}$—one simply checks if a given graph contains some forbidden minor for $\mathcal{M}$.

For instance, any of the following questions can be solved in $O(n^3)$ time, where n is the order of the input graph G:

(1) Let Σ be a fixed surface and G a graph. Can G be embedded in Σ?

(2) Is G linklessly embeddable in $\mathbb{R}^3$

(3) Is G knotlessly embeddable in $\mathbb{R}^3$?

A linkless embedding of a graph is one where no two cycles are linked, and a knotless embedding is one where no cycle is knotted. It is known that the set of forbidden minors for linkless embeddings consists of precisely seven graphs which can be obtained from the Petersen graph by a sequence of $Y\Delta$ and ΔY operations (the "Petersen family"). On the other hand, only sporadic examples of forbidden minors for knotless embeddings are known (one of them is K_7).

The set of forbidden minors is not known for embeddings of graphs in general surfaces. Exceptions are the sphere, whose forbidden minors are K_5 and $K_{3,3}$ by the Kuratowski Theorem, and the projective plane, which has 35 forbidden minors. Computer searches have shown that there are several thousands of forbidden minors for embeddings in the torus, and there is no indication how to obtain the complete list. Despite these difficulties, in some specific cases, like testing embeddability of graphs in a fixed surface, efficient algorithms have been devised using different approaches.

Another significant consequence of Robertson and Seymour's work concerns the related *structure*. Wagner proved that graphs that do not contain K_5 as a minor are precisely those that can be built from planar graphs and subgraphs of a certain cubic graph on 8 vertices by pasting these together in a tree-like manner. The paper "Graph Minors XVI" shows that a similar structure is obtained if we exclude any fixed graph M as a minor. Every graph that does not contain M as a minor can be built from building blocks, each of which is a graph that can be "nearly embedded" in some surface in which M cannot be embedded. Different building blocks intersect each other in bounded size subgraphs and form a tree-like structure. This is a "rough structure result" in the sense that none of its ingredients can be eliminated, and in the sense that the structure itself describes a minor-closed family such that certain other graphs are excluded from this family.

To be slightly more precise, each building block B is a graph that contains a "small" vertex set A such that $B - A$ can be written as the union of a graph G embedded in some surface S of bounded genus (so that M itself cannot be embedded in S), and a bounded number of other "path-like" graphs that are attached to G only along the face boundaries on S. This shows that up to "small" and "well-behaved" perturbations, graphs with minor M excluded are essentially 2-dimensional. This is a surprising fact—graph minors have been introduced with the aim of proving and generalizing the Kuratowski theorem, and yet, the structure theory of graph minors cannot be done without studying graphs embedded in surfaces.

The theory of graph minors has several powerful consequences for theoretical computer science, most notably in computational complexity and in the theory of algorithms. Besides the aforementioned consequences of well-quasi-ordering, the concept of "tree-like" structure, formally giving rise to the notion of the tree-width of graphs, has been widely applied in the theory of algorithms.

Extensive research is being carried out today to extend graph minor results to matroids. The goal in this area is to prove a conjecture of Gian-Carlo Rota, claiming that matroids representable over any finite field are well-quasi-ordered with respect to matroid minors. Part of this extensive project has already been accomplished by Jim Geelen, Bert Gerards, and Geoff Whittle.

In closing, let us briefly return to edge-deletions and edge-contractions since they are closely related to some other aspects of graph minors. In the 1950s, Tutte introduced a two-variable polynomial, now known as the *Tutte polynomial*, defined for a graph G recursively as follows. If G has no edges, then $T(G;x,y) = 1$. Otherwise, let e be an edge of G. If e is a loop, then $T(G;x,y) = y\,T(G-e;x,y)$; if e is a cutedge, then $T(G;x,y) = x\,T(G/e;x,y)$; otherwise $T(G;x,y) = T(G-e;x,y) + T(G/e;x,y)$. It can be shown that the Tutte polynomial is well-defined for arbitrary graphs. Its coefficients are nonnegative integers, and it turns out that $T(G;x,y)$ is essentially the same as the partition function of the Potts model, studied in theoretical physics in relation to phase transition and critical phenomena. The special case when $y = -1/x$ is equivalent to the Jones polynomial of an alternating link via a projection of the link in the plane and medial graph construction.

Proofs of graph minor results are usually hard and technical. This explains why most textbooks cover only the basic facts about minors. An exception is [1], which is also a good reference for further reading.

Further Reading

[1] R. Diestel, *Graph Theory*, 3rd Edition, Springer, 2005.

[2] N. Robertson and P. D. Seymour, Graph minors. I. Excluding a forest, *J. Combin. Theory Ser. B* **35** (1983) 39–61.

[3] ____ , Graph minors. XX. Wagner's conjecture, *J. Combin. Theory Ser. B* **92** (2004) 325–357.