Improved Upper Bounds on the Crossing Number

Vida Dujmović*
School of Computer Science, Carleton University, Ottawa, Canada
vida@cs.mcgill.ca

Ken-ichi Kawarabayashi†
National Institute of Informatics
2-1-2 Hitotsubashi, Chiyoda-ku Tokyo 101-8430, Japan
k_keniti@nii.ac.jp

Bojan Mohar‡
Department of Mathematics, Simon Fraser University, Burnaby, Canada; and Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia
mohar@sfu.ca

David R. Wood§
Departament de Matemática Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain
david.wood@upc.es

ABSTRACT

The crossing number of a graph is the minimum number of crossings in a drawing of the graph in the plane. Our main result is that every graph G that does not contain a fixed graph as a minor has crossing number $\mathcal{O}(\Delta n)$, where G has n vertices and maximum degree Δ. This dependence on n and Δ is best possible. This result answers an open question of Wood and Telle [*New York J. Mathematics*, 2007], who proved the best previous bound of $\mathcal{O}(\Delta^2 n)$.

In addition, we prove that every K_5-minor-free graph G has crossing number at most $2\sum_v \deg(v)^2$, which again is the best possible dependence on the degrees of G. We also study the convex and rectilinear crossing numbers, and prove an $\mathcal{O}(\Delta n)$ bound for the convex crossing number of bounded pathwidth graphs, and a $\sum_v \deg(v)^2$ bound for the rectilinear crossing number of $K_{3,3}$-minor-free graphs.

*Research supported by E.B. Carty Memorial Foundation.

†Research partly supported by JSPS Postdoctoral Fellowship for Research Abroad. Visiting Department of Mathematics, Simon Fraser University, Burnaby, B.C.

‡Research supported by an NSERC Dicsovery Grant, CRC program, and in part by ARRS, Research Program P1-0297.

§Research supported by a Marie Curie Fellowship of the European Community under contract MEIF-CT-2006-023865, and by the projects MEC MTM2006-01267 and DURSI 2005SGR00692.

SCG'08, June 9–11, 2008, College Park, Maryland, USA.

Categories and Subject Descriptors

G.2 [**DISCRETE MATHEMATICS**]: Graph Theory

General Terms

Algorithms, Theory

Keywords

graph drawing, crossing number, rectilinear crossing number, convex crossing number, graph minors, maximum degree, pathwidth, treewidth

1. INTRODUCTION

The *crossing number* of a graph[1] G, denoted by $\mathsf{cr}(G)$, is the minimum number of crossings in a drawing[2] of G in the plane; see [16, 33, 51] for surveys. The crossing number is an important measure of non-planarity of a graph [50], with applications in discrete and computational geometry

[1]We consider graphs G that are undirected, simple, and finite. Let $V(G)$ and $E(G)$ respectively be the vertex and edge sets of G. Let $|G| := |V(G)|$ and $\|G\| := |E(G)|$. For each vertex v of G, let $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ be the neighbourhood of v in G. The *degree* of v, denoted by $\deg_G(v)$, is $|N_G(v)|$. When the graph is clear from the context, we write $\deg(v)$. Let $\Delta(G)$ be the maximum degree of G.

[2]A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A drawing is *rectilinear* if each edge is a line-segment, and is *convex* if, in addition, the vertices are in convex position. A *crossing* is a point of intersection between two edges (other than a common endpoint). A drawing with no crossings is *crossing-free*. A graph is *planar* if it has a crossing-free drawing.

[32, 49], VLSI circuit design [3, 26, 27], and in several other areas of mathematics and theoretical computer science; see [50] for details. In information visualisation, one of the most important measures of the quality of a graph drawing is the number of crossings [37, 36, 38].

Computing the crossing number is $\mathcal{NP}$-hard [18], and remains so for simple cubic graphs [22, 35]. Moreover, the exact or even asymptotic crossing number is not known for specific graph families, such as complete graphs [42], complete bipartite graphs [29, 40, 42], and cartesian products [1, 5, 20, 41]. On the other hand, for every fixed k, Kawarabayashi and Reed [25] developed a linear-time algorithm that decides whether a given graph has crossing number at most k, and if this is the case, produces a drawing of the graph with at most k crossings.

Given that the crossing number seems so difficult, it is natural to focus on asymptotic bounds rather than exact values. The 'crossing lemma', conjectured by Erdős and Guy [16] and first proved by Leighton [26] and Ajtai et al. [2], gives such a lower bound. It states that every graph G with average degree greater than $6 + \epsilon$ has

$$\mathsf{cr}(G) \geq c_\epsilon \frac{\|G\|^3}{|G|^2}.$$

Other general lower bound techniques that arose out of the work of Leighton [26, 27] include the bisection/cutwidth method [14, 31, 47, 48] and the embedding method [46, 47].

Upper bounds on the crossing number of general families of graphs have been less studied, and are the focus of this paper. Obviously $\mathsf{cr}(G) \leq \binom{\|G\|}{2}$ for every graph G. A family of graphs has *linear* crossing number if $\mathsf{cr}(G) \leq c\,|G|$ for some constant c and for every graph G in the family. The following theorem of Pach and Tóth [34] shows that graphs of bounded genus[3] and bounded degree have linear crossing number.

Theorem 1.1 ([34]). *For every integer $\gamma \geq 0$, there are constants c and c', such that every graph G with orientable genus γ has crossing number*

$$\mathsf{cr}(G) \leq c \sum_{v \in V(G)} \deg(v)^2 \leq c' \Delta(G) \cdot |G|.$$

Böröczky et al. [9] extended Theorem 1.1 to graphs of bounded non-orientable genus. Djidjev and Vrt'o [15] greatly improved the dependence on γ in Theorem 1.1, by proving that $\mathsf{cr}(G) \leq c\gamma \cdot \Delta(G) \cdot |G|$. Wood and Telle [52] proved that bounded-degree graphs that exclude a fixed graph as a minor[4] have linear crossing number.

Theorem 1.2 ([52]). *For every graph H, there is a constant $c = c(H)$, such that every H-minor-free graph G has crossing number*

$$\mathsf{cr}(G) \leq c\,\Delta(G)^2 \cdot |G|.$$

Theorem 1.2 is stronger than Theorem 1.1 in the sense that graphs of bounded genus exclude a fixed graph as a minor, but there are graphs with a fixed excluded minor and arbitrarily large genus. On the other hand, Theorem 1.1 has better dependence on Δ than Theorem 1.2. For other recent work on minors and crossing number see [6, 7, 8, 17, 19, 21, 22, 30, 35].

Note that for any reasonably general class of graphs to have linear crossing number, excluding a fixed minor and bounding the maximum degree (as in Theorem 1.2) is unavoidable. For example, $K_{3,n}$ has no K_5-minor, yet its crossing number is $\Omega(n^2)$ [40, 29]. Conversely, bounded degree does not by itself guarantee linear crossing number. For example, a random cubic graph on n vertices has $\Omega(n)$ bisection width [10, 12], which implies that its crossing number is $\Omega(n^2)$ [14, 26].

Pach and Tóth [34] proved that the upper bound in Theorem 1.1 is best possible, in the sense that for all Δ and n, there is a toroidal graph with n vertices and maximum degree Δ whose crossing number is $\Omega(\Delta n)$. In Section 2 we extend this $\Omega(\Delta n)$ lower bound to graphs with no $K_{3,3}$-minor, no K_5-minor, and more generally, with no K_h-minor. Our main result is to prove a matching upper bound for all graphs excluding a fixed minor. That is, we improve the quadratic dependence on $\Delta(G)$ in Theorem 1.2 to linear.

Theorem 1.3. *For every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has crossing number*

$$\mathsf{cr}(G) \leq c\,\Delta(G) \cdot |G|.$$

For a graph G, let $D^2(G) := \sum_{v \in V(G)} \deg(v)^2$. While our upper bound in Theorem 1.3 is optimal in terms of $\Delta(G)$ and $|G|$, it remains open whether every graph excluding a fixed minor has $\mathcal{O}(D^2(G))$ crossing number, as is the case for graphs of bounded genus. Note that a $D^2(G)$ upper bound is stronger than a $\Delta(G) \cdot |G|$ upper bound. In particular, for every graph G with bounded average degree (such as graphs with bounded genus or those excluding a fixed minor),

$$D^2(G) \leq \Delta(G) \sum_{v \in V(G)} \deg(v) = 2\Delta(G) \cdot \|G\| \leq c\,\Delta(G) \cdot |G|.$$

Wood and Telle [52] conjectured that every graph excluding a fixed minor has crossing number $\mathcal{O}(D^2(G))$. In Section 4, we establish this conjecture for K_5-minor-free graphs, and prove the same bound on the rectilinear crossing number[5]

[3] Let $\mathbb{S}_\gamma$ be the orientable surface with $\gamma \geq 0$ handles. An *embedding* of a graph in $\mathbb{S}_\gamma$ is a crossing-free drawing in $\mathbb{S}_\gamma$. A 2-*cell embedding* is an embedding in which each region of the surface (bounded by edges of the graph) is an open disk. The (*orientable*) *genus* of a graph G is the minimum γ such that G has a 2-cell embedding in $\mathbb{S}_\gamma$. In what follows, by a *face* we mean the set of vertices on the boundary of the face. Let $F(G)$ be the set of faces in an embedded graph G. See the monograph by Mohar and Thomassen [28] for a thorough treatment of graphs on surfaces.

[4] Let vw be an edge of a graph G. Let G' be the graph obtained by identifying the vertices v and w, deleting loops, and replacing parallel edges by a single edge. Then G' is obtained from G by *contracting* vw. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. A family of graphs $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies that every minor of G is in $\mathcal{F}$. $\mathcal{F}$ is *proper* if it is not the family of all graphs. A deep theorem of Robertson and Seymour [45] states that every proper minor-closed family can be characterised by a finite family of excluded minors. Every proper minor-closed family is a subset of the H-minor-free graphs for some graph H. We thus focus on minor-closed families with one excluded minor.

[5] The *rectilinear crossing number* of a graph G, denoted by $\overline{\mathsf{cr}}(G)$, is the minimum number of crossings in a rectilinear drawing of G. The *convex crossing number*, denoted by $\mathsf{cr}^\star(G)$, is the minimum number of crossings in a convex drawing of G.

of $K_{3,3}$-minor-free graphs. In addition to these results, we establish in Section 5 optimal bounds on the convex crossing number of interval graphs, chordal graphs, and bounded pathwidth graphs.

It is worth noting that our proof is constructive, assuming a structural decomposition (Theorem 6.2) by Robertson and Seymour [44] is given. Demaine et al. [11] gave a polynomial-time algorithm to compute this decomposition. Consequently, our proof can be converted into a polynomial-time algorithm that, given a graph G excluding a fixed minor, finds a drawing of G with the claimed number of crossings.

2. LOWER BOUNDS

In this section we describe graphs that provide lower bounds on the crossing number. The constructions are variations on those by Pach and Tóth [34]. We include them here to motivate our interest in matching upper bounds in later sections.

LEMMA 2.1. *For all positive integers Δ and n, such that $\Delta \equiv 0 \pmod 4$ and $n \equiv 0 \pmod{5(\Delta/2-1)}$, there is a (chordal) $K_{3,3}$-minor-free graph G with $|G| = n$, $\Delta(G) = \Delta$, and*

$$\mathsf{cr}(G) = \frac{\Delta n}{40}\Big(1 + \frac{2}{\Delta-2}\Big) > \frac{\Delta n}{40}.$$

PROOF SKETCH. Start with K_5 as the base graph. For each edge vw of K_5, add $\Delta/4 - 1$ new vertices, each adjacent to v and w. The resulting graph G' is chordal and $K_{3,3}$-minor-free, $\Delta(G') = \Delta$, and $|G'| = 5(\Delta/2-1)$. Take $\frac{n}{5(\Delta/2-1)}$ disjoint copies of G' to obtain a $K_{3,3}$-minor-free graph G on n vertices and maximum degree Δ. Thus $\mathsf{cr}(G) = \mathsf{cr}(G')\frac{n}{5(\Delta/2-1)}$. A standard technique proves that $\mathsf{cr}(G') = (\Delta/4)^2$. Thus $\mathsf{cr}(G) = (\Delta/4)^2\frac{n}{5(\Delta/2-1)} = \frac{\Delta n}{40}(1+\frac{2}{\Delta-2})$, as claimed. □

A similar technique gives the following lemma.

LEMMA 2.2. *For every set $D = \{2, d_1, \dots, d_p\}$ of positive integers such that $d_i \equiv 0 \pmod 4$ for $i = 1, \dots, p$, there are infinitely many (chordal) $K_{3,3}$-minor-free graphs G such that the degree set of G is D and*

$$\mathsf{cr}(G) > \frac{D^2(G)}{200}.$$

PROOF. For each $d_i \in D \setminus \{2\}$, let $n_i = \frac{5}{2}d_i - 5$. By Lemma 2.1, there is a (chordal) $K_{3,3}$-minor-free graph G_i with five vertices of degree d_i and $n_i - 5$ vertices of degree 2, such that

$$\mathsf{cr}(G_i) > \frac{d_i n_i}{40} > \frac{5d_i^2 + (n_i-5)2^2}{200} = \frac{D^2(G_i)}{200}.$$

Every graph G created by taking one or more disjoint copies of each of $G_1, \dots, G_p$ is $K_{3,3}$-minor-free with degree set D, and $\mathsf{cr}(G) \ge \frac{1}{200}D^2(G)$. □

The above results generalize to K_h-minor-free graphs, for $h \ge 5$.

LEMMA 2.3. *For every integer $h \ge 5$ and every Δ such that $\Delta \equiv 0 \pmod{h-2}$ for $h \ge 6$ and $\Delta \equiv 0 \pmod 3$ for $h = 5$, there exists infinitely many K_h-minor-free graphs G with $\Delta(G) = \Delta$ and*

$$\mathsf{cr}(G) \ge c\, h\, \Delta \cdot |G|,$$

for some absolute constant c. Moreover, G is chordal for $h \ge 6$.

PROOF SKETCH. For $h = 5$, use $K_{3,3}$ as the starting graph. For $h \ge 6$, use K_{h-1}. The remaining arguments follow the proof of Lemma 2.1 and use the fact that $\mathsf{cr}(K_{3,3}) = 1$ and $\mathsf{cr}(K_{h-1}) \in \Theta(h^4)$. □

3. LINEAR BOUNDING FUNCTIONS

In this section we give some sufficient conditions for a graph to satisfy certain linear bounds on the crossing number. The derived bounds will be used in subsequent sections.

LEMMA 3.1. *Let X be a class of graphs closed under taking subdivisions. Suppose that*

$$\mathsf{cr}(G) \le c \sum_{vw \in E(G)} \deg(v)\deg(w)$$

for every graph $G \in X$. Then

$$\mathsf{cr}(G) \le 2c\, D^2(G)$$

for every graph $G \in X$.

PROOF. Let $G \in X$. Let G' be the graph obtained from G by subdividing every edge once. By assumption, $G' \in X$ and

$$\begin{aligned}
\mathsf{cr}(G') &\le c \sum_{vw \in E(G')} \deg(v)\deg(w) \\
&= c \sum_{vw \in E(G)} (2\deg(v) + 2\deg(w)) \\
&= 2c \sum_{vw \in E(G)} (\deg(v) + \deg(w)) \\
&= 2c \sum_{v \in V(G)} \deg(v)^2.
\end{aligned}$$

The result follows since $\mathsf{cr}(G) = \mathsf{cr}(G')$. □

We can also conclude a $\mathcal{O}(\Delta(G) \cdot |G|)$ bound from $\sum_{vw \in E(G)} \deg(v)\deg(w)$.

LEMMA 3.2. *Let G be a graph with bounded arboricity. In particular, every subgraph of G on n vertices has at most kn edges. Then*

$$\sum_{vw \in E(G)} \deg(v)\deg(w) \;\le\; 16k\cdot\Delta(G)\cdot\|G\| \;\le\; 16k^2\cdot\Delta(G)\cdot|G|.$$

PROOF. Let $i, j \ge 0$ be integers. Let

$$\begin{aligned}
\Delta_i &:= \Delta(G)/2^i \\
V_i &:= \{v \in V(G) : \Delta_{i+1} < \deg(v) \le \Delta_i\} \\
n_i &:= |V_i| \\
E_{i,j} &:= \{vw \in E(G) : v \in V_i, w \in V_j\} \\
e_{i,j} &:= |E_{i,j}| \;.
\end{aligned}$$

Let $S_i := \{j \geq 0 : n_j \leq n_i\}$. Thus

$$\begin{aligned}\sum_{vw \in E(G)} \deg(v)\deg(w) &\leq \sum_{i \geq 0}\sum_{j \in S_i}\sum_{vw \in E_{i,j}} \deg(v)\deg(w) \\ &\leq \sum_{i \geq 0}\sum_{j \in S_i} e_{i,j}\Delta_i\Delta_j \\ &\leq k\sum_{i \geq 0}\sum_{j \in S_i}(n_i + n_j)\Delta_i\Delta_j \\ &\leq 2k\sum_{i \geq 0}\sum_{j \geq 0} n_i\Delta_i\Delta_j \\ &\leq 2k\sum_{i \geq 0} n_i\Delta_i\sum_{j \geq 0}\Delta_j \ .\end{aligned}$$

Since $\sum_{j \geq 0}\Delta_j < 2 \cdot \Delta(G)$,

$$\sum_{vw \in E(G)} \deg(v)\deg(w) < 4k \cdot \Delta(G)\sum_{i \geq 0} n_i\Delta_i \ .$$

Observe that

$$2\|G\| = \sum_{i \geq 0}\sum_{v \in V_i} \deg(v) > \sum_{i \geq 0} n_i\Delta_{i+1} = \tfrac{1}{2}\sum_{i \geq 0} n_i\Delta_i \ .$$

Thus

$$\sum_{vw \in E(G)} \deg(v)\deg(w) < 16k \cdot \Delta(G) \cdot \|G\| \ .$$

□

4. DRAWINGS BASED ON PLANAR DECOMPOSITIONS

Let G and D be graphs, such that each vertex of D is a set of vertices of G (called a *bag*). For each vertex v of G, let $D(v)$ be the subgraph of D induced by the bags that contain v. Then D is a *decomposition* of G if:

- $D(v)$ is connected and nonempty for each vertex v of G, and
- $D(v)$ and $D(w)$ touch[6] for each edge vw of G.

Decompositions, when D is a tree, were introduced by Robertson and Seymour [43]. Diestel and Kühn [13] first generalised the definition for arbitrary graphs D.

Let D be a decomposition of a graph G. The *width* of D is the maximum cardinality of a bag. Let v be a vertex of G. The number of bags in D that contain v is the *spread* of v in D. The *spread* of D is the maximum spread of a vertex of G. A decomposition D of G is a *partition* if every vertex of G has spread 1. The *order* of D is the number of bags. D has *linear order* if $|D| \leq c\,|G|$ for some constant c. If the graph D is a tree, then the decomposition D is a *tree decomposition*. If the graph D is a path, then the decomposition D is a *path decomposition*. The decomposition D is *planar* if the graph D is planar.

A decomposition D of a graph G is *strong* if $D(v)$ and $D(w)$ intersect for each edge vw of G. The *treewidth* (*pathwidth*) of G, is 1 less than the minimum width of a strong tree (path) decomposition of G. Treewidth is particularly important in structural and algorithmic graph theory; see the surveys [4, 39].

[6]Let A and B be subgraphs of a graph G. Then A and B *intersect* if $V(A) \cap V(B) \neq \emptyset$, and A and B *touch* if they intersect or $v \in V(A)$ and $w \in V(B)$ for some edge vw of G.

Wood and Telle [52] showed that planar decompositions were closely related to crossing number. The next result improves a bound in [52] from $(p-1)\,\Delta(G)\,\|G\|$ to $(p-1)D^2(G)$.

LEMMA 4.1. *Every graph G with a planar partition H of width p has a rectilinear drawing in which each edge crosses at most $2\,\Delta(G)\,(p-1)$ other edges. The total number of crossings,*

$$\overline{\mathsf{cr}}(G) \leq (p-1)D^2(G).$$

PROOF. The following drawing algorithm is in [52]. By the Fáry-Wagner Theorem, H has a rectilinear drawing with no crossings. Let $\epsilon > 0$. Let $D_\epsilon(B)$ be the disc of radius ϵ centred at each bag B of H. For each edge BC of H, let $D_\epsilon(BC)$ be the union of all line-segments with one endpoint in $D_\epsilon(B)$ and one endpoint in $D_\epsilon(C)$. For some $\epsilon > 0$, we have $D_\epsilon(B) \cap D_\epsilon(C) = \emptyset$ for all distinct bags B and C of H, and $D_\epsilon(BC) \cap D_\epsilon(PQ) = \emptyset$ for all edges BC and PQ of H that have no endpoint in common. For each vertex v of G in bag B of H, position v inside $D_\epsilon(B)$ so that $V(G)$ is in general position (no three collinear). Draw every edge of G straight. Thus no edge passes through a vertex.Suppose that two edges e and f cross. Then e and f have distinct endpoints in a common bag, as otherwise two edges in H would cross. (The analysis that follows is new.) Say v_i is an endpoint of e and v_j is an endpoint of f, where $\{v_1, \dots, v_p\}$ is some bag with $\deg(v_1) \leq \dots \leq \deg(v_p)$. Without loss of generality $i < j$. Charge the crossing to v_j. The number of crossings charged to v_j is at most

$$\sum_{i<j} \deg(v_i) \cdot \deg(v_j) \leq (p-1)\deg(v_j)^2$$

So the total number of crossings is as claimed. □

Wood and Telle [52] proved that every $K_{3,3}$-minor-free graph has a planar partition of width 2. Thus Lemma 4.1 implies the following theorem.

THEOREM 4.2. *Every graph G with no $K_{3,3}$-minor has rectilinear crossing number*

$$\overline{\mathsf{cr}}(G) \leq D^2(G).$$

We now extend Lemma 4.1 from planar partitions to planar decompositions.

LEMMA 4.3. *Suppose that D is a planar decomposition of a graph G with width p, in which each vertex v of G has spread at most $s(v)$. Then G has crossing number*

$$\mathsf{cr}(G) \leq 4p\sum_{v \in V(G)} s(v) \cdot \deg(v)^2 \ .$$

Moreover, G has a drawing with the claimed number of crossings, in which each edge vw is represented by a polyline with at most $s(v) + s(w) - 2$ bends.

PROOF. For each vertex v of G, let $X(v)$ be a bag of D that contains v. For each edge vw of G, let $P(vw)$ be a minimum length path in D between $X(v)$ and $X(w)$, such that v or w is in every bag in $P(vw)$. Let G' be the subdivision of G obtained by subdividing each edge vw of G once for each internal bag in $P(vw)$. Then D defines a planar partition D' of G', where each original vertex v is in $X(v)$, and each division vertex is in the corresponding bag. We say a division vertex x of vw *belongs* to v and v *owns* x, if x corresponds to a bag in D that contains v. If x corresponds to a bag that contains both v and w, then arbitrarily choose v or w to be the owner of x.

Apply the drawing algorithm in Lemma 4.1 to the planar partition D' of G'. We obtain a rectilinear drawing of G', which defines a drawing of G since G' is a subdivision of G. Each edge vw of G is represented by a polyline with $\max\{|P(vw)|-2,0\}$ bends, which is at most $s(v)+s(w)-2$. We now bound the number of crossings in the drawing of G', which in turn bounds the number of crossings in the drawing of G.

Let $\preceq$ be a total order on $V(G)$ such that if $\deg(v) < \deg(w)$ then $v \prec w$ for all $v, w \in V(G)$.

Say edges e and f of G' cross. As proved in Lemma 4.1, e and f have distinct endpoints in a common bag B'. Let x and y be these endpoints of e and f respectively. Let v and w be the vertices of G that own x and y respectively. Without loss of generality, $v \preceq w$. Charge the crossing to the pair (w, B), where B is the bag in D corresponding to B'.

Consider a bag $B = \{v_1, \dots, v_p\}$ in D, where $v_1 \prec \cdots \prec v_p$. Thus $\deg(v_1) \leq \cdots \leq \deg(v_p)$. Consider a vertex $v_i \in B$. If $X(v_i) = B$ then $\deg(v_i)$ edges of G' are incident to v_i, which is the only vertex in B' that belongs to v_i. If $X(v_i) \neq B$ then there are at most $\deg(v_i)$ division vertices in B' that belong to v_i, and there are at most $2\deg(v_i)$ edges of G' incident to a division vertex in B' that belongs to v_i (since each division vertex has degree 2 in G'). Thus the number of crossings charged to (v_i, B) is at most

$$\sum_{j=1}^{i} 2\deg(v_j) \cdot 2\deg(v_i) \leq 4i \deg(v_i)^2 \leq 4p \deg(v_i)^2.$$

For each vertex v of G, since v is in at most $s(v)$ bags of D, the number of crossings charged to some pair (v, B) is at most $4p \cdot s(v) \cdot \deg(v)^2$. Hence the total number of crossings is at most

$$4p \sum_{v \in V(G)} s(v) \cdot \deg(v)^2.$$

□

Lemma 4.4. *Let D be a planar decomposition of a graph G, such that every bag in D is a clique in G, and every pair of adjacent vertices in G are in at most c common bags in D. Then*

$$\mathsf{cr}(G) \leq c \sum_{vw \in E(G)} \deg(v)\deg(w).$$

Proof. Draw G as in the proof of Lemma 4.3. We now count the crossings in G between edges vw and xy that have no common endpoint. Each crossing between vw and xy can be charged to a bag B that contains distinct vertices p and q, where $p \in \{v, w\}$ and $q \in \{x, y\}$. Since B is a clique, pq is an edge of G. Charge the crossing to the pair (pq, B). At most one crossing between vw and xy is charged to (pq, B). Thus at most $\deg(p)\deg(q)$ crossings are charged to (pq, B). Since p and q are in at most c common bags, the number of crossings charged to pq is at most $c\deg(p)\deg(q)$. Thus the total number of crossings between edges with no common endpoint is at most $c\sum_{pq}\deg(p)\deg(q)$. It is folklore that $\mathsf{cr}(G)$ equals the minimum, taken over all drawings of G, of the number of crossings between pairs of edges of G with no endpoint in common. Hence $\mathsf{cr}(G) \leq c\sum_{pq}\deg(p)\deg(q)$. □

Wood and Telle [52] constructed planar decompositions of K_5-minor-free graphs as follows.

Lemma 4.5 ([52]). *Let G be a K_5-minor-free graph. Then G has a set of at most $|G|-2$ edges E such that if V is the set of vertices of G that are not incident to an edge in E, then G has a planar decomposition D of width 2 with $V(D) = \{\{v\} : v \in V\} \cup \{\{v, w\} : vw \in E\}$ with no duplicate bags.*

Since the bags of D correspond to vertices and edges of G (with no duplicates) each vertex of G has spread $s(v) \leq \deg(v)$. Thus Lemmas 4.3 and 4.5 imply that every graph G with no K_5-minor has crossing number

$$\mathsf{cr}(G) \leq 8 \sum_{v \in V(G)} \deg(v)^3.$$

This result represents a qualitative improvement over the $\mathcal{O}(\Delta(G)^2|G|)$ bound in [52]. But we can do better. In particular, Lemmas 4.5 and 4.4 with $c = 1$ imply that

$$\mathsf{cr}(G) \leq \sum_{vw \in E(G)} \deg(v)\deg(w).$$

Thus Lemma 3.1 implies:

Theorem 4.6. *Every graph G with no K_5-minor has crossing number*

$$\mathsf{cr}(G) \leq 2D^2(G).$$

5. INTERVAL GRAPHS AND CHORDAL GRAPHS

A graph is *chordal* if every induced cycle is a triangle. An *interval graph* is the intersection graph of a set of intervals in $\mathbb{R}$. Every interval graph is chordal.

Theorem 5.1. *Every interval graph G has convex crossing number*

$$\begin{aligned}\mathsf{cr}^\star(G) &\leq \tfrac{1}{2}(\omega(G)-2)\sum_{v \in V(G)} \deg(v)(\deg(v)-1)\\ &\leq (\omega(G)-2)(\omega(G)-1)(\Delta(G)-1)|G|.\end{aligned}$$

Proof. Jamison and Laskar [23] proved that G is an interval graph if and only if there is a linear order $\preceq$ of $V(G)$ such that if $u \prec v \prec w$ and $uw \in E(G)$ then $uv \in E(G)$. Orient the edges of G left to right in $\preceq$. Position $V(G)$ on a circle in the order of $\preceq$, with the edges drawn straight. Say edges xy and vw cross. Without loss of generality, $x \prec v \prec y \prec w$. Thus $vy \in E(G)$. Charge the crossing to vy. Say the out-neighbours of v are $w_1, \dots, w_d$. The in-neighbourhood of each w_i is a clique including v. Hence each w_i has at most $\omega(G)-2$ in-neighbours to the left of v. Now v has $d-i$ neighbours to the right of w_i. Thus the number of crossings charged to vw_i is at most $(\omega(G)-2)(d-i)$. Hence the number of crossings charged to outgoing edges at v is at most $\frac{1}{2}(\omega(G)-2)(d-1)d$. Therefore the total number of crossings is at most $\frac{1}{2}\sum_v(\omega(G)-2)(d_v-1)d_v$, where d_v is the out-degree of v. The other claims follow since $\|G\| < (\omega(G)-1)|G|$. □

It is well known that the pathwidth of a graph G equals the minimum k such that G is a spanning subgraph of an interval graph G' with $\omega(G') \leq k+1$.

THEOREM 5.2. *Every graph G with pathwidth k has convex crossing number*

$$\mathsf{cr}^\star(G) \leq k^2 \cdot \Delta(G) \cdot |G|.$$

PROOF. G is a spanning subgraph of an interval graph G' with $\omega(G') \leq k+1$. Apply the drawing algorithm in the proof of Theorem 5.1 to G'. Say edges xy and vw of G cross. Without loss of generality, $x \prec v \prec y \prec w$. Thus $vy \in E(G')$. Charge the crossing to vy. Now v has at most $\Delta(G)$ neighbours in G to the right of y. The in-neighbourhood of y is a clique in G' including v. Hence y has at most k neighbours to the left of v. Thus the number of crossings charged to vy is at most $k \cdot \Delta(G)$. Since G' has less than $k \cdot |G|$ edges, the total number of crossings is at most $k^2 \cdot \Delta(G) \cdot |G|$. □

LEMMA 5.3. *Let D be an outerplanar decomposition of a graph G. Then G has a convex drawing such that if two edges e and f cross then some bag of D contains both an endpoint of e and an endpoint of f.*

PROOF. Assign each vertex v of G to a bag $B(v)$ that contains v. Fix a crossing-free convex drawing of D. Replace each bag B of D by the set of vertices of G assigned to B. Draw the edges of G straight. Consider two edges vw and xy of G. Thus there is a path P in D between $B(v)$ and $B(w)$ and every bag in P contains v or w. Similarly, there is a path Q in D between $B(x)$ and $B(y)$ and every bag in Q contains x or y. Now suppose that vw and xy cross. Without loss of generality, the endpoints are in the cyclic order (v,x,w,y). Thus in the crossing-free convex drawing of D, the vertices $(B(v),B(x),B(w),B(y))$ appear in this cyclic order. Since D is crossing-free, P and Q have a bag X of D in common. Thus X contains v or w, and x or y. □

THEOREM 5.4. *Every chordal graph G has convex crossing number*

$$\mathsf{cr}^\star(G) \leq \sum_{vw \in E(G)} \deg(v)\deg(w).$$

PROOF. It is well known that every chordal graph has a strong tree decomposition in which each bag is a clique. By Lemma 5.3, G has a convex drawing such that if two edges vw and xy of G cross then some bag B of D contains v or w, and x or y. Say B contains v and x. Since B is a clique, vx is an edge. Charge the crossing to vx. In every crossing charged to vx, one edge is incident to v and the other edge is incident to x. Since edges are drawn straight, no two edges cross twice. Thus the number of crossings charged to vx is at most $\deg(v)\deg(x)$. Hence the total number of crossings is as claimed. □

THEOREM 5.5. *Every chordal graph G with no $(k+2)$-clique (which includes every k-tree) has convex crossing number*

$$\mathsf{cr}^\star(G) \leq 16k^2 \cdot \Delta(G) \cdot |G|.$$

PROOF. It is well known that G has less than kn edges. Thus the claim follows from Lemma 3.2 and Theorem 5.4. □

6. EXCLUDING A FIXED MINOR

In this section we prove our main result (Theorem 1.3): for every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has a crossing number at most $c\,\Delta(G)\cdot|G|$. The proof is based on Robertson and Seymour's rough characterization of H-minor-free graphs, which we now introduce. For an integer $h \geq 1$ and a surface S, Robertson and Seymour [44] defined a graph G to be *h-almost embeddable* in S if G has a set X of at most h vertices (called *apices*) such that $G - X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$ such that:

- G_0 has an embedding in S.
- The graphs $G_1, \ldots, G_h$ (called *vortices*) are pairwise disjoint.
- There are faces[7] $F_1, \ldots, F_h$ of the embedding of G_0 in S, such that each $F_i = V(G_0) \cap V(G_i)$.
- If $F_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,|F_i|})$ in clockwise order about the face, then G_i has a strong $|F_i|$-path decomposition Q_i of width at most h, such that each vertex $u_{i,j}$ is in the j-th bag of Q_i.

THEOREM 6.1. *For all integers $h \geq 1$ and $\gamma \geq 0$, there is a constant $k = k(h,\gamma) \geq h$, such that every graph G that is h-almost embeddable in some surface whose Euler genus is at most γ, has crossing number at most $k\,\Delta(G)\cdot|G|$.*

PROOF. Let X and $\{G_0, G_1, \ldots, G_h\}$ be the parts of G as specified in the definition of h-almost embeddable graphs. Let $\Delta := \Delta(G)$ and $n := |G|$. Start with an embedding of G_0 in S. For each $i \in \{1,\ldots,h\}$, draw vortex G_i inside of the face F_i on S, as prescribed in Theorem 5.2. Then the resulting drawing of $G - X$ in S has at most $h^2\Delta n$ crossings. Replace each crossing by a dummy degree-4 vertex. The resulting graph G' has Euler genus at most γ. By Theorem 1.1, $\mathsf{cr}(G') \leq c\,D^2(G') \leq c\,D^2(G) + c4^2h^2\Delta n$. Since $\mathsf{cr}(G-X) \leq h^2\Delta n + \mathsf{cr}(G')$, we conclude that $\mathsf{cr}(G-X) \leq c\,D^2(G) + (16c+1)h^2\Delta n$.

Consider a drawing of $G - X$ in the plane that achieves at most this many crossings. Add each vertex of X to the drawing at some arbitrary position and draw its incident edges to obtain a drawing of G. Since $|X| \leq h$, there are at most $h\Delta$ edges in G that are not in $G - X$. Each such edge crosses at most $\|G\|$ edges in the drawing of G. Recall that in the H-minor-free graph G, the number of edges is at most $c'\,|G|$, where $c' = c'(H)$ is a constant. Thus $\mathsf{cr}(G) \leq \mathsf{cr}(G-X) + h\Delta\|G\| \leq k\Delta(G)|G|$. □

Let G_1 and G_2 be disjoint graphs. Suppose that C_1 and C_2 are cliques of G_1 and G_2 respectively, each of size k, for some integer $k \geq 0$. Let $C_1 = \{v_1, v_2, \ldots, v_k\}$ and $C_2 = \{w_1, w_2, \ldots, w_k\}$. Let G be a graph obtained from $G_1 \cup G_2$ by identifying v_i and w_i for each $i \in \{1,\ldots,k\}$, and possibly deleting some of the edges v_iv_j. Then G is a *k-clique-sum* of G_1 and G_2 *joined* at $C_1 = C_2$. An ℓ-clique-sum for some $\ell \leq k$ is called a $(\leq k)$-clique-sum.

The following rough characterization of H-minor-free graphs is a deep theorem by Robertson and Seymour [44]; see the recent survey [24].

[7] Recall that we identify a face with the set of vertices on its boundary.

Theorem 6.2. (Graph Minor Structure Theorem [44]) *For every graph H, there is a positive integer $h = h(H)$, such that every H-minor-free graph G can be obtained by $(\leq h)$-clique-sums of graphs that are h-almost embeddable in some surface in which H cannot be embedded.*

By the graph minor structure theorem, Theorem 1.3 is directly implied by the following theorem.

Theorem 6.3. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $c = c(h, \gamma) \geq h$, such that every graph G that can be obtained by $(\leq h)$-clique-sums of graphs that are h-almost embeddable in some surface of Euler genus at most γ has crossing number at most $c\,\Delta(G) \cdot |G|$.*

The remainder of this section is dedicated to proving Theorem 6.3. Let $\Delta := \Delta(G)$. Let U be the set of integers $\{1, 2, \ldots, |U|\}$, such that $\{G_i : i \in U\}$ is the set (of the minimum cardinality) of graphs such that for all $i \in U$, G_i is h-almost embeddable in some surface of Euler genus $\leq \gamma$, and G is obtained by $(\leq h)$-clique-sums of graphs in the set. These graphs can be ordered $G_1, \ldots, G_{|U|}$, such that for all $j \geq 2$, there is a minimum integer $i < j$, such that G_i and G_j are joined at some clique C in the construction of G. We say G_j is a *child* of G_i, G_i is a *parent* of G_j, and $P_j := V(C)$ is the *parent clique* of G_j. We consider the parent clique of G_1 to be the empty set; that is, $P_1 = \emptyset$. This defines a rooted tree T with vertex set U where ij is an edge of T if and only if G_j is a child of G_i. Let U_i denote the set of children of i in T. Let T_i denote the subtree of T rooted at i. For $S \subseteq V(T)$, let $G[S]$ be the graph induced in G by $\bigcup\{V(G_\ell) : \ell \in S\}$. For example, for $S = \{i\}$, then $G[S]$ is a spanning subgraph of G_i.

The proof outline is as follows. For each G_i, $i \in U$, we define an auxiliary graph K_i (closely related to G_i), such that

$$\|K_i\| = \mathcal{O}\Big(\sum_{v \in V(G_i)\setminus P_i} \deg_G(v)\Big).$$

We draw each K_i in the plane with at most $f(h)\Delta\|K_i\|$ crossings, where f is some function of the parameter h. We then join the drawings of $K_1, \ldots, K_{|U|}$ into a drawing of G, where the price of the joining is at most an additional $f(h)\Delta$ crossings for each edge of K_i, $i \in U$. Thus the crossing number of G is at most $f_1(h)\Delta \sum_{i\in U} \|K_i\|$, which, by the above claim on the number of edges of K_i, is at most

$$\begin{aligned} & f_2(h)\,\Delta \sum_{i\in U} \sum_{v\in V(G_i)\setminus P_i} \deg_G(v) \\ \leq\ & f_2(h)\,\Delta \sum_{v \in V(G)} \deg_G(v) \\ =\ & 2 f_2(h)\,\Delta\,\|G\| \\ \leq\ & f_3(h)\,\Delta\,|G|, \end{aligned}$$

which is the desired result.

Defining K_i. For each $i \in U$, let $G_i^- := G_i - P_i$. Note that, for each $v \in V(G)$, there is precisely one value $t \in U$ for which $v \in G_t^-$. Thus $\{V(G_1^-), \ldots, V(G_{|U|}^-)\}$ is a partition of $V(G)$. For each $i \in U$, define K_i as follows. Start with G_i^-. For each child G_j of G_i (that is, for each $j \in U_i$), add a new vertex c_j to G_i^-. For each edge $vw \in E(G)$ such that $v \in V(G_i^-) \cap P_j$ (that is, $v \in P_j \setminus P_i$) and $w \in G_\ell^-$ where $\ell \in V(T_j)$, connect v and c_j by an edge. Subdivide that edge once and label the subdivision vertex by the triple $(v, w, \mathcal{P}_{vw})$, where $\mathcal{P}_{vw}$ is the path in T from i to ℓ (thus, $\mathcal{P}_{vw} = (i, j, \ldots, \ell)$). The resulting graph is K_i. Note that for each v in G_i^-, $\deg_{K_i}(v) = \deg_{G-P_i}(v)$.

Drawing K_i. Suppose that for each $i \in U$, we remove each c_j, $j \in U_i$, from K_i. Consider the union of the resulting graphs, over all $i \in U$. Suppose that, for each vertex labelled $(v, w, \mathcal{P}_{vw})$ in the union, we connect this vertex and w by an edge. The resulting graph is a subdivision of G. This is the strategy that we will follow when constructing a drawing of G. Namely, first draw each K_i, and then take the (disjoint) union of all the drawings. Next, remove all c_j's. Finally, to obtain a drawing of G, route each missing edge of G. In particular, for a missing edge between $(v, w, \mathcal{P}_{vw})$ and w with $\mathcal{P}_{vw} = (i, j, \ldots, \ell)$, we route that edge from $(v, w, \mathcal{P}_{vw})$ in the drawing of K_i, through the drawing of K_j, etc., until we finally reach w in the drawing of K_ℓ.

We first claim that the number of edges in K_i is as stated in the outline. In addition to the edges in $E(G_i^-)$, K_i contains two edges for each edge $vw \in E(G)$, such that $v \in G_i^-$ and $w \in G_\ell^-$, where $\ell \in V(T_i) \setminus i$. Thus

$$\|K_i\| \leq 2 \sum_{v \in V(G_i^-)} \deg_G(v) = 2 \sum_{v \in V(G_i)\setminus P_i} \deg_G(v).$$

Lemma 6.4. *For each $i \in U$, the crossing number of K_i is at most $f(h)\Delta\|K_i\|$.*

Proof. For each G_i, let A_i denote the set of apex vertices of G_i that are not in P_i. Remove all the vertices of A_i from K_i. We now prove that the resulting graph $K_i - A_i$ can be drawn in some surface S of Euler genus at most γ with at most $f(h)\Delta\|K_i - A_i\|$ crossings. That will complete the proof since Theorem 1.1 implies that $\mathsf{cr}(K_i - A_i) \leq f(h)\Delta\|K_i - A_i\|$, the same way it did in the proof of Theorem 6.1. Then we add back each vertex of A_i to the drawing of $K_i - A_i$ at some arbitrary position in the plane and draw its incident edges to obtain a drawing of K_i. As in the proof of Theorem 6.1, $\mathsf{cr}(K_i) \leq \mathsf{cr}(K_i - A_i) + h\Delta\|K_i\| \leq f_2(h)\Delta\|K_i\|$.

Thus it remains to prove that $K_i - A_i$ can be drawn in S with at most $f(h)\Delta\|K_i - A_i\|$ crossings. The graph $Q := G_i^- - A_i$ is an apex-free h-almost embeddable graph on S, with parts $\{Q_0, Q_1, \ldots, Q_h\}$, where Q_0 is the subgraph of Q embedded in S and $\{Q_1, \ldots, Q_h\}$ are its vortices. For each $j \in U_i$, let C_j denote the subgraph of $K_i - A_i$ induced by c_j and the vertices at distance at most two from c_j. The vertices at distance 2 from c_j form a clique $C \subseteq (P_j \setminus P_i) \setminus A_i \subseteq K_i - A_i$. It is simple to verify that C_j has a strong tree decomposition J of width at most $h + 2$, where J is a rooted star whose root bag contains $C \cup \{c_j\}$; for each $(v, w, \mathcal{P}_{vw}) \in C_j$ (where $v \in C$), J contains a leaf bag with $\{w, c_j, (v, w, \mathcal{P}_{vw})\}$; if $v \notin C$, then v is in A_i and the leaf bag contains $\{c_j, (v, w, \mathcal{P}_{vw})\}$.

We now add the vortices and C_j's to Q_0 to obtain a drawing of $K_i - A_i$ in S while creating at most $f(h)\Delta\|K_i - A_i\|$ crossings in S.

For each $j \in U_i$, C_j is joined to a clique C of Q. If C contains a vertex v of a vortex Q_ℓ, where $\ell \in \{1, \ldots, h\}$, then each vertex of C is in Q_ℓ. In that case, we say that C_j *belongs to* the face F_ℓ of the embedding of Q_0 in S. Otherwise, all the vertices of C are in Q_0. In that case, an extended version of the graph minor decomposition theorem (see [24]) states that $|C| \leq 3$ and moreover, if $|C| = 3$, then the 3-cycle induced by C is a face in Q_0. In that case, we say that C_j *belongs to* that face. If $|C| \leq 2$ we assign C_j to any face of Q_0 incident to all the vertices of C.

Now consider a face F of Q_0. If $F = F_\ell$ for some ℓ ($1 \leq \ell \leq h$), take its vortex Q_ℓ, and all C_j, $j \in U_i$, that belong to F. Let F' be the subgraph of $K_i - A_i$ induced by the union of F and all of these. If F is not one of the vortex faces, then we define F' similarly by taking the union of F and all C_j, $j \in U_i$, that belong to F. If F' contains a vortex Q_ℓ, consider a strong path decomposition P_F of $F \cup Q_\ell$, as defined by the h-almost embedding. If F has no vortex, then its strong path decomposition P_F is just a bag containing $|F| \leq 3$ vertices of F in it. For each C_j in F', the join clique C of C_j is in some bag of P_F. Extend the decomposition P_F and J by adding an edge between that bag of P_F and the root of J. It is simple to verify that the resulting strong tree decomposition of F' can be converted into a strong path decomposition of width at most $h + 3$. Thus by Theorem 5.2, F' can be drawn inside of F with at most $(h+3)^2\Delta|F'|$ crossings. Accounting for all the faces of Q_0 gives $f_4(h)\Delta\|K_i - A_i\|$ bound on the number of crossings in the resulting drawing of $K_i - A_i$ in S, as required. □

In addition to having at most as many crossings as proved in Lemma 6.4, we will need a drawing of K_i that has the following additional properties.

LEMMA 6.5. *For each $i \in U$, there is a drawing of K_i with at most $f(h)\Delta\|K_i\|$ crossings such that:*

(1) No pair of vertices in K_i has the same x-coordinate.
(2) For each $j \in U_i$, there is a square[8] D_j such that $D_j \cap K_i = c_j$, and c_j is an internal point of the top side of D_j, and no vertex in $V(K_i) \setminus \{c_j\}$ has the same x-coordinate as any point of D_j.
(3) For any two $j, t \in U_i$, there is no line parallel to the y-axis that intersects both D_j and D_t.
(4) Moreover, given a circular ordering σ_j of the edges incident to each vertex c_j in K_i, $j \in U_i$, there is a drawing of K_i that satisfies (1)–(3) such that the circular ordering of the edges incident to each c_j respects σ_j.

PROOF. Apply Lemma 6.4 to K_i to obtain a drawing of K_i with at most $s := f(h)\Delta\|K_i\|$ crossings. Clearly, the edges incident to c_j can be bent without changing the number of crossings such that there is a small enough square D_j that satisfies all the properties imposed on D_j, as stated in (2). Similarly, condition (3) is satisfied by shrinking the squares further, if necessary. By an appropriate rotation, the conditions on the x- and y-coordinates imposed in (1)–(3) are satisfied.

Consider a disk C_j centered at c_j, such that c_j is the only vertex of K_i that intersects C_j, and the only edges of K_i that intersect C_j are the edges incident to c_j. Order the edges around c_j with respect to σ_j by moving (that is, bending) the edges incident to c_j within $C_j \setminus D_j$. This may introduce new crossings. Each new crossing point is in $C_j \setminus D_j$ and thus it occurs between a pair of edges incident to c_j. There are at most $h\Delta$ edges incident to c_j. Thus each edge incident to c_j gets at most $h\Delta$ new crossings. Therefore, the resulting drawing of K_i satisfies conditions (1)–(4) and has at most $s + h\Delta\|K_i\| \leq f'(h)\Delta\|K_i\|$ crossings. □

[8]By a *square*, we mean a 4-sided regular polygon together with its interior.

Joining the K_i's into a drawing of G. We obtain a drawing of G from the union of the drawings of K_i, $i \in U$, as follows. Join the drawings of these graphs in the order determined by a breath-first search on T, as follows. For each G_i, consider a drawing of K_i together with the squares incident to its children, as defined in Lemma 6.5. For each $j \in U_i$, place the drawing of K_j strictly inside of the square D_j of K_i (while scaling the drawing of K_j, if necessary). Denote by K the resulting drawing of $\bigcup_i K_i$. This procedure introduces no new crossings, thus by Lemma 6.5, the number of crossings in K is at most $\sum_{i\in U} f'(h)\Delta\|K_i\|$.

We still have the freedom to choose an arbitrary ordering σ_j (cf. Lemma 6.5(4)) to be used in the drawing of K_j. Define the ordering σ_j of edges around each vertex c_j ($j \in U \setminus \{1\}$) as follows. Consider an edge e_1 joining c_j and $(v, w, \mathcal{P}_{vw})$, and an edge e_2 joining c_j and $(a, b, \mathcal{P}_{ab})$. Define $e_1 \leq_{\sigma_j} e_2$ if the x-coordinate of w in K is less than the x-coordinate of b in K. If $w = b$, order e_1 and e_2 according to the x-coordinates of v and a. Since no pair of vertices in K have the same x-coordinate, σ_j is a linear order of the edges incident to c_j.

For each $j \in U \setminus \{1\}$, we may assume that the graph induced in K by c_j and its neighbours (the subdivision vertices), is a crossing-free star in K; that is, no edge of this star is crossed by any other edge of K.

For each $i \in U$, remove each c_j, $j \in U_i$, from K. The subdivision vertices of K become degree-1 vertices. For each such subdivision vertex $(v, w, \mathcal{P}_{vw})$, where $\mathcal{P}_{vw} = (i, j, \dots, \ell)$, draw an edge from $(v, w, \mathcal{P}_{vw})$ to the point on the top side of the square D_j that has the same x-coordinate as the vertex w in K. Since $w \in G[T_j] - P_j$, it is drawn inside D_j, and thus such a point on the top side of D_j exists. If w is an endpoint of $s \geq 2$ such edges, draw s points very close together on the top side of D_j and connect each of the s edges to one of these s points in the order σ_j. (In fact, imagine that these points are almost overlapping; that is, their x-coordinates are almost the same as that of w in K). Since the star incident to c_j is crossing-free in K, this can be done so that the resulting drawing K_i^- has the same number of crossings as K_i. Label each point on the top side of D_j by the same label as the subdivision vertex it is adjacent to. (In fact, consider that point on the top side of D_j to be the subdivision vertex instead of the old one). Draw a line-segment between each subdivision vertex $(v, w, \mathcal{P}_{vw})$ on the top side of D_j and w. Call these segments *vertical segments.* This defines a drawing of G. We now prove that the number of crossings in G does not increase much compared to the number of crossings in K. Specifically, it increases by at most $f(h)\Delta\sum_{i\in U}\|K_i\|$.

Note that Lemma 6.5 does not define the square D_1. Let D_1 be the whole plane. For each $i \in U$, let D_i^- be the region $D_i \setminus \{\bigcup_{j\in U_i} D_j\}$. Denote by d_i the number of crossings in the drawing of G restricted to D_i^-. Then $\mathsf{cr}(G) \leq \sum_i d_i$.

We now prove that for each $i \in U$, $d_i \leq f(h)\Delta\|K_i\|$, which will complete the proof. Quantity d_i is at most the number of crossings in K_i^- plus the number of crossings caused by the vertical segments intersecting D_i^-. By construction (cf. properties (2) and (3) of Lemma 6.5), each vertical segment that intersects D_i^- is a part of an edge that has one endpoint in G_s^- where $i \in V(T_s) \setminus s$ (that is, G_s^- is an ancestor of G_i^-) and its other endpoint is either in G_i^- (and, thus in K_i^-) or is in a descendent G_ℓ^- of G_i^-. Thus the number of vertical segments that cross D_i^- is at most $f(h)\Delta$. No pair of vertical segments cross in D_i^- due to their ordering. Thus each new crossing in D_i^- (that is, a crossing not present in the drawing of K_i^-) occurs between a vertical segment and an edge of K_i^-. Thus each edge of K_i^- accounts for at most $f(h)\Delta$ new crossings, and thus $d_i \leq f(h)\Delta\|K_i^-\| \leq f(h)\Delta\|K_i\|$, as desired. This completes the proof of Theorem 6.3.

7. REFERENCES

[1] Jay Adamsson and R. Bruce Richter. Arrangements, circular arrangements and the crossing number of $C_7 \times C_n$. *J. Combin. Theory Ser. B*, 90(1):21–39, 2004.

[2] Miklós Ajtai, Vašek Chvátal, Monroe M. Newborn, and Endre Szemerédi. Crossing-free subgraphs. In *Theory and practice of combinatorics*, vol. 60 of *North-Holland Math. Stud.*, pp. 9–12. North-Holland, 1982.

[3] Sandeep N. Bhatt and F. Thomson Leighton. A framework for solving VLSI graph layout problems. *J. Comput. System Sci.*, 28(2):300–343, 1984.

[4] Hans L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

[5] Drago Bokal. On the crossing numbers of cartesian products with paths. *J. Combin. Theory Ser. B*, 97(3):381–384, 2007.

[6] Drago Bokal, Éva Czabarka, László A. Székely, and Imrich Vrt'o. Graph minors and the crossing number of graphs. *Electron. Notes Discrete Math.*, 28:169–175, 2007.

[7] Drago Bokal, Gašper Fijavž, and Bojan Mohar. The minor crossing number. *SIAM J. Discrete Math.*, 20(2):344–356, 2006.

[8] Drago Bokal, Gašper Fijavž, and David R. Wood. The minor crossing number of graphs with an excluded minor. *Electron. J. Combin.*, 15(R4), 2008.

[9] Károly Böröczky, János Pach, and Géza Tóth. Planar crossing numbers of graphs embeddable in another surface. *Int.. J. Found. Comput. Sci.*, 17(5):1005–1015, 2006.

[10] Lane H. Clark and Roger C. Entringer. The bisection width of cubic graphs. *Bull. Austral. Math. Soc.*, 39(3):389–396, 1989.

[11] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Ken-ichi Kawarabayashi. Algorithmic graph minor theory: Decomposition, approximation, and coloring. In *Proc. 46th Annual IEEE Symposium on Foundations of Computer Science* (FOCS '05), pp. 637–646. IEEE, 2005.

[12] Josep Díaz, Norman Do, Maria J. Serna, and Nicholas C. Wormald. Bounds on the max and min bisection of random cubic and random 4-regular graphs. *Theoret. Comput. Sci.*, 307(3):531–547, 2003.

[13] Reinhard Diestel and Daniela Kühn. Graph minor hierarchies. *Discrete Appl. Math.*, 145(2):167–182, 2005.

[14] Hristo N. Djidjev and Imrich Vrt'o. Crossing numbers and cutwidths. *J. Graph Algorithms Appl.*, 7(3):245–251, 2003.

[15] Hristo N. Djidjev and Imrich Vrt'o. Planar crossing numbers of genus g graphs. In Michele Bugliesi, Bart Preneel, Vladimiro Sassone, and Ingo Wegener, editors, *Proc. 33rd Int. Colloquium on Automata, Languages and Programming* (ICALP '06), vol. 4051 of *Lecture Notes in Comput. Sci.*, pp. 419–430, 2006.

[16] Paul Erdős and Richard K. Guy. Crossing number problems. *Amer. Math. Monthly*, 80:52–58, 1973.

[17] Enrique Garcia-Moreno and Gelasio Salazar. Bounding the crossing number of a graph in terms of the crossing number of a minor with small maximum degree. *J. Graph Theory*, 36(3):168–173, 2001.

[18] Micahel R. Garey and David S. Johnson. Crossing number is NP-complete. *SIAM J. Algebraic Discrete Methods*, 4(3):312–316, 1983.

[19] James F. Geelen, R. Bruce Richter, and Gelasio Salazar. Embedding grids in surfaces. *European J. Combin.*, 25(6):785–792, 2004.

[20] Lev Yu. Glebsky and Gelasio Salazar. The crossing number of $C_m \times C_n$ is as conjectured for $n \geq m(m+1)$. *J. Graph Theory*, 47(1):53–72, 2004.

[21] Petr Hliněny. Crossing-number critical graphs have bounded path-width. *J. Combin. Theory Ser. B*, 88(2):347–367, 2003.

[22] Petr Hliněný. Crossing number is hard for cubic graphs. *J. Combin. Theory Ser. B*, 96(4):455–471, 2006.

[23] Robert E. Jamison and Renu Laskar. Elimination orderings of chordal graphs. In *Combinatorics and Applications*, pp. 192–200. Indian Statist. Inst., 1984.

[24] Ken-ichi Kawarabayashi and Bojan Mohar. Some recent progress and applications in graph minor theory. *Graphs Combin.*, 23(1):1–46, 2007.

[25] Ken-ichi Kawarabayashi and Bruce Reed. Computing crossing number in linear time. In *Proc. 39th Annual ACM Symposium on Theory of Computing* (STOC '07), pp. 382–390. ACM, 2007.

[26] F. Thomson Leighton. *Complexity Issues in VLSI*. MIT Press, 1983.

[27] F. Thomson Leighton. New lower bound techniques for VLSI. *Math. Systems Theory*, 17(1):47–70, 1984.

[28] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*. Johns Hopkins University Press, 2001.

[29] Nagi H. Nahas. On the crossing number of $K_{m,n}$. *Electron. J. Combin.*, 10:N8, 2003.

[30] Seiya Negami. Crossing numbers of graph embedding pairs on closed surfaces. *J. Graph Theory*, 36(1):8–23, 2001.

[31] János Pach, Farhad Shahrokhi, and Mario Szegedy. Applications of the crossing number. *Algorithmica*, 16(1):111–117, 1996.

[32] János Pach and Micha Sharir. On the number of incidences between points and curves. *Combin. Probab. Comput.*, 7(1):121–127, 1998.

[33] János Pach and Géza Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000.

[34] János Pach and Géza Tóth. Crossing number of toroidal graphs. In Patrick Healy and Nikola S. Nikolov, editors, *Proc. 13th Int. Symp. on Graph Drawing* (GD '05), vol. 3843 of *Lecture Notes in Comput. Sci.*, pp. 334–342, 2006.

[35] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Crossing number of graphs with rotation systems. In Seok-Hee Hong, Takao Nishizeki, and Wu Quan, editors, *Proc. 15th Int. Symp. on Graph Drawing* (GD '07), vol. 4875 of *Lecture Notes in Comput. Sci.*, pp. 3–12, 2007.

[36] Helen C. Purchase. Which aesthetic has the greatest effect on human understanding? In Giuseppe Di Battista, editor, *Proc. 5th Int. Symp. on Graph Drawing* (GD '97), vol. 1353 of *Lecture Notes in Comput. Sci.*, pp. 248–261, 1997.

[37] Helen C. Purchase. Performance of layout algorithms: Comprehension, not computation. *J. Visual Languages and Computing*, 9:647–657, 1998.

[38] Helen C. Purchase, Robert F. Cohen, and Murray I. James. An experimental study of the basis for graph drawing algorithms. *ACM Journal of Experimental Algorithmics*, 2(4), 1997.

[39] Bruce A. Reed. Algorithmic aspects of tree width. In Bruce A. Reed and Cláudia L. Sales, editors, *Recent Advances in Algorithms and Combinatorics*, pp. 85–107, 2003.

[40] R. Bruce Richter and Jozef Širáň. The crossing number of $K_{3,n}$ in a surface. *J. Graph Theory*, 21(1):51–54, 1996.

[41] R. Bruce Richter and Carsten Thomassen. Intersections of curve systems and the crossing number of $C_5 \times C_5$. *Discrete Comput. Geom.*, 13(2):149–159, 1995.

[42] R. Bruce Richter and Carsten Thomassen. Relations between crossing numbers of complete and complete bipartite graphs. *Amer. Math. Monthly*, 104(2):131–137, 1997.

[43] Neil Robertson and Paul D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986.

[44] Neil Robertson and Paul D. Seymour. Graph minors. XVI. Excluding a non-planar graph. *J. Combin. Theory Ser. B*, 89(1):43–76, 2003.

[45] Neil Robertson and Paul D. Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[46] Farhad Shahrokhi, Ondrej Sýkora, László A. Székely, and Imrich Vrt'o. The crossing number of a graph on a compact 2-manifold. *Adv. Math.*, 123(2):105–119, 1996.

[47] Farhad Shahrokhi and László A. Székely. On canonical concurrent flows, crossing number and graph expansion. *Combin. Probab. Comput.*, 3(4):523–543, 1994.

[48] Farhad Shahrokhi, László A. Székely, Ondrej Sýkora, and Imrich Vrt'o. Drawings of graphs on surfaces with few crossings. *Algorithmica*, 16(1):118–131, 1996.

[49] László A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.*, 6(3):353–358, 1997.

[50] László A. Székely. A successful concept for measuring non-planarity of graphs: the crossing number. *Discrete Math.*, 276(1–3):331–352, 2004.

[51] Imrich Vrt'o. Crossing numbers of graphs: A bibliography, 2007. ftp://ftp.ifi.savba.sk/pub/imrich/crobib.pdf.

[52] David R. Wood and Jan Arne Telle. Planar decompositions and the crossing number of graphs with an excluded minor. *New York J. Math.*, 13:117–146, 2007.