University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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MINIMAL OBSTRUCTIONS FOR 1-IMMERSIONS AND HARDNESS OF 1-PLANARITY TESTING

Vladimir P. Korzhik   Bojan Mohar

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Minimal obstructions for 1-immersions and hardness of 1-planarity testing

Vladimir P. Korzhik*
National University of Chernivtsi, Chernivtsi
and
Institute of Applied Problems of Mechanics and Mathematics
of National Academy of Science of Ukraine, Lviv
Ukraine

Bojan Mohar†‡
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
Canada

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‡On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

Abstract

A graph is 1-*planar* if it can be drawn on the plane so that each edge is crossed by no more than one other edge. A non-1-planar graph G is *minimal* if the graph $G - e$ is 1-planar for every edge e of G. We construct two infinite families of minimal non-1-planar graphs and show that for every integer $n \geq 63$, there are at least $2^{(n-54)/4}$ nonisomorphic minimal non-1-planar graphs of order n. It is also proved that testing 1-planarity is NP-complete.

Running head:
Obstructions for 1-immersions

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1 Introduction

A graph drawn in the plane is 1-*immersed* in the plane if any edge is crossed by at most one other edge. A graph is 1-*planar* if it can be 1-immersed into the plane. It is easy to see that if a graph has 1-immersion in which two edges e, f with a common endvertex cross, then the drawing of e and f can be changed so that these two edges no longer cross. Consequently, we may assume that adjacent edges are never crossing each other and that no edge is crossing itself. We take this assumption as a part of the definition of 1-immersions since this limits the number of possible cases when discussing 1-immersions.

The notion of 1-immersion of a graph was introduced by Ringel [14] when trying to color the vertices and faces of a plane graph so that adjacent or incident elements receive distinct colors. In the last two decade this class of graphs received additional attention because of its relationship to the family of *map graphs*, see [6, 7] for further details.

Little is known about 1-planar graphs. Borodin [1, 2] proved that every 1-planar graph is 6-colorable. Some properties of maximal 1-planar graphs are considered in [15]. It was shown in [3] that every 1-planar graph is acyclically 20-colorable. The existence of subgraphs of bounded vertex degrees in 1-planar graphs is investigated in [9]. It was shown in [4, 5] that a 1-planar graph with n vertices has at most $4n - 8$ edges and that this upper bound is tight. In the paper [8] it was observed that the class of 1-planar graphs is not closed under the operation of edge-contraction.

Much less is known about non-1-planar graphs. The basic question is how to recognize 1-planar graphs. This problem is clearly in NP, but it is not clear at all if there is a polynomial time recognition algorithm. We shall answer this question by proving that 1-planarity testing problem is NP-complete.

The recognition problem is closely related to the study of minimal obstructions for 1-planarity. A graph G is said to be a *minimal* non-1-planar graph (MN-*graph*, for short) if G is not 1-planar, but $G - e$ is 1-planar for every edge e of G. An obvious question is:

> How many MN-graphs are there? Is their number finite? If not, can they be characterized?

The answer to the first question is not hard: there are infinitely many. This was first proved in [12]. Here we present two additional simple arguments implying the same conclusion.

Example 1. Let G be a graph such that $t = \lceil \mathrm{cr}(G)/|E(G)| \rceil - 1 \geq 1$, where $\mathrm{cr}(G)$ denotes the crossing number of G. Let G_t be the graph obtained from G by replacing each edge of G by a path of length t. Then $|E(G_t)| = t|E(G)| < \mathrm{cr}(G) = \mathrm{cr}(G_t)$. This implies that G_t is not 1-planar. However, G_t contains an MN-subgraph H. Clearly, H contains at least one subdivided edge of G in its entirety, so $|V(H)| > t$. Since t can be arbitrarily large, this shows that there are infinitely many MN-graphs.

Example 2. Let $K \in \{K_5, K_{3,3}\}$ be one of Kuratowski graphs. For each edge $xy \in E(K)$, let L_{xy} be a 5-connected triangulation of the plane and u, v be adjacent vertices of L_{xy} whose degree is at least 6. Let $L'_{xy} = L_{xy} - uv$. Now replace each edge xy of K with L'_{xy} by identifying x with u and y with v. It is not hard to see that the resulting graph G is not 1-planar (since two of graphs L'_{xy} must "cross each other", but that is not possible since they come from 5-connected triangulations). Again, one can argue that they contain large MN-graphs.

The paper [12] and the above examples prove the existence of infinitely many MN-graphs but do not give any concrete examples. They provide no information on properties of MN-graphs. Even the most basic question if there are infinitely many MN-graphs whose minimum degree is at least three cannot be answered by considering these constructions. In [12], two specific MN-graphs of order 7 and 8, respectively, are given. One of them, the graph $K_7 - E(K_3)$, is the unique 7-vertex MN-graph and since all 6-vertex graphs are 1-planar, the graph $K_7 - E(K_3)$ is the MN-graph with the minimum number of vertices. Surprisingly enough, the two MN-graphs in [12] are the only explicit MN-graphs known in the literature.

The main problem when trying to construct 1-planar graphs is that we have no characterization of 1-planar graphs. The set of 1-planar graphs is not closed under taking minors, so 1-planarity can not be characterized by forbidding some minors.

In the present paper we construct two explicit infinite families of MN-graphs whose minimum degree is at least three and, correspondingly, we give two different approaches how to prove that a graph has no plane 1-immersion.

In Sect. 2 we construct MN-graphs based on the Kuratowski graph $K_{3,3}$. To obtain them, we replace six edges of $K_{3,3}$ by some special subgraphs. The minimality of these examples is easy to verify, but their non-1-planarity needs long and somewhat technical arguments. Using these MN-graphs, we show that for every integer $n \geq 63$, there are at least $2^{(n-54)/4}$ nonisomorphic minimal non-1-planar graphs of order n. In Sect. 3 we describe a class of 3-connected planar graphs that have no plane 1-immersions with at least one crossing point (PN-*graphs*, for short). Every 3-connected PN-graph has a unique plane 1-immersion, namely, its unique plane embedding. Hence, if a 1-planar graph G contains a PN-graph H as a subgraph, then in every plane 1-immersion of G the subgraph H is 1-immersed in the plane in the same way. Having constructions of PN-graphs, we can construct 1-planar and non-1-planar graphs with some desired properties: 1-planar graphs that have exactly $k > 0$ different plane 1-immersions; MN-graphs, etc.

In Sect. 4 we construct MN-graphs based on PN-graphs. Each of these MN-graphs G has as a subgraph a PN-graph H and the unique plane 1-immersion of H prevents to 1-immerse the remaining part of G in the plane.

Despite the fact that minimal obstructions for 1-planarity (i.e., the MN-graphs) have diverse structure, and despite the fact that discovering 1-immersions of specific graphs can be very tricky, it turned out to be a hard problem to establish hardness of 1-planarity testing. A solution is given in Sect. 5, where we show that 1-planarity testing is NP-complete, see Theorem 5. The proof is geometric in the sense that the reduction is from 3-colorability of planar graphs (or similarly, from planar 3-satisfiability).

An extended abstract of this paper was published in Graph Drawing 2008 [13].

2 Chain graphs based on $K_{3,3}$

Two cycles of a graph are *adjacent* if they share a common edge. If a graph G is drawn in the plane, then we say that a vertex x lies *inside* (resp. *outside*) an embedded (that is, non-self-intersecting) cycle C, if x lies in the interior (resp. exterior) of C, and does not lie on C. Having two embedded adjacent cycles C and C', we say that C lies inside (resp. outside) C' if every point of C either lies inside (resp. outside) C' or lies on C'. From this point on, by a 1-immersion of a graph we mean a plane 1-immersion. We assume that in 1-immersions, adjacent edges do not cross each other and no edge crosses itself. Thus, every 3-cycle of a 1-immersed graph is embedded in the plane. Hence, given a 3-cycle of a 1-immersed graph, we can speak about its interior and exterior. We say that an embedded cycle *separates* two vertices x and y on the plane, if one of the vertices lies inside and the other one lies outside the cycle. Two edges e and e' of a graph G *separate* vertices x and y of

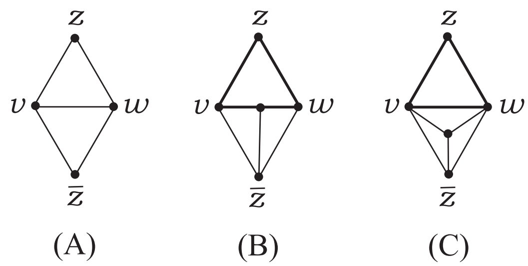


Figure 1: Links.

the graph if x and y belong to different connected components of the graph $G - e - e'$.

Throughout the paper we will deal with 1-immersed graphs. When an immersion of a graph G is clear from the context, we shall identify vertices, edges, cycles and subgraphs of G with their image in $\mathbb{R}^2$ under the 1-immersion. Then by a face of a 1-immersion of G we mean any connected component of $\mathbb{R}^2 \setminus G$.

By using Möbius transformations combined with homeomorphisms of the plane it is always possible to exchange the interior and exterior of any embedded cycle and it is possible to change any face of a given 1-immersion into the outer face of a 1-immersion. Formally, we have the following observation (which we will use without referring to it every time):

(A) Let C be a cycle of a graph G. If G has a 1-immersion φ in which C is embedded, then G has a 1-immersion φ' with the same number of crossings as φ, in which C is embedded and all vertices of G, which lie inside C in φ, lie outside C in φ' and vice versa.

Now we begin describing a family of MN-graphs based on the graph $K_{3,3}$.

By a *link* $L(x, y)$ connecting two vertices x and y we mean any of the graphs shown in Fig. 1 where $\{z, \overline{z}\} = \{x, y\}$. We say that the vertices x and y are incident with the link. The links in Figs. 1(A) and (B) are called *A-link* and *B-link*, respectively, and the one in Fig. 1(C) is called a *base link*. Every link has a *free cycle*: both 3-cycles in an A-link are its free cycles, while every B-link or base link has exactly one free cycle (the cycle indicated by thick lines in Fig. 1).

By an A-*chain* of length $n \geq 2$ we mean the graph shown in Fig. 2(a). By a B-*chain* of length $n \geq 2$ we mean the graph shown in Fig. 2(c) and, for $n \geq 3$, every graph obtained from that graph in the following way: for some

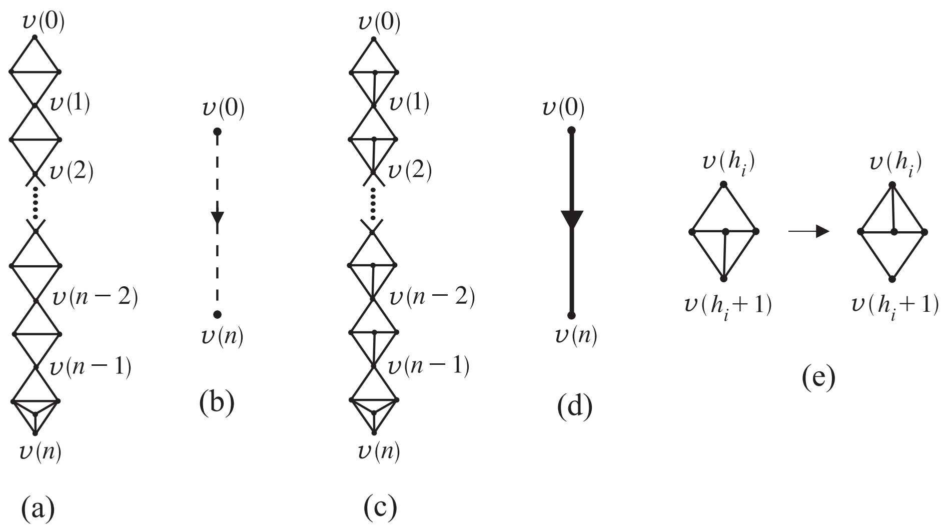


Figure 2: A- and B-chains.

integers $h_1, h_2, \dots, h_t$, where $t \geq 1$ and $1 \leq h_1 < h_2 < \cdots < h_t \leq n-2$, we replace the link at the left of Fig. 2(e) by the link shown at the right, for $i = 1, 2, \dots, t$. Note that, by definition, A- and B-chains have length at least 2. We say that the chains in Figs. 2(a) and (c) connect the vertices $v(0)$ and $v(n)$ which are called the *end vertices* of the chain. Two chains are *adjacent* if they share a common end vertex. A-chains and B-chains will be represented as shown in Figs. 2(b) and (d), respectively, where the arrow points to the end vertex incident with the base link. The vertices $v(0), v(1), v(2), \dots, v(n)$ are the *core vertices* of the chains. Every free cycle of a link contains exactly one core vertex. The two edges of a free cycle C incident to the core vertex are the *core-adjacent* edges of C. It is easy to see that two edges e and e' of a chain separate the end vertices of the chain if and only if the edges are the core-adjacent edges of a free cycle of a link of the chain.

By a *subchain* of a chain shown in Figs. 2(a) and (c) we mean a subgraph of the chain consisting of links incident with $v(i)$ and $v(i+1)$ for all $i = m, m+1, \dots, m'-1$ for some $0 \leq m < m' \leq n$. We say that the subchain connects the vertices $v(m)$ and $v(m')$.

A *chain graph* is any graph obtained from $K_{3,3}$ by replacing three of its edges incident with the same vertex by A-chains and three edges incident with another vertex by B-chains, where the chains can have arbitrary lengths ≥ 2. These changes are to be made as shown in Fig. 3(a). The vertices $\Omega(1)$,

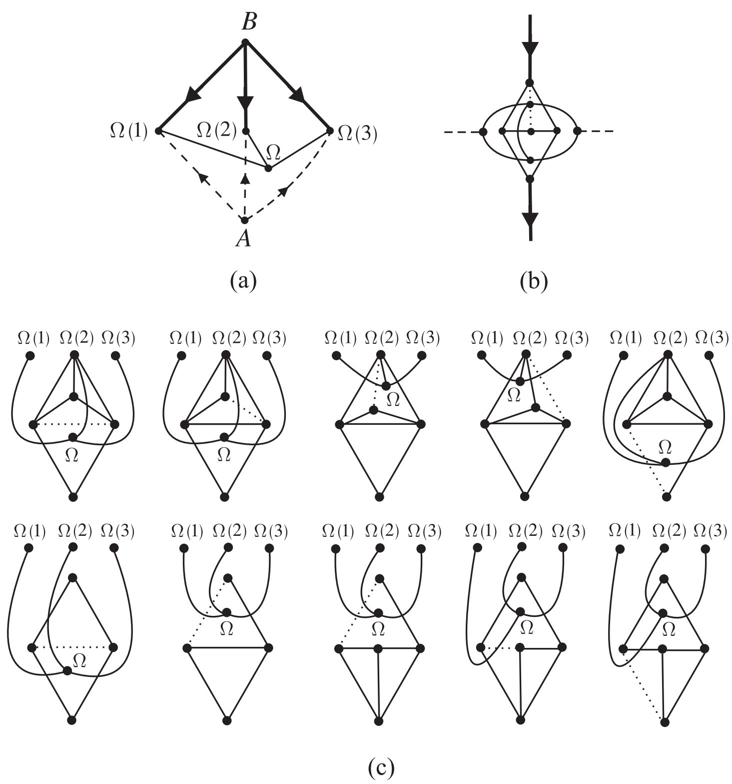


Figure 3: A chain graph G and 1-planarity of the graph $G - e$.

$\Omega(2)$, and $\Omega(3)$ are the *base vertices* of the chain graph. The edges joining the vertex Ω to the base vertices are called the Ω-*edges*.

We will show that every chain graph is an MN-graph.

Lemma 1 *Let G be a chain graph and $e \in E(G)$. Then $G - e$ is 1-planar.*

Proof. If e is an Ω-edge, then $G - e$ is planar and hence 1-planar. Suppose now that e is not an Ω-edge. By symmetry, we may assume that e belongs to an A- or B-chain incident to $\Omega(2)$. If e is the "middle" edge of a B-link, then Fig. 3(b) shows that the corresponding B-chain can be crossed by an A-chain, and it is easy to see that this can be made into a 1-immersion of $G-e$. In all other cases, 1-immersions are made by crossing the link L whose edge e is deleted with the edges incident with the vertex Ω. The upper row in Fig. 3(c) shows the cases when L is a base link. The lower row covers the cases when L is an A-link or a B-link. The edge e is shown in all cases as the dotted edge. □

Our next goal is to show that chain graphs are not 1-planar. In what follows we let G be a chain graph and φ a (hypothetical) 1-immersion of G.

Lemma 2 *Let φ be a 1-immersion of a chain graph G such that the number of crossings in φ is minimal among all 1-immersions of G. If L is a link in an A- or B-chain of G, then no two edges of L cross each other in φ.*

Proof. The first thing to observe is that whenever edges ab and bc cross, there is a disk D having a, c, b, d on its boundary, and D contains these two edges but no other points of G. In 1-immersions with minimum number of crossings this implies that no other edges between the vertices a, c, b, d are crossed. Similarly, if $L = L(z, \bar{z})$ is a link in a chain, and an edge incident with z crosses an edge incident with $\bar{z}$, the whole link L can be drawn in D without making any crossings. This shows that the only possible cases for a crossing of two edges e, f in L are the following ones, where we take the notation from Figure 1 and we let u be the vertex of L that is not labeled in the figure:
(a) L is a B-link and $e = zv$, $f = uw$.
(b) L is a B-link and $e = \bar{z}v$, $f = uw$.
(c) L is a base link and $e = zv$, $f = uw$.
(d) L is a base link and $e = vw$, $f = \bar{z}u$.
(e) L is a base link and $e = vu$, $f = \bar{z}w$.
Let D be a disk as discussed above corresponding to the crossing of e and f. In cases (b), (d) and (e), the vertex u has all neigbors on the boundary of D, so the crossing between e and f can be eliminated by moving u inside D onto the other side of the edge f.

It remains to consider cases (a) and (c). Observe that the boundary of D contains vertices z, u, v, w in this order and that u and v both have precisely one additional neighbor $\bar{z}$ outside of D. Therefore, we can turn φ into another 1-immersion of G by switching u and v and only redraw the edges inside D. However, this eliminates the crossing in D and yields a 1-immersion with fewer crossings, a contradiction. □

Lemma 3 *Let G and φ be as in Lemma 2. If Π and Π' are nonadjacent A- and B-chain, respectively, then in φ the following holds for every 3-cycle C of Π:*

(i) *The core vertices of* Π' *either all lie inside or all lie outside* C.

(ii) *If all core vertices of* Π' *lie inside* (*resp. outside*) C, *then at most one vertex of* Π' *lies outside* (*resp. inside*) C.

Proof. First we show (ii). By (A), we may assume that all core vertices of Π' lie inside C. By inspecting Fig. 1, it is easy to check that for every link L, for every set W of noncore vertices of L such that $|W| \geq 2$, there are at least four edges joining W with of $V(L) \setminus W$. Hence, if at least two noncore vertices belonging to the same link of Π' lie outside C, then at least four edges join them with the vertices of Π' lying inside C, a contradiction. Every noncore vertex of Π' has valence at least 3. Hence if exactly n ($n \geq 2$) noncore vertices of Π' lie outside C and if they all belong to different links, then at least $3n \geq 6$ edges join them with the vertices of Π' lying inside C, a contradiction.

Now we prove (i). If C does not contain the vertex A, then every two core vertices of Π' are connected by four edge-disjoint paths not passing through the vertices of C, hence (i) holds for C.

Suppose now that C contains the vertex A and that core vertices of Π' lie inside and outside C. Then there is a link $L(z, \overline{z})$ of Π' such that the vertex z lies inside and the vertex $\overline{z}$ lies outside C. We may assume without loss of generality that Π and Π' are incident to the base vertices $\Omega(1)$ and $\Omega(2)$, respectively, and (taking (A) into account) that the vertex z (if $z \neq B$) separates the vertices B and $\overline{z}$ in Π' (see Fig. 4(a), where in $L(z, \overline{z})$ the dotted line indicates that the link has either edge εz or $\varepsilon \overline{z}$; also if $\bar{z} = \Omega(2)$, then the link indicated in Fig. 4(a) is a base link).

The 3-cycle C crosses at least two edges of $L(z, \overline{z})$. The vertex z (resp. $\overline{z}$) is connected to each of the vertices $\Omega(1)$ and $\Omega(3)$ (resp. to the vertex $\Omega(2)$) by two edge-disjoint paths not passing through $V(C)$ or through the noncore vertices of $L(z, \overline{z})$. Hence, $\Omega(1)$ and $\Omega(3)$ lie inside C (resp. $\Omega(2)$ lies outside C). It follows that the vertex Ω lies inside C and the edge $(\Omega, \Omega(2))$ is the third edge that crosses C. We conclude that C crosses exactly two edges of $L(z, \overline{z})$ and the two edges separate z from $\overline{z}$ in $L(z, \overline{z})$. Thus, the two edges are the core-adjacent edges of the free cycle of $L(z, \overline{z})$. Hence, in φ, the link $L(z, \overline{z})$ is 1-immersed as shown in Fig. 4(b), where the dotted edges indicate alternative possibilities.

Let $v, \bar{v}$ be the vertices of C different from A and let x be the fourth vertex of the link containing C. The vertex x is connected to $\Omega(1)$ by two

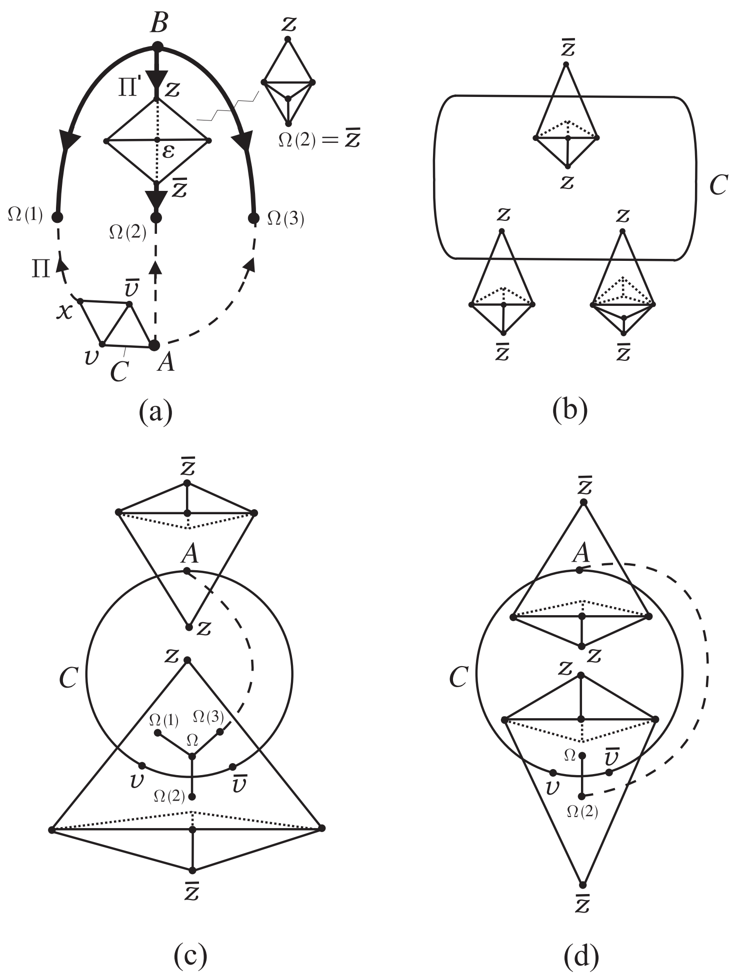
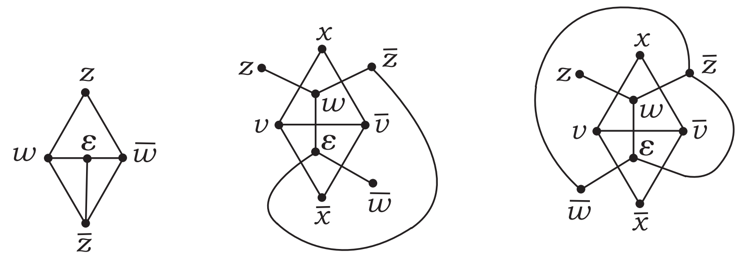


Figure 4: Cases in the proof of Lemma 3.

edge-disjoint paths not passing through the vertices of C, hence x lies inside C. At most two vertices of C lie inside the free cycle of $L(z, \overline{z})$. If only one of the vertices v and $\overline{v}$ of C lies inside the free cycle, then we see that in the case of $L(z, \overline{z})$ at the bottom of Fig. 4(b), the path $vx\overline{v}$ can not lie inside C, a contradiction. In the case of $L(z, \overline{z})$ at the top of Fig. 4(b), if the path $vx\overline{v}$ lies inside C, then x must lie inside a 3-cycle Q of $L(z, \overline{z})$ incident to z, whereas A lies outside Q, a contradiction, since Q is not incident to B and in G there are two edge-disjoint paths connecting x to A and not passing through the vertices v, $\overline{v}$, and the vertices of Q.



(a) (b) (c)

Figure 5: Cases in the proof of Lemma 4.

If either both v and $\overline{v}$ or none of them lie inside the free 4-cycle, then in the case of Fig. 4(c) (resp. (d)), where we depict the two possible placements of the nonbase link $L(z, \overline{z})$, there are two edge-disjoint paths of G connecting A and $\Omega(3)$ (resp. A and $\Omega(2)$) and not passing through z (resp. $\overline{z}$), a contradiction. Reasoning exactly in the same way, we also obtain a contradiction when $L(z, \overline{z})$ is a base link. □

Lemma 4 *Let G and φ be as in Lemma 2. If Π and Π' are nonadjacent A- and B-chain, respectively, then Π does not cross Π' in φ.*

Proof. Suppose, for a contradiction, that Π crosses Π'. Then an edge of a link $L(z, \overline{z})$ of Π' crosses a 3-cycle $C = xv\bar{v}$ of a link L of Π. Let $\overline{C} = \bar{x}v\bar{v}$ be a 3-cycle that is adjacent to C in L. (If L is not a base link, then $L = C \cup \overline{C}$ and $x, \bar{x}$ are the core vertices of L.) By Lemma 3, we may assume that all core vertices of Π' lie outside C and that exactly one vertex u of Π' lies inside C. The vertex u is 3-valent. By (A) we may also assume that all core vertices of Π' lie outside $\overline{C}$. Since the edge $v\bar{v}$ in $C \cap \overline{C}$ is crossed by an edge of $L(z, \bar{z})$, also $\overline{C}$ contains precisely one vertex u' of $L(z, \bar{z})$ and u' has degree 3 and is not a core vertex. In particular, C lies outside $\overline{C}$, and $\overline{C}$ lies outside C.

Adjacent trivalent vertices u, u' cannot be contained in a base link. Therefore, $L(z, \overline{z})$ is not a base link. Let $L(z, \overline{z})$ be depicted as shown in Fig. 5(a). Because of symmetry, we may assume that $u = w$ and $u' = \varepsilon$, and that the crossings are as shown in Fig. 5(b) and (c).

In the case of Fig. 5(b) the adjacent vertices z and $\overline{w}$ of $L(z, \overline{z})$ are separated by the 3-cycle $\overline{z}w\varepsilon$, whose edges are crossed by three edges different from the edge $z\overline{w}$, a contradiction.

Consider the case in Fig. 5(c). If x and $\overline{x}$ are core vertices of Π, then they are separated by the 3-cycle $\overline{zw}\varepsilon$ of Π', a contradiction.

Suppose that x and $\overline{x}$ are not two core vertices. This is possible only when L is a base link. The vertex $\overline{v}$ is separated by the 3-cycle $\overline{z}w\varepsilon$ from the three vertices $x, v, \overline{x}$ of L, hence $\overline{v}$ is not a core vertex and has valence 3. The link L has exactly one noncore vertex $\overline{v}$ of valence 3 and the vertices x and $\overline{x}$ are adjacent. Hence the 3-cycle $x\overline{vx}$ separates two core vertices z and $\overline{z}$ of Π', a contradiction. □

Theorem 1 *Every chain graph is an MN-graph.*

Proof. Let G be a chain graph. By Lemma 1, it suffices to prove that G is not 1-planar. Consider, for a contradiction, a 1-immersion φ of G and suppose that φ has minimum number of crossings.

We know by Lemma 4 that non-adjacent chains do not cross each other. In the sequel we will consider possible ways that the Ω-edges cross with one of the chains. Let us first show that such a crossing is inevitable.

Claim 1 *At least one of the chains contains a link $L = L(x, y)$ such that every (x, y)-path in L is crossed by an Ω-edge.*

Proof. Suppose that for every link $L = L(x, y)$, an (x, y)-path in L is not crossed by any Ω-edge. Then every chain contains a path joining its end vertices that is not crossed by the Ω-edges. All six such paths plus the Ω-edges form a subgraph of G that is homeomorphic to $K_{3,3}$. By Lemma 4, the only crossings between subdivided edges of this $K_{3,3}$-subgraph are among adjacent paths. However, it is easy to eliminate crossings between adjacent paths and obtain an embedding of $K_{3,3}$ in the plane. This contradiction completes the proof of the claim. □

Let $L = L(x, y)$ be a link in an A- or B-chain Π whose (x, y)-paths are all crossed by the Ω-edges. We may assume that L is contained in a chain connecting the vertex $\Omega(1)$ with A or B and that x separates y and $\Omega(1)$ in Π. By Lemma 2, the induced 1-immersion of L is an embedding. The vertex Ω lies inside a face of L and all Ω-edges that cross L cross the edges of the boundary of the face. Considering the possible embeddings of L, it is easy to see that all (x, y)-paths are crossed by Ω-edges only in the case when Ω lies inside a face of L whose boundary contains two core-adjacent edges of a free cycle C of L, and two Ω-edges cross the two core-adjacent edges. By (A), we may assume that Ω lies inside C.

If C is a k-cycle, $k \in \{3, 4\}$, then L has another cycle C' that shares with C exactly $k - 2$ edges and contains a core vertex not belonging to C. If C lies inside C', then we consider the plane as the complex plane and apply the Möbius transformation $f(z) = 1/(z-a)$ with the point a taken inside C' but outside C. This yields a 1-immersion of G such that C does not lie inside C' and Ω lies inside C. Hence, we may assume that C does not lie inside C', that Ω lies inside C and two Ω-edges h and h' cross two core-adjacent edges of C. Note that any two among the vertices $A, B, \Omega(2), \Omega(3)$ are joined by four edge-disjoint paths not using any edges in the chain Π containing L. Therefore, these four vertices of G are all immersed in the same face of L.

Let the Ω-edges h and h' join the vertex Ω with basic vertices $\Omega(i)$ and $\Omega(j)$, respectively. If the third basic vertex $\Omega(\ell)$ is $\Omega(2)$ or $\Omega(3)$, then $\Omega(2)$ and $\Omega(3)$ lie inside different faces of L, a contradiction, hence $\Omega(\ell) = \Omega(1)$.

The vertex $\Omega(1)$ is connected to one of the vertices A and B by two edge-disjoint paths, not passing the vertices of C. Hence $\Omega(1)$ does not lie inside the 3-cycle C.

Now the embeddings of possible links L (so that we can join the vertices Ω and $\Omega(1)$ by an edge not violating the 1-planarity) are shown in Figure 6.

Let us now consider particular cases (a)–(f) of Figure 6.

(a): In this case, L is a base link. Consider two edge-disjoint paths in a chain $\overline{\Pi}$ joining $\Omega(1)$ with the vertex A or B which is not incident with Π. These paths must cross the edges e and f indicated in the figure. Let a be the edge crossing e and b be the edge crossing f. It is easy to see that a and b cannot be both incident with $\Omega(1)$ since $\Omega(1)$ is incident with three edges of the base link in the chain $\overline{\Pi}$. The 1-planarity implies that the edges a, b and the vertex $\Omega(1)$ separate the graph G. Therefore, a and b are core-adjacent edges of a link in the chain $\overline{\Pi}$. If the edge g (shown in the figure) is crossed by an edge c of $\overline{\Pi}$, then also a, c and $\Omega(1)$ separate the graph. Thus a, c would be core-adjacent edges in a link in $\overline{\Pi}$ as well, a contradiction. But if g is not crossed and a is not incident with $\Omega(1)$, then the edge a and the vertex $\Omega(1)$ separate G, a contradiction. If a is incident with $\Omega(1)$, then b is not (as proved above). Now we get a contradiction by considering the separation of G by the edge b and the vertex $\Omega(1)$.

(b): In this case, L is a B-link, the cycle C' lies inside C and $\Omega(1)$ is inside C. Again, consider two edge-disjoint paths in the A-chain $\overline{\Pi}$ joining $\Omega(1)$ with the vertex A. Their edges a and b (say) must cross the edges e

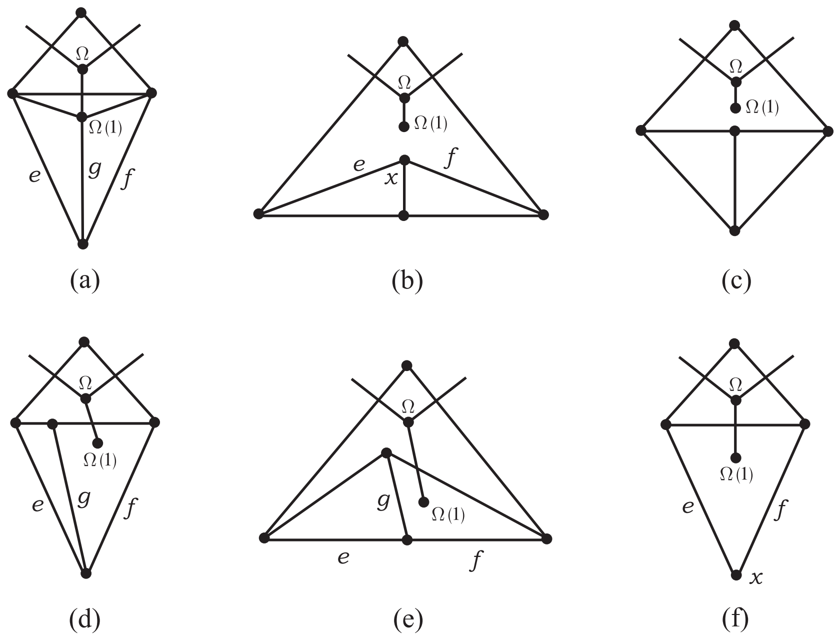


Figure 6: The Ω-edges crossing a link

and f, respectively. As in the proof of case (a), we see that the edges a, b and the vertex $\Omega(1)$ separate G, thus a and b are core-adjacent edges of a free cycle of a link of $\overline{\Pi}$. This free cycle has length 3 and separates x from $\Omega(1)$, a contradiction.

(c): In this case, L is a B-link, the cycle C' lies outside C and $\Omega(1)$ is inside C. The vertex $\Omega(1)$ is connected by 4 edge-disjoint paths with vertices x and A such that the paths do not cross noncore vertices of L, a contradiction.

(d) and (e): In this case, L is a B-link and $\Omega(1)$ lies inside a 3-cycle of L. Two edge-disjoint paths cross the edges e and f of L. One of the paths also crosses the edge g. Then the edges crossing e and g separate G, a contradiction, since G is 3-edge-connected.

(f): In this case, L is an A-link and the chain Π is joining $\Omega(1)$ with the vertex A. Again, consider two edge-disjoint paths in the B-chain $\overline{\Pi}$ joining $\Omega(1)$ with B. Their edges a and b (say) must cross the edges e and f, respectively. As in the proof of case (a), we see that the edges a, b and the vertex $\Omega(1)$ separate G, thus a, b are core-adjacent edges in a link $L' = L(z, z')$ in $\overline{\Pi}$. Let z be the core vertex of L' incident with the edges $a = zp$ and $b = zq$. Note that in L', vertices p, q have two common neighbors, a vertex r of degree 3 in G and the core vertex z', and that z' is adjacent to r. Inside L we have the subchain Π' of the A-chain Π connecting x with $\Omega(1)$. Hence, the free cycle of L' containing the edges a and b must have length at least 4 (that is, L' is not a base link) and z is immersed outside L, while p, q, r, z' are inside. The subchain Π' has two edge-disjoint paths that are crossed by the paths prq and $pz'q$. Each of the paths prq and $pz'q$ crosses core-adjacent edges in Π since the crossed edges and $\Omega(1)$ separate G. Thus, they enter faces of these links that are bounded by free cycles of the links. However, the edge rz' would need to cross one edge of each of these two free cycles, hence the two free cycles are adjacent, that is, they are the two free cycles of an A-link of Π. Then the vertex z' lies inside one of the two free cycles and the free cycle separates z' from $\Omega(1)$, a contradiction, since z' is connected with $\Omega(1)$ by a B-subchain. $\square$

The following theorem shows how chain graphs can be used to construct exponentially many nonisomorphic MN-graphs of order n.

Theorem 2 *For every integer $n \geq 63$, there are at least $2^{(n-54)/4}$ nonisomorphic MN-graphs of order n.*

Proof. The A-chain of length t has $3t+2$ vertices and a B-chain of length t has $4t+1$ vertices. Consider a chain graph whose three A-chains have length 2, 2, and $\ell \geq 2$, respectively, and whose B-chains have length 2, 3, and $t \geq 4$, respectively. The graph has $35 + 3\ell + 4t$ vertices. Applying the modification shown in Fig. 2(e) to links of the two B-chains of the graph which have length at least 3, we obtain 2^{t-1} nonisomorphic chain graphs of order $35 + 3\ell + 4t$, where $\ell \geq 2$ and $t \geq 4$. We claim that for every integer $n \geq 63$, there are integers $2 \leq \ell \leq 5$ and $t \geq 4$ such that $n = 35 + 3\ell + 4t$. Indeed, if $n \equiv 0, 1, 2, 3 \pmod 4$, put $\ell = 3, 2, 5, 4$, respectively. If $n = 35 + 3\ell + 4t$, where $2 \leq \ell \leq 5$, then $t \geq n/4 - 50/4$. Hence, there are at least $2^{\frac{n}{4} - \frac{54}{4}}$ nonisomorphic chain graphs of order $n \geq 63$. Since every chain graph is a MN-graph, the theorem follows. $\square$

3 PN-graphs

By a *proper* 1-immersion of a graph we mean a 1-immersion with at least one crossing point. Let us recall that a PN-*graph* is a planar graph that does not have proper 1-immersions. In this section we describe a class of PN-graphs and construct some graphs of the class. They will be used in Section 4 to construct MN-graphs.

Two disjoint edges vw and $v'w'$ of a graph G are *paired* if the four vertices v, w, v', w' are all four vertices of two adjacent 3-cycles. For every cycle C of G, denote by $N(C)$ the set of all vertices of the graph not belonging to C but adjacent to C.

Following Tutte, we call a cycle C of a graph G *peripheral* if it is an induced cycle in G and $G - V(C)$ is connected. If G is 3-connected and planar, then the face boundaries in its (combinatorially unique) embedding in the plane are precisely its peripheral cycles.

Theorem 3 *Suppose that a 3-connected planar graph G satisfies the following conditions:*

(C1) *Every vertex has degree at least* 4 *and at most* 6.

(C2) *Every edge belongs to at least one* 3*-cycle.*

(C3) *Every* 3*-cycle is peripheral.*

(C4) *Every* 3*-cycle is adjacent to at most one other* 3*-cycle.*

(C5) *No vertex belongs to three mutually edge-disjoint* 3*-cycles.*

(C6) *Every* 4*-cycle is either peripheral or is the boundary of two adjacent triangular faces.*

(C7) *For every* 3*-cycle C, any two vertices of $V(G) \setminus (V(C) \cup N(C))$ are connected by four edge-disjoint paths not passing through the vertices of C.*

(C8) *If an edge vw of a nontriangular peripheral cycle C is paired with an edge $v'w'$ of a nontriangular peripheral cycle C', then:*

   (i) *C and C' have no vertices in common;*

   (ii) *any two vertices a and a' of C and C', respectively, such that $\{a, a'\} \not\subseteq \{v, w, v', w'\}$ are non-adjacent and are not connected by a path aba' of length* 2*, where b does not belong to C and C'.*

(C9) *G does not contain the subgraphs shown in Fig. 7 (in this figure,* 4*-valent (resp.* 5*-valent) vertices of G are encircled (resp. encircled twice)).*

Then G has no proper 1*-immersion.*

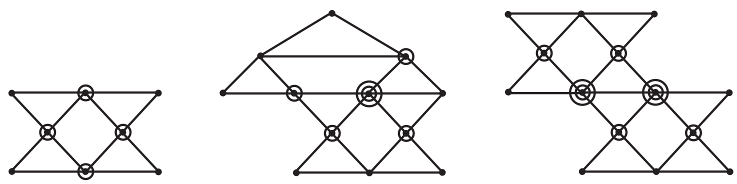


Figure 7: Forbidden subgraphs.

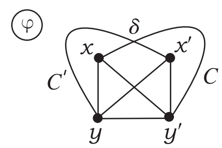


Figure 8: Crossing two adjacent 3-cycles.

Proof. Denote by f the unique plane embedding of G. Suppose, for a contradiction, that there is a proper 1-immersion φ of G. Below we consider the 1-immersion and show that then G has a subgraph which is excluded by (C8) and (C9), thereby obtaining a contradiction. In the figures below, the encircled letter f (resp. φ) at the top left of a figure means that the figure shows a fragment of the plane embedding f (resp. 1-immersion φ).

Lemma 5 *In φ, there is a 3-cycle such that there is a vertex inside and a vertex outside the cycle.*

Proof. The 1-immersion φ has crossing edges e and e'. By (C2), the crossing edges belong to different 3-cycles. If the 3-cycles are nonadjacent, then we apply the following obvious observation:

(a) If two nonadjacent 3-cycles D and D' cross each other, then there is a vertex of D inside and outside D'.

If $e = xy$ and $e' = x'y'$ belong to adjacent 3-cycles xyy' and $x'yy'$, respectively (see Fig. 8), then, by (C4), there are nontriangular peripheral cycles C and C' containing e and e', respectively. The cycles C and C' intersect at some point δ different from the intersection point of edges e and e'. By (C8)(i), the two cycles do not have a common vertex, hence δ is the intersection point of two edges. By (C2), these two edges belong to some 3-cycles, D and D'. Property (C8)(ii) implies that D and D' are nonadjacent 3-cycles. By (a), the proof is complete. $\square$

Lemma 6 *If $C = u_1u_2u_3$ is a 3-cycle such that there is a vertex inside and a vertex outside C, then there is only one vertex inside C or only one vertex outside C, respectively, and this vertex belongs to $N(C)$.*

Proof. By (C7), we may assume that all vertices of $V(G) \setminus (V(C) \cup N(C))$ lie outside C. Then there can be only vertices of $N(C)$ inside C. To prove the lemma, it suffices to show the following:

(a) For every $Q \subseteq N(C)$, $|Q| \geq 2$, at least four edges join vertices of Q to vertices in $V(G) \setminus (V(C) \cup Q)$.

By (C1), every vertex of Q has valence at least 4. By (C4), every vertex of $N(C)$ is adjacent to at most two vertices of C. We claim that if a vertex $v \in N(C)$ is adjacent to two vertices u_1 and u_2 of C, then v is not adjacent to other vertices of $N(C)$. Suppose, for a contradiction, that v is adjacent to a vertex $w \in N(C)$. Then, by (C4), the vertex w can be adjacent only to u_3 and the 4-cycle vwu_3u_2 is not the boundary of two adjacent 3-cycles, hence, by (C6), the 4-cycle is peripheral. Then, by (C3), any two of the three edges u_2v, u_2u_1, and u_2u_3 are two edges of a peripheral cycle, hence u_2 has valence 3, contrary to (C1).

Now to prove (a) it suffices to prove the following claim:

(b) For every $Q \subseteq N(C)$, $|Q| = 1$ (resp. $|Q| \geq 2$), such that every vertex of Q is adjacent to exactly one vertex of C, at least 2 (resp. 4) edges join vertices of Q to vertices of $V(G) \setminus (V(C) \cup Q)$.

The claim is obvious for $|Q| \in \{1, 2\}$. For $|Q| = 3$, it suffices to show that the three vertices of Q are not pairwise adjacent. Suppose, for a contradiction, that the vertices v_1, v_2, and v_3 of Q are pairwise adjacent. Then, by (C4), the three vertices of Q are not adjacent to the same vertex of C. Let v_1 and v_2 be adjacent to the vertices u_1 and u_2 of C, respectively. Then any two of the edges v_1u_1, v_1u_2, and v_1u_3 are two edges of a 3-cycle (peripheral cycle) or a 4-cycle which is not the boundary of two adjacent 3-cycles, so by (C6), that 4-cycle is also peripheral. Hence, v_1 has valence 3, contrary to (C1).

For $|Q| \geq 4$, it suffices to show that no vertex of Q is adjacent to three other vertices in Q. Suppose, for a contradiction, that $v \in Q$ is adjacent to $w_1, w_2, w_3 \in Q$. If v is adjacent to u_1, then the edge vu_1 belongs to three cycles D_1, D_2, and D_3 such that for $i = 1, 2, 3$, the cycle D_i contains edges vu_1 and vw_i, has length 3 or 4, and if D_i has length 4, then D_i is not the boundary of two adjacent 3-cycles. By (C3) and (C6), these three cycles are peripheral. This contradiction completes the proof of (b). □

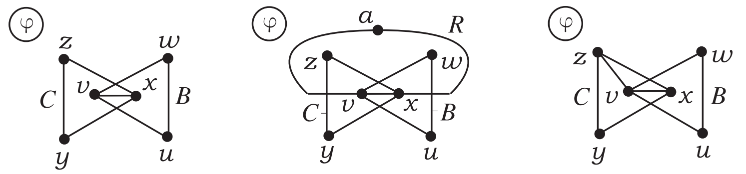


Figure 9: The 3-cycle B separates x from y and z.

Suppose that a vertex h belongs to two adjacent 3-cycles hvw and hvw'. Since $\deg(h) \geq 4$, h is adjacent to a vertex $u \notin \{v, w, w'\}$. By (C2), the edge hu belongs to a 3-cycle huu'. By (C4), $u' \notin \{v, w, w', u\}$. Hence, we have the following:

(B) If an edge e is contained in two 3-cycles of G, then both endvertices of e have valence at least 5.

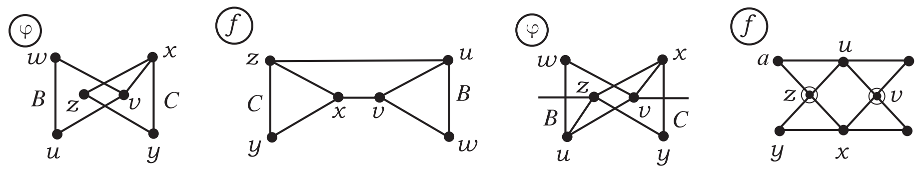
In the remainder of the proof of Theorem 3, we will show that any two crossing edges of the proper 1-immersion φ belong to a subgraph that is excluded by (C8) and (C9).

By Lemma 5, there is a 3-cycle $C = xyz$ such that there is a vertex inside and a vertex outside C. By Lemma 6, there is only one vertex v inside C and v is adjacent to x.

Now we show that there is a 3-cycle $B = vuw$ disjoint from C. Let $x, a_1, a_2, \ldots, a_t$ ($t \geq 3$) be all vertices adjacent to v. Suppose there is a 3-cycle $D = va_ib$, where $b \in \{x, y, z\}$. If $b \in \{y, z\}$, then the 3-cycle xvb is adjacent to two 3-cycles C and D, contrary to (C4). Hence $D = va_ix$. By (C4), at most two vertices of $\{a_1, a_2, \ldots, a_t\}$ are adjacent to x. Hence, there is a vertex a_j such that a 3-cycle containing the edge va_j is disjoint from C.

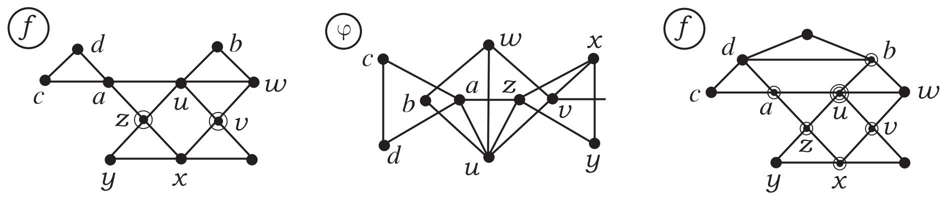
Since there is only one vertex v inside C, exactly two edges of B cross edges of C. First, suppose that B separates x from y and z (Fig. 9(a)). By (C2), the edge xv belongs to a 3-cycle $R = xva$. If $a \notin \{y, z, u, v\}$, then two of the vertices y, z, u, w lie inside R and the other two vertices lie outside R (see Fig. 9(b)), contrary to Lemma 6. So, we may assume, without loss of generality, that $a = z$ (Fig. 9(c)). Then the vertex x belongs to two adjacent 3-cycles, C and vxz, hence, by (B), $\deg(x) \geq 5$. By (C4), x is not adjacent to u or w. Since x is the only vertex inside B, x has valence at most 4, a contradiction. Hence, B can not separate x from y and z.

Now suppose that B separates the vertices y and z, and, without loss of



(a) (b) (c) (d)

Figure 10: The 3-cycle B separates y and z.



(a) (b) (c)

Figure 11: The edge uw belongs to two 3-cycles.

generality, let z lie inside B (Fig. 10(a)). If z is adjacent to v, then, by (B), z has valence at least 5, hence z is adjacent to a vertex $b \in \{u, w\}$. But then the 3-cycle xzv is adjacent to two 3-cycles, C and vuz, contrary to (C4). If z is adjacent to u and w, then the edge xz belongs to three peripheral cycles, C, $xvuz$, and $xvwz$, a contradiction, since every edge belongs to at most two peripheral cycles. Hence, since z is the only vertex inside B, z has valence 4, z is not adjacent to v and is adjacent to exactly one of the vertices u and w, say u (Fig. 10(b)).

Consider the vertex v. If v is adjacent to y, then (see Fig. 10(b)) x is incident with exactly three peripheral cycles and has valence 3, a contradiction. Hence, v is not adjacent to y and since v is the only vertex inside C, v has valence 4 (see Fig. 10(c)). By (C2), we obtain a subgraph shown in Fig. 10(d). By (C9), at least one of the vertices u and x, say u, is not 4-valent. Then, by (C5), at least one of the edges au and uw belongs to two 3-cycles. Here we have two cases to consider.

Case 1: The edge uw belongs to two 3-cycles uwv and uwb (Fig. 11(a)).

Now, a is the only vertex inside the 3-cycle uwb (Fig. 11(b)). By (C4), a is non-adjacent to w and b, hence a has valence 4. The edges yz and za (see Fig. 11(a)) belong to a nontriangular peripheral cycle $yzac\ldots$. The edge ac

belongs to a 3-cycle acd and b is the only vertex inside the 3-cycle acd. The vertex b is not adjacent to c, since the 4-cycle $bcau$ can not be peripheral (see Fig. 11(a)). Since b has valence at least 4, b is adjacent to d and has valence 4. The 4-cycle $dbua$ is peripheral, so we obtain a subgraph of G shown in Fig. 11(c). Note that, by (C3)–(C6), the vertex at the top of Figure 11(c) is different from all other vertices shown in the figure. This contradicts (C9).

Case 2: The edge au belongs to two 3-cycles auz and aub (Fig. 12(a)).

Since b has valence at least 4, b lies outside the 3-cycle auz (Fig. 12(b)). The edges yz and za (resp. wu and ub) belong to a nontriangular peripheral cycle C_1 (resp. C_2). The cycles C_1 and C_2 are paired. In φ, the crossing point of edges uw and az is an intersection point of C_1 and C_2. The cycles C_1 and C_2 have at least one other crossing point, denote this intersection point by δ. By (C8)(i), C_1 and C_2 are vertex-disjoint, hence, δ is the crossing point of C_1 and an edge h_1h_2 of C_2 (Fig. 12(c)). The edge h_1h_2 is not the edge uw and belongs to a 3-cycle $h_1h_2h_3$. If h_3 belongs to C_2, then in the embedding f the edge h_1h_3 is a chord of the embedded peripheral cycle C_2 and thus $\{h_1, h_3\}$ is a separating vertex set of G. But G is 3-connected, a contradiction. Hence h_3 does not belong to C_2.

Now suppose that $h_1h_2 \neq bu$. We have $h_2 \notin \{a, z, b, u\}$. By (C8)(ii), $h_3 \neq a$, a is not adjacent to h_2 and h_3, and C_1 does not pass through h_3 (that is, h_3 does not belong to C_1 and C_2). Hence, a vertex s of C_1 lies inside the 3-cycle $h_1h_2h_3$ (see Fig. 12(c)). By (B), $\deg(a) \geq 5$. If $s = a$, then, since s is the only vertex inside the 3-cycle $h_1h_2h_3$, a is adjacent to at least one of h_2 and h_3, a contradiction. Hence, $s \neq a$. Since z is the only vertex inside the 3-cycle B, and the 3-cycle $h_1h_2h_3$ is not B (since $h_1h_2 \neq uw$), we have $s \neq z$. Now, since s has valence at least 4, s is adjacent to at least one of the vertices h_1, h_2, and h_3, contrary to (C8)(ii). Hence $h_1h_2 = bu$ and $h_3 = a$.

We have $C_1 = yzah\ldots$ and the edge ah belongs to a 3-cycle ahd (see Figs. 12(d) and (e)). Considering Fig. 12(e), if there is a vertex inside the 3-cycle auz, the vertex has valence at most 3, a contradiction. Hence no edge crosses the edge au. If h lies inside the 3-cycle aub, then, by (C4), h is not adjacent to d and u, h has valence at most 3, a contradiction. Hence, h lies outside the 3-cycle aub and b is the only vertex inside the 3-cycle ahd (see Fig. 12(e)).

By (C4), b is not adjacent to d and h, hence b has valence 4 and belongs to a 3-cycle btt' disjoint from aub (Fig. 12(f)), where $C_2 = wubt\ldots$. Now d is the only vertex inside the 3-cycle btt' (Fig. 12(e)). By (C8)(ii), d is not adjacent to t. Since d has valence at least 4, d is adjacent to t' and has

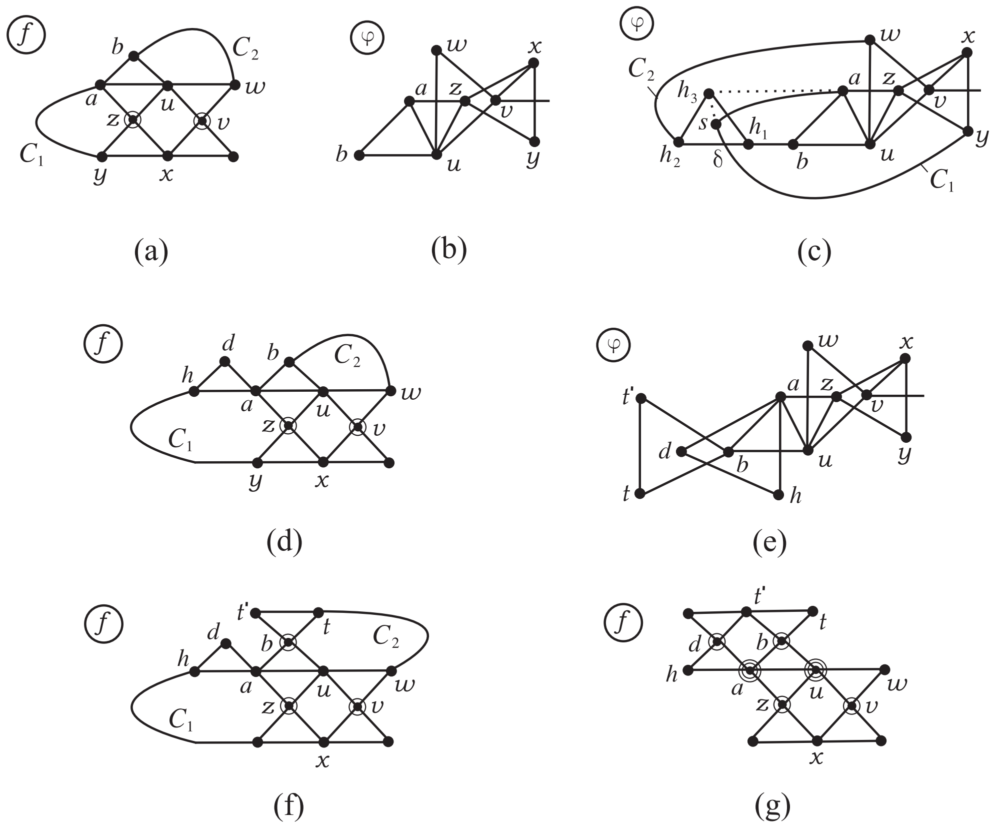


Figure 12: The edge au belongs to two 3-cycles.

valence 4. The 4-cycle $dt'ba$ is peripheral. We obtain a subgraph of G shown in Fig. 12(g), contrary to (C9). The obtained contradiction completes the proof of the theorem. □

Denote by $\mathcal{A}$ the class of all 3-connected plane graphs G satisfying the conditions (C1)–(C9) of Theorem 3. In what follows we show how to construct some graphs in $\mathcal{A}$ and, as an example, we shall give two infinite families of graphs in $\mathcal{A}$, one of which will be used in Section 4 to construct MN-graphs.

First we describe a large family of 3-connected plane graphs satisfying the conditions (C1)–(C6) and (C8) of Theorem 3.

Given a 4-valent vertex v of a 3-connected plane graph, two peripheral cycles C and C' containing v are *opposite* peripheral cycles at v if C and C' have no edges incident with v in common.

Denote by $\mathcal{H}$ the class of all 3-connected (simple) planar graphs H satisfying the following conditions (H1)–(H4):

(H1) Every vertex has valence 3 or 4.

(H2) H has no 3-cycles.

(H3) Every 4-cycle is peripheral.

(H4) For every 4-valent vertex v and for any two opposite peripheral cycles C and C' at v, no edge joins a vertex of $C - v$ to a vertex of $C' - v$.

A plane graph G is a *medial extension* of a graph $H \in \mathcal{H}$ if G is obtained from H in the following way. The vertex set of G is the set $\{v(e) : e \in E(H)\}$. The edge set of G is defined as follows. For every 3-valent vertex v of H, if e_1, e_2, e_3 are the edges incident with v, then in G the vertices $v(e_1)$, $v(e_2)$, and $v(e_3)$ are pairwise adjacent (the three edges of G are said to be *associated with* v). For every 4-valent vertex w of H, if (e_1, e_2, e_3, e_4) is the cyclic order of edges incident to w around w in the plane, then G contains the edges of the 4-cycle $v(e_1)v(e_2)v(e_3)v(e_4)$, and contains either the edge $v(e_1)v(e_3)$ or the edge $v(e_2)v(e_4)$; these five edges of G are said to be *associated with* the 4-valent vertex of H. Note that G can be obtained from the medial graph of H by adding a diagonal to every 4-cycle associated with a 4-valent vertex of H.

Lemma 7 *Every medial extension G of any graph $H \in \mathcal{H}$ is a 3-connected planar graph satisfying the conditions* (C1)–(C6) *and* (C8) *of Theorem 3.*

Proof. By the construction of G, if $\{v(e), v(e')\}$ is a separating vertex set of G, then the graph $H - e - e'$ is disconnected, a contradiction, since H is 3-connected. Hence G is 3-connected. Every peripheral cycle of H *induces* a peripheral cycle of G. It is easy to see that all peripheral cycles of G that are not induced by the peripheral cycles of H are the 3-cycles formed by the edges associated with the vertices of H.

It is easy to see that G satisfies (C1)–(C5). To show that G satisfies (C6), let $J = v(e_1)v(e_2)v(e_3)v(e_4)$ be a 4-cycle of G. By the construction of G, if vertices $v(e)$ and $v(e')$ of G are adjacent, then the edges e and e' of H are adjacent, too. Since in H no three edges among e_1, e_2, e_3, e_4 form a cycle (by (H2)), these four edges either form a 4-cycle (in this case J is peripheral) or are the edges incident to a 4-valent vertex of H (in this case, by the construction of G, J is the boundary of two adjacent faces of G). Hence, G satisfies (C6).

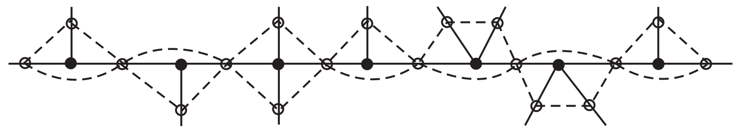


Figure 13: Two paths of G associated with a path of $\overline{H}$.

It remains to show that G satisfies (C8). Let C and C' be nontriangular peripheral cycles of G such that an edge a of C is paired with an edge a' of C'. Then, by the construction of G, the peripheral cycles C and C' are induced by peripheral cycles $\overline{C}$ and $\overline{C}'$ of H, respectively, that are opposite at some 4-valent vertex u. If C and C' have a common vertex $v(e)$, then $\overline{C}$ and $\overline{C}'$ have a common edge e, hence H has a separating vertex set $\{u, w\}$, where w is a vertex incident to e, a contradiction, since H is 3-connected. Now suppose that G has an edge joining a vertex $v(e)$ of C to a vertex $v(e')$ of C' such that at least one of the vertices $v(e)$ and $v(e')$ is not incident to a and a'. Then the edges e and e' are incident to the same vertex w of H and the cycles $\overline{C}$ and $\overline{C}'$ pass through w. It follows that $\{u, w\}$ is a separating vertex set of H, a contradiction. Next suppose that G has a path $v(e)v(b)v(e')$ connecting a vertex $v(e)$ of C to a vertex $v(e')$ of C' such that $v(b)$ does not belong to C and C', and at least one of the vertices $v(e)$ and $v(e')$ is not incident to a or a'. If in H the edges e, b, and e' are incident to the same vertex w, then $\{u, w\}$ is a separating vertex set of H, a contradiction. If in H the edges a and b (resp. b and e') are incident to a vertex w (resp. w') such that $w \neq w'$, then the edge b joins the vertex w of $\overline{C}$ with the vertex w' of $\overline{C}'$, contrary to (H4). Hence G satisfies (C8). The proof is complete. $\square$

There are medial extensions of graphs in $\mathcal{H}$ that do not satisfy conditions (C7) and (C9). In the sequel we shall describe a way how to verify conditions (C7) and (C9), and henceforth give examples of graphs satisfying (C1)–(C9). To show that a medial extension G of $H \in \mathcal{H}$ satisfies (C7) it is convenient to proceed in the following way. Subdivide every edge e of H by a two-valent vertex $v(e)$ of G. We obtain a graph $\overline{H}$ whose vertex set is $V(H) \cup V(G)$ where the vertices of $V(G)$ are all 2-valent vertices of $\overline{H}$. We will consider paths of G associated with paths of $\overline{H}$ connecting 2-valent vertices.

Two paths P and P' of $\overline{H}$ are *H-disjoint* if $P \cap P' \cap V(H) = \emptyset$, i.e., $P \cap P' \subseteq V(G)$.

Consider a path $P = v(e_1)w_1v(e_2)w_2v(e_3)\dots w_{n-1}v(e_n)$ in $\overline{H}$ where w_1,

$w_2, \ldots, w_{n-1}$ are the H-vertices on P. It is easy to see that the edges of G associated with the vertices $w_1, w_2, \ldots, w_{n-1}$ of H contain two edge-disjoint paths connecting in G the vertices $v(e_1)$ and $v(e_n)$ (see Fig. 13); any two such paths in G are said to be *associated* with the path P of $\overline{H}$. Since H has no multiple edges, every edge of G is associated with exactly one vertex of H. Hence, if P and P' are H-disjoint paths in $\overline{H}$, each of which is connecting 2-valent vertices, then every path in G associated with P is edge-disjoint from every path in G associated with P'. As a consequence, we have the following conclusion:

(C) If $\overline{H}$ has a cycle containing 2-valent vertices $v(e)$ and $v(e')$, then G has four edge-disjoint paths connecting $v(e)$ and $v(e')$.

The fact that a path in H gives rise to two edge-disjoint paths in G (paths associated with the path of H) can be used to check the property (C7) of G.

For a 3-cycle C of G, a path of $\overline{H}$ is C*-independent* if the path does not contain vertices of C. When checking (C7) for a medial extension G of $H \in \mathcal{H}$, given a 3-cycle C of G and two 2-valent vertices $x, y \in V(G) \setminus (V(C) \cup N(C)) \subset V(\overline{H})$, four C-independent edge-disjoint paths P_1, P_2, P_3, and P_4 of G connecting x and y in G will be represented in some subsequent figures (see, e.g., Figure 15) in the following way. The edges of the paths incident to vertices of $N(C)$ are depicted as dashed edges joining 2-valent vertices, the dashed edges are not edges of $\overline{H}$ (see, for example, Fig. 15(b), where the edges of H are given as solid lines). All other edges of the paths are represented by paths in $\overline{H}$. If X is a subpath of P_i such that X is associated with a path $\overline{X}$ in $\overline{H}$, then X is represented by a dashed line passing near the edges of $\overline{X}$ in $\overline{H}$. If X_i and X_j are subpaths of P_i and P_j (where possibly $i = j$), respectively, such that X_i and X_j are edge-disjoint paths associated with a path $\overline{X}$ of $\overline{H}$, then X_i and X_j are represented by two (parallel) dashed lines passing near the edges of $\overline{X}$ in $\overline{H}$. Using these conventions, the reader will be able to check that the depicted dashed paths and edges in the figure of $\overline{H}$ represent four C-independent edge-disjoint paths of G connecting x and y.

Now we describe some graphs in $\mathcal{A}$. Let us recall that graphs in $\mathcal{A}$ are precisely those 3-connected planar graphs that satisfy conditions (C1)–(C9). To simplify the arguments, we construct graphs with lots of symmetries so that, for example, to check the condition (C7) we will have to consider only two 3-cycles of a graph.

For $n \geq 6$, let $H_n \in \mathcal{H}$ be the Cartesian product of the path P_3 of length 2

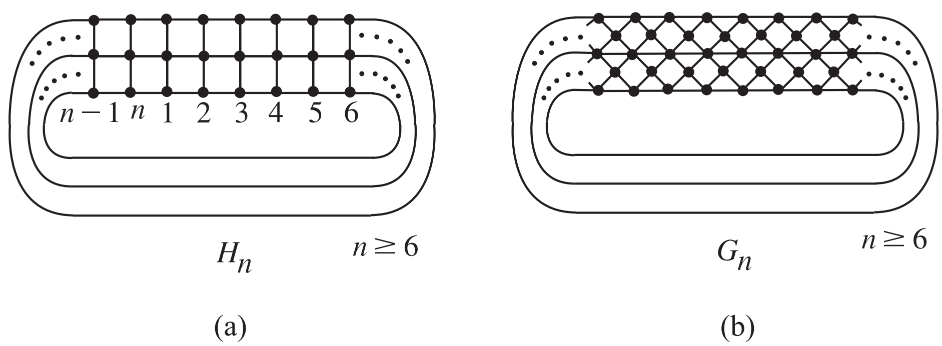


Figure 14: The graph $H_n \in \mathcal{H}$ and its extension G_n.

and the cycle C_n of length n (see 14(a)). Fig. 14(b) shows a medial extension G_n of H_n. By Lemma 7, G_n satisfies (C1)–(C6) and (C8). Every 4-gonal face of G_n is adjacent to a 6-valent vertex and G_n has no 5-valent vertices, hence G_n satisfies (C9).

Now we show that G_n satisfies (C7). Consider a fragment of $\overline{H_n}$ shown in Fig. 15(a) (in the figure we introduce notation for some vertices and also depict in dashed lines some edges of G_n). Because of the symmetries of G_n, it suffices to consider the following two cases for a 3-cycle C, when checking (C7):

Case 1. $V(C) = \{6, 9, 10\}$.

Then $N(C) = \{2, 3, 5, 7, 8, 11, 12, 13, 14\}$. If we delete from $\overline{H_n}$ the vertices $1, 2, \ldots, 14$, then the obtained graph has only one connected component U with a vertex in H_n, and U is 2-connected. Hence, any two 2-valent vertices x and y of U belong to a cycle in U and then, by (C), G_n has four C-independent edge-disjoint paths connecting x and y. Fig. 15(b) shows four C-independent edge-disjoint paths in G_n connecting vertices 1 and 4. If we delete from $\overline{H_n}$ the vertices $\{1, 2, \ldots, 18\} \setminus \{1, 4\}$, then in the resulting non-trivial connected component, for every vertex $x \in V(H_n) \setminus \{1, 4\}$, there is a path P connecting the vertices 1 and 4, and passing through x; combining two edge-disjoint paths of G_n associated with P and the two edge-disjoint paths connecting 1 and 4, shown in Fig. 15(b), we obtain four C-independent edge-disjoint paths connecting the vertices 4 and x (and, analogously, for the vertices 1 and x). Now, because of the symmetries of $\overline{H_n}$, it remains to show that there are four C-independent edge-disjoint paths connecting the vertex 4 with each of the vertices $15, 16, 17, 18$; Fig. 15(c) shows the paths (since

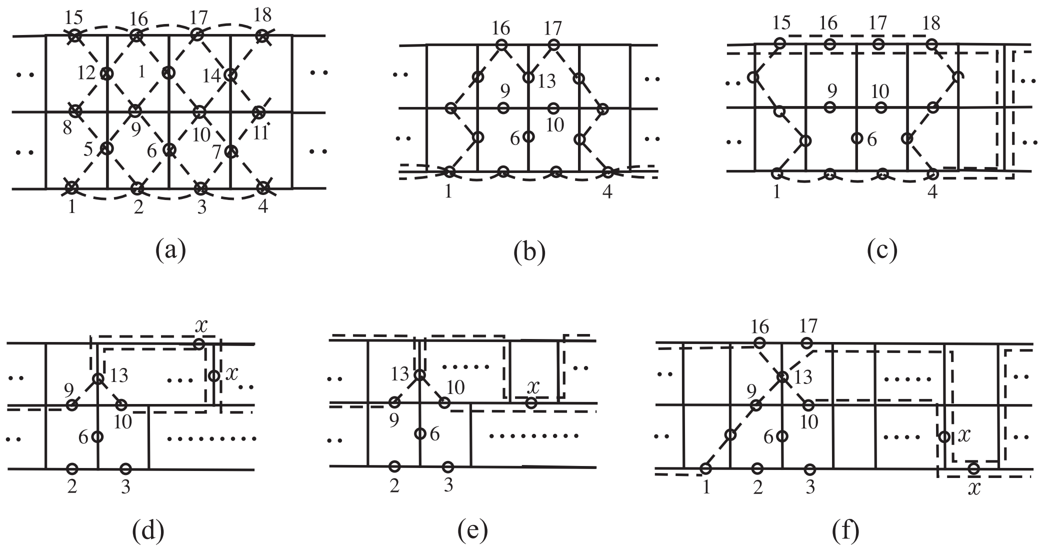


Figure 15: Verifying (C7) for the graph G_n.

$n \geq 6$).

Case 2: $V(C) = \{2, 3, 6\}$.

We have $N(C) = \{1, 4, 5, 7, 9, 10\}$. If we delete from $\overline{H_n}$ the vertices $1, 2, \ldots, 7$ and $9, 10, 13$, then the obtained graph has only one connected component U with at least two vertices and U is 2-connected. Hence any two vertices x and y in $\overline{H_n}$ that are both in U belong to a cycle of U and then, by (C), G_n has four C-independent edge-disjoint paths connecting x and y. It remains to show that for every vertex x of $\overline{H_n}$ belonging to U, there are four C-independent edge-disjoint paths connecting x and the vertex 13. These four paths are shown in Figs. 15(d)–(f), depending on the choice of x. We conclude that G_n satisfies (C1)–(C9), hence G_n is a PN-graph for every $n \geq 6$.

Fig. 16 gives another example of an extension G'_n of a graph $H'_n \in \mathcal{H}$. By Lemma 7, G'_n satisfies (C1)–(C6) and (C8). Since G'_n has no 4-valent vertices, it satisfies (C9). Using the symmetry of $\overline{H'_n}$, one can easily check that for every 3-cycle C of G'_n, if we delete from $\overline{H'_n}$ the vertices $V(C) \cup N(C)$ of G'_n, then the obtained graph has only one connected component U with at least two vertices and U is 2-connected. Then, by (C), any two vertices of G'_n in U are connected by four C-independent edge-disjoint paths. Hence, G'_n satisfies (C7) and is a PN-graph.

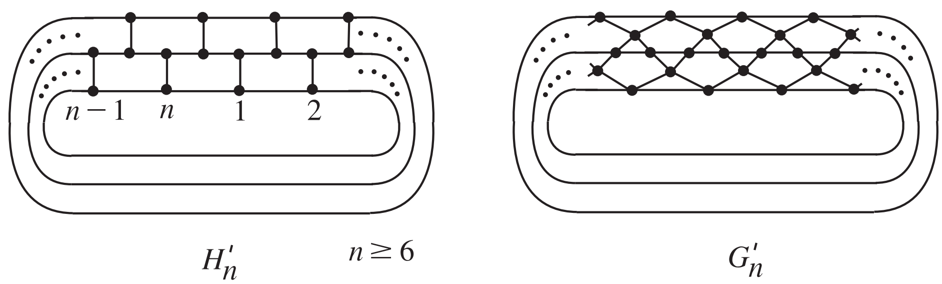


Figure 16: The graph $H_n' \in \mathcal{H}$ and its extension G_n'.

4 MN-graphs based on PN-graphs

In this section we construct MN-graphs based on the PN-graphs G_n described in Section 3.

For $m \geq 2$, denote by S_m, the graph shown in Fig. 17. The graph has $m+1$ disjoint cycles of length $12m-2$ labelled by $B_0, B_1, \dots, B_m$ as shown in the figure. The vertices of B_0 are called the *central vertices* of S_m and are labelled by $1, 2, \dots, 12m-2$ (see Fig. 17). For every central vertex $x \in \{1, 2, \dots, 12m-2\}$, denote by x^* its "opposite" vertex $x+(6m-1)$ if $x \in \{1, 2, \dots, 6m-1\}$ and the vertex $x-(6m-1)$ if $x \in \{6m, 6m+1, \dots, 12m-2\}$. In S_m, any pair $\{x, x^*\}$ of central vertices is connected by a *central path* $P(x, x^*)$ of length $6m-3$ with $6m-4$ two-valent vertices. There are exactly $6m-1$ central paths.

For any integers $m \geq 4$ and $n \geq 0$, denote by $\Phi_m(n)$ the set of all $(12m-2)$-tuples $(n_1, n_2, \dots, n_{12m-2})$ of nonnegative integers such that $n_1 + n_2 + \dots + n_{12m-2} = n$. For every $\lambda \in \Phi_m(n)$, denote by $S_m(\lambda)$ the graph obtained from S_m by replacing, for every central vertex $x \in \{1, 2, \dots, 12m-2\}$, the eight edges marked by short crossings in Fig. 18(a) by $8(1+n_x)$ new edges marked by crossings in Fig. 18(b) (the value $x+1$ in that figure is to be considered modulo $12m-2$). The graph $S_m(\lambda)$ has $m-2$ $(12m-2)$-cycles $B_0, B_1, \dots, B_{m-3}$ and three $(12m-2+n)$-cycles B_{m-2}, B_{m-1}, B_m.

We want to show that for every $m \geq 4$ and for every $\lambda \in \Phi_m(n)$, $n \geq 0$, the graph $S_m(\lambda)$ is an MN-graph.

Lemma 8 *For every $m \geq 4$ and $\lambda \in \Phi_m(n)$, the graph $S_m(\lambda) - e$ is 1-planar for every edge e.*

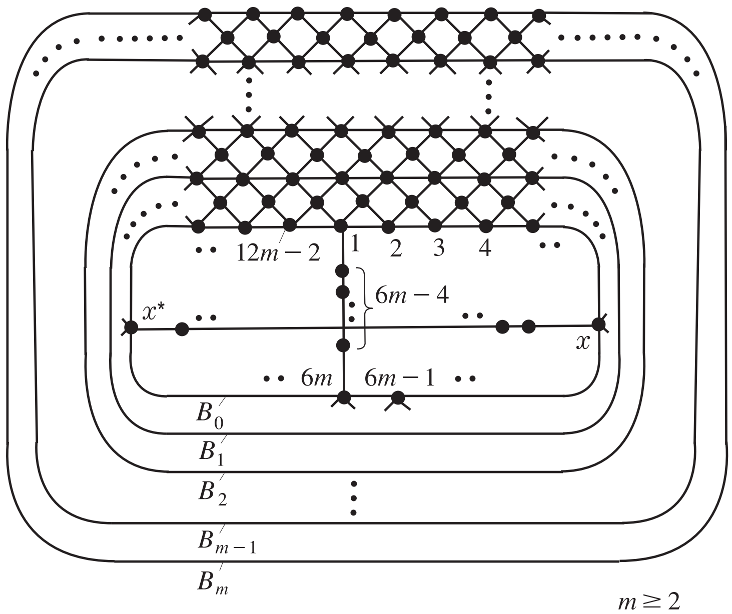


Figure 17: The graph S_m.

Proof. If we delete an edge of a central path, then the remaining $6m-2$ central paths, each with $6m-3$ edges, can be 1-immersed inside B_0 in Fig. 17. If we delete one of the edges shown in Fig. 19(a) by a thick line, then the central path $P(x, x^*)$ can be drawn outside B_0 with $6m-3$ crossing points as shown in the figure and then the remaining $6m-2$ central paths can be 1-immersed inside B_0. If we delete one of the two edges depicted in Fig. 19(a) by a dotted line, then Fig. 19(b) shows how to place the central vertex x so that the path $P(x, x^*)$ can be drawn outside B_0 with $6m-3$ crossing points (analogously to Fig. 19(a)) and then the remaining $6m-2$ central paths can be 1-immersed inside B_0. This exhibits all possibilities for the edge e (up to symmetries of S_m) and henceforth completes the proof. $\square$

Given a 1-immersion of a graph G and an embedded cycle C, we say that G lies *inside* (resp. *outside*) C, if the exterior (resp. interior) of C does not contain vertices and edges of G.

Denote by J_{m-2} the graph obtained from the graph S_m in Fig. 17 by deleting the 2-valent vertices of all central paths and by deleting all vertices lying outside the cycle B_{m-2}.

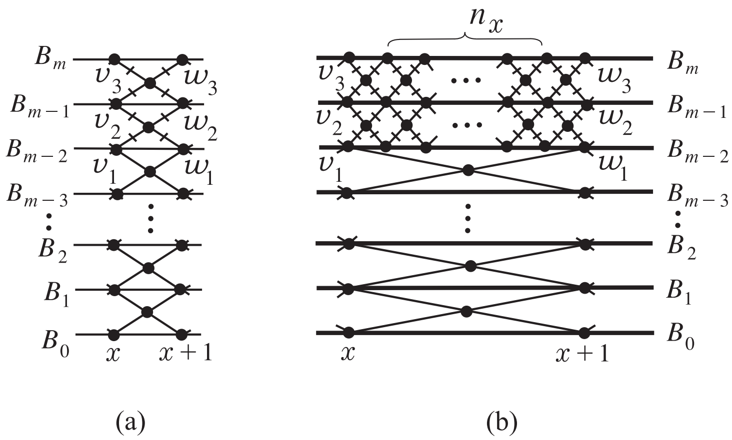


Figure 18: Obtaining the graph $S_m(\lambda)$.

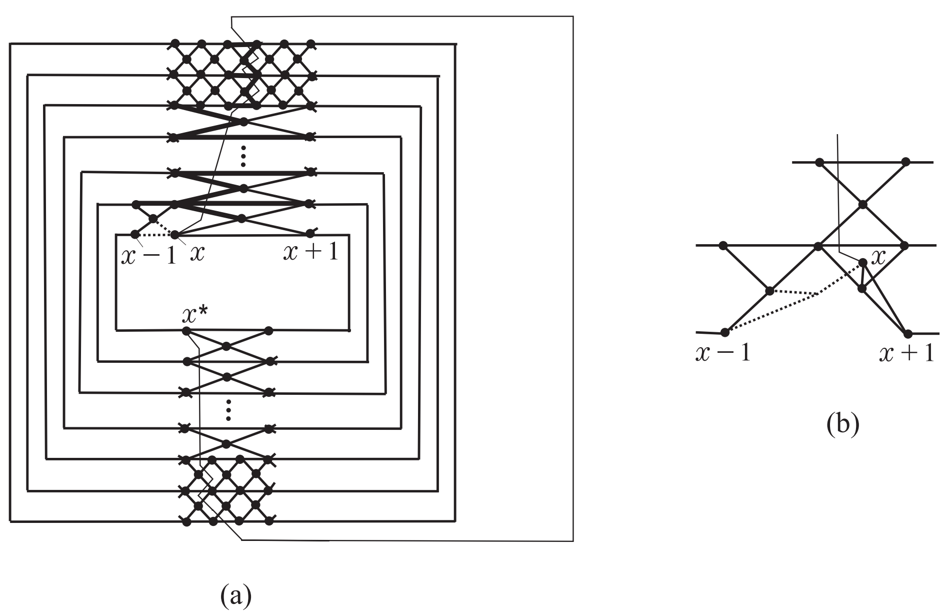


Figure 19: The central path $P(x, x^*)$ immersed outside B_0.

Lemma 9 *For every $m \geq 4$, J_{m-2} is a PN-graph.*

Proof. The graph J_{m-2} contains $m-3$ subgraphs $L_1, L_2, \ldots, L_{m-3}$ isomorphic to the PN-graph G_{12m-2} such that for $i = 1, 2, \ldots, m-1$, the graph L_i contains the cycles B_{i-1}, B_i, and B_{i+1}. Consider an arbitrary 1-immersion φ of J_{m-2}. Suppose that in the plane embedding of the PN-graph L_1 in φ, the cycle B_2 is the boundary cycle of the outer $(12m-2)$-gonal face of the embedding. Then the embedding of L_1 determines an embedding of the subgraph of L_2 bounded by the cycles B_1 and B_2. Since L_2 is a PN-graph, the subgraph of L_2 bounded by B_2 and B_3 lies outside the cycle B_2. Reasoning similarly, we obtain that for $i = 3, 4, \ldots, m-3$, the subgraph of the PN-graph L_i bounded by B_i and B_{i+1} lies outside B_i. As a result, φ is a plane embedding of J_{m-2}, hence J_{m-2} is a PN-graph. □

Denote by $\overline{S}_m(\lambda)$ the graph obtained from $S_m(\lambda)$, where $m \geq 4$ and $\lambda \in \Phi_m(n)$, by deleting the 2-valent vertices of all central paths.

Lemma 10 *For every $m \geq 4$ and $\lambda \in \Phi_m(n)$, $\overline{S}_m(\lambda)$ is a PN-graph.*

Proof. The graph $\overline{S}_m(\lambda)$ contains a subgraph G isomorphic to the PN-graph $G_{12m-2+n}$ and contains a subgraph G' homeomorphic to the PN-graph J_{m-2}. The graph G contains the cycles B_{m-2}, B_{m-1}, and B_m of $\overline{S}_m(\lambda)$, and the graph G' contains the cycles $B_0, B_1, \ldots, B_{m-2}$ of $\overline{S}_m(\lambda)$ and is obtained from J_{m-2} by subdividing the edges of the cycle B_{m-2} (by using n 2-valent vertices in total).

Consider, for a contradiction, a proper 1-immersion φ of $\overline{S}_m(\lambda)$. In φ, the graph G has a plane embedding and we shall investigate in which faces of the embedding of G lie the vertices of G'. We shall show that they all lie in the face of G bounded by the (subdivided) cycle B_{m-2}.

In the graph $\overline{S}_m(\lambda)$ the cycles B_{m-2} and B_i, $i \in \{0, 1, \ldots, m-3\}$ are connected by $24m-4$ edge-disjoint paths. This implies that no 3- or 4-gonal face of G contains all vertices of B_i in its interior.

Any two vertices of B_i are connected by six edge-disjoint paths in $G' - B_{m-2}$. Therefore:

(a) No 3- or 4-gonal face of G contains any vertex of the cycles B_i, $i = 0, 1, \ldots, m-3$, in its interior.

Suppose that a vertex v of G' does not belong to the cycles B_i, $i = 1, 2, \ldots, m-2$, and lies inside a 3- or 4-gonal face F of G. By construction

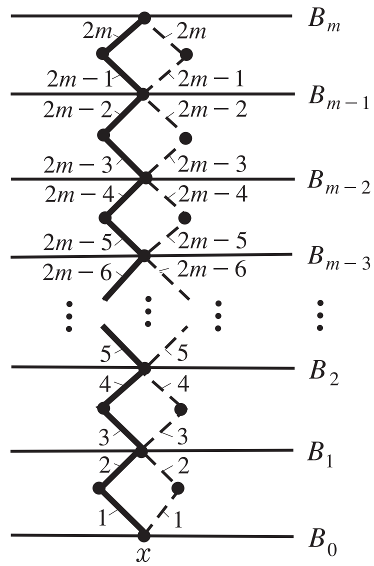


Figure 20: The paths associated with a central vertex and the types of edges.

of G', the vertex v is adjacent to two vertices w and w' of some B_j, $j \in \{0, 1, \ldots, m-3\}$. By (a), w and w' do not lie inside F, hence they lie, respectively, in faces F_1 and F_2 of G adjacent to F. However, at least one of F_1 and F_2 is 3- or 4-gonal, contrary to (a). This implies that all vertices of $G' - B_{m-2}$ lie inside the face of G bounded by B_{m-2}. Hence G' lies inside B_{m-2} and has a proper 1-immersion in φ. If in this 1-immersion of G' we ignore the 2-valent vertices on the cycle B_{m-2} of $\overline{S}_m(\lambda)$, then we obtain a proper 1-immersion of the PN-graph J_{m-2}, a contradiction. $\square$

By the paths of $\overline{S}_m(\lambda)$ associated with any central vertex x we mean the two paths shown in Fig. 20; one of them is depicted in thick line and the other in dashed line. Every edge of $\overline{S}_m(\lambda)$ not belonging to the cycles $B_0, B_1, \ldots, B_m$ is assigned a *type* $t \in \{1, 2, \ldots, 2m\}$ as shown in Fig. 20 such that for $i = 1, 2, \ldots, m$, the edges of type $2i-1$ and $2i$ are all edges lying between the cycles B_{i-1} and B_i, and the edges of type $2i-1$ (resp. $2i$) are incident to vertices of B_{i-1} (resp. B_i).

Suppose that there is a 1-immersion φ of $S_m(\lambda)$. By Lemma 10, $\overline{S}_m(\lambda)$ is a PN-graph. Thus, φ induces an embedding of this graph. We shall assume that the outer face F_0 of this embedding is bounded by the cycle B_m. We shall first show that F_0 is also a face of φ. To prove this, it suffices to see that no central cycle can enter F_0.

Any central vertex x is separated from F_0 by $3m-1$ edge-disjoint cycles: m cycles $B_1, B_2, \dots, B_m$ and $2m-1$ cycles $C_2, C_3, \dots, C_{2m}$, where the cycle C_i consists of all edges of type i ($i = 2, 3, \dots, 2m$). The central path $P = P(x, x^*)$ can have at most $6m-3$ crossing points, hence P cannot enter F_0. If P lies between B_0 and B_m in φ, then it must cross $2(6m-2)$ paths associated either with $6m-2$ central vertices $x+1, x+2, \dots, x^*-1$ or with $6m-2$ central vertices $x^*+1, x^*+2, \dots, x-1$ (here we interpret all additions modulo $12m-2$), a contradiction. Hence, in φ any central path either lies inside B_0 or crosses some edges of $\overline{S}_m(\lambda)$ but does not lie entirely between B_0 and B_m.

The main goal of this section is to show that $S_m(\lambda)$ has no 1-immersions (see Theorem 4 in the sequel). Roughly speaking, the main idea of the proof is as follows. Suppose, for a contradiction, that $S_m(\lambda)$ has a 1-immersion. Every central path can have at most $6m-3$ crossing points, hence, all $6m-1$ central paths can not be 1-immersed inside B_0. Then there is a central path which crosses some edges of $\overline{S}_m(\lambda)$. Let P be a central path with maximum number of such crossings. Since P can have at most $6m-3$ crossing points, some of the other $6m-2$ central paths do not cross P and have to "go around" P and, in doing so, one of the paths has to cross more edges of $\overline{S}_m(\lambda)$ than P does, a contradiction.

Before proving Theorem 4, we need some definitions and preliminary Lemmas 11 and 12.

Consider a 1-immersion of $S_m(\lambda)$ (if it exists). If a central path $P = P(x, x^*)$ does not lie inside B_0, consider the sequence $\delta_1, \delta_2, \dots, \delta_r$ ($r \geq 2$), where $\delta_1 = x$ and $\delta_r = x^*$, obtained by listing the intersection points of the path and B_0 when traversing the path from the vertex x to the vertex x^* (here $\delta_2, \delta_3, \dots, \delta_{r-1}$ are crossing points). By a *piece* of P we mean the segment of P from δ_i to δ_{i+1} for some $i \in \{1, 2, \dots, r-1\}$; denote the piece by $P(\delta_i, \delta_{i+1})$. A piece of P with an end point x or x^* is called an *end piece* of P at the vertex x or x^*, respectively. An *outer piece* of P is every piece of P that is immersed outside B_0. Clearly, either $P(\delta_1, \delta_2), P(\delta_3, \delta_4), P(\delta_5, \delta_6), \dots$ or $P(\delta_2, \delta_3), P(\delta_4, \delta_5), P(\delta_6, \delta_7), \dots$ are all outer pieces of P. The end points δ and δ' of an outer piece Π of P partition B_0 into two curves A and A' such that the curve A lies inside the closed curve consisting of Π and A' (see Fig. 21). The central vertices belonging to A and different from δ and δ' are said to be *bypassed* by Π and P (cf. Fig. 21).

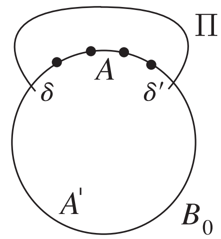


Figure 21: The central vertices bypassed by an outer piece Π.

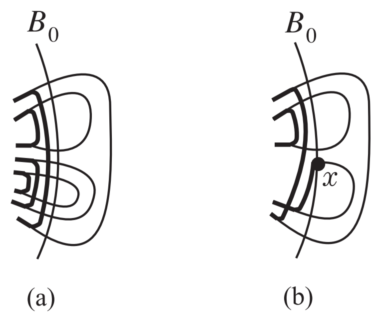


Figure 22: Transforming paths $P(x, x^*)$ into paths $P'(x, x^*)$.

Lemma 11 *If $P(x, x^*)$ bypasses neither a central vertex y nor its opposite vertex y^*, and $P(y, y^*)$ bypasses neither x nor x^*, then $P(x, x^*)$ crosses $P(y, y^*)$.*

Proof. Suppose, for a contradiction, that $P(x, x^*)$ does not cross $P(y, y^*)$. For every outer piece of the two paths we can replace a curve of a path containing the piece by a new curve lying inside B_0 so that the path $P(x, x^*)$ (resp. $P(y, y^*)$) becomes a new path $P'(x, x^*)$ (resp. $P'(y, y^*)$) connecting the vertices x and x^* (resp. y and y^*) such that the two new paths lie inside B_0 and do not cross each other, a contradiction. How the replacements can be done is shown in Fig. 22, where the new curves are depicted in thick line. (Note that in Fig. 22(b), since $P(y, y^*)$ does not bypass x, the depicted pieces bypassing x belong to $P(x, x^*)$.) □

By a *type* of an outer piece of a central path we mean the maximal type of an edge of $\overline{S}_m(\lambda)$ crossed by the path.

For an outer piece Π of a central path $P(x, x^*)$, denote by $b(\Pi)$ the number of central vertices bypassed by Π, and by $\Delta(\Pi)$ the number of intersection points of Π and $\overline{S}_m(\lambda)$, including the crossings at the end points of Π (except if an end point is x or x^*).

Lemma 12 *If Π is an outer piece of type t of a central path P, then*

$$\Delta(\Pi) - b(\Pi) \geq 2t - \tau,$$

where $\tau = 1$ if Π is an end piece and $\tau = 0$ otherwise.

Proof. The piece Π crossing an edge of type t has a point separated from the interior of B_0 by $\lfloor \frac{t+1}{2} \rfloor + t$ edge-disjoint cycles: $B_0, B_1, \ldots, B_{\lfloor \frac{t+1}{2} \rfloor - 1}$, and the cycles $C_1, C_2, \ldots, C_t$, where the cycle C_i consists of all edges of type i ($i = 1, 2, \ldots, t$). Thus, Π crosses each of these cycles twice, except that for an end piece, we may miss one crossing with C_1. The piece Π bypasses $b(\Pi)$ central vertices, hence Π crosses $2b(\Pi)$ paths associated with those $b(\Pi)$ vertices. Hence we obtain two inequalities

$$\Delta(\Pi) \geq 2 \left\lfloor \tfrac{t+1}{2} \right\rfloor + 2t - \tau, \tag{1}$$

and

$$\Delta(\Pi) \geq 2 \left\lfloor \tfrac{t+1}{2} \right\rfloor + 2b(\Pi) - \tau. \tag{2}$$

By (1), we have

$$\Delta(\Pi) - b(\Pi) \geq 2t - \tau + 2 \left\lfloor \tfrac{t+1}{2} \right\rfloor - b(\Pi) \geq 2t - \tau$$

for $2\lfloor \frac{t+1}{2} \rfloor \geq b(\Pi)$. By (2), we have

$$\Delta(\Pi) - b(\Pi) \geq 2t - \tau + 2 \left\lfloor \tfrac{t+1}{2} \right\rfloor - 2t + b(\Pi) \geq 2t - \tau$$

for $b(\Pi) \geq 2t - 2\lfloor \frac{t+1}{2} \rfloor$. Since $2t - 2\lfloor \frac{t+1}{2} \rfloor \leq 2\lfloor \frac{t+1}{2} \rfloor + 1$, the lemma follows. $\square$

By the *type* of a central path not lying (entirely) inside B_0 we mean the maximal type of the outer pieces of the path. If t different central paths bypass a central vertex x, then all the paths cross edges of the same path T associated with x and since the edges of T have pairwise different types, we obtain that one of the central paths crosses an edge of type at least t. Hence we have:

(D) If t different central paths bypass the same central vertex, then one of the paths has type at least t.

Theorem 4 *For every $m \geq 4$ and $\lambda \in \Phi_m(n)$, the graph $S_m(\lambda)$ is not 1-planar.*

Proof. Consider, for a contradiction, a 1-immersion φ of $S_m(\lambda)$ and a path $P = P(x, x^*)$ of maximal type $t > 0$.

As above, let $\Delta(P)$ be the number of crossing points of P and $\overline{S}_m(\lambda)$, and let $b(P)$ be the number of distinct central vertices bypassed by P and different from x and x^*.

There are $6m - 2 - b(P)$ different pairs $\{y, y^*\}$ ($\{y, y^*\} \neq \{x, x^*\}$) of central vertices such that P does not bypass y and y^*; denote by $\mathcal{P}$ the set of the corresponding $6m - 2 - b(P)$ paths $P(y, y^*)$. If $P(x, x^*)$ does not bypass y and y^*, then $P(y, y^*)$ either does not bypass x and x^* (in this case $P(y, y^*)$ crosses $P(x, x^*)$ by Lemma 11) or bypasses at least one of the vertices x and x^*. Hence, we have

$$6m - 2 - b(P) = \beta + \gamma + \varepsilon, \tag{3}$$

where: β (resp. γ) is the number of paths of $\mathcal{P}$ that cross P and do not bypass x or x^* (resp. that bypass x or x^* and do not cross P); ε is the number of paths of $\mathcal{P}$ that cross P and bypass x or x^*. We are interested in the number $\gamma + \varepsilon$ of paths of $\mathcal{P}$ that bypass x or x^*.

The path P has at most $6m - 3$ crossing points, hence

$$6m - 3 - \Delta(P) \geq \beta + \varepsilon$$

and, by (3), we obtain

$$\gamma = 6m - 2 - b(P) - (\beta + \varepsilon) \geq \Lambda(P) - b(P) + 1,$$

whence

$$\gamma + \varepsilon \geq \Delta(P) - b(P) + 1 + \varepsilon. \tag{4}$$

Let $\Pi_1, \Pi_2, \ldots, \Pi_\ell$ ($\ell \geq 1$) be all outer pieces of P, and let Π_1 be of the maximal type t. We have $\Delta(P) = \sum_{i=1}^{\ell} \Delta(\Pi_i)$ and $b(P) \leq \sum_{i=1}^{\ell} b(\Pi_i)$ (the vertices x and x^* can be bypassed by P, and some central vertices can be bypassed by P more than once). By (2), $\Delta(P_i) - b(\Pi_i) \geq 0$ for every $i = 1, 2, \ldots, \ell$. Hence, by (4) and Lemma 12, we obtain

$$\begin{aligned}
\gamma + \varepsilon &\geq \sum_{i=1}^{\ell} \Delta(\Pi_i) - \sum_{i=1}^{\ell} b(\Pi_i) + 1 + \varepsilon \\
&\geq \Delta(\Pi_1) - b(\Pi_1) + 1 + \varepsilon \geq (2t + 1) - \tau + \varepsilon,
\end{aligned} \tag{5}$$

where $\tau = 1$ if Π_1 is an end piece and $\tau = 0$ otherwise. If Π_1 is not an end piece or $\varepsilon \geq 1$, then, by (5), $\gamma + \varepsilon \geq 2t + 1$, hence one of the vertices x and x^* is bypassed by at least $t+1$ paths of $\mathcal{P}$. Now, by (D), one of the $t+1$ paths has type at least $t+1$, a contradiction. Now suppose that Π_1 is an end piece at the vertex x and $\varepsilon = 0$. Then every path of $\mathcal{P}$ either crosses P or bypasses x and x^*, and at least $2t$ paths of $\mathcal{P}$ bypass x or x^*. If no one of the $2t$ paths bypasses x, then all the $2t \geq t+1$ paths bypass x^* and, by (D), one of the paths has type at least $t+1$. If one of the $2t$ paths, say, P', bypasses x, then P' has an outer piece Π' that bypasses x and does not cross Π_1 (since $\varepsilon = 0$, P' does not cross P). The piece Π_1 has type t and is an end piece at x, hence Π' has type at least $t+1$, a contradiction. $\square$

We have shown that every graph $S_m(\lambda)$, where $m \geq 4$ and $\lambda \in \Phi_m(n)$, is an MN-graph. These graphs have order $(5m-1)(12m-2)+5n$. Clearly, graphs $S_{m_1}(\lambda_1)$ and $S_{m_2}(\lambda_2)$, where $\lambda_1 \in \Phi_{m_1}(n_1)$ and $\lambda_2 \in \Phi_{m_2}(n_2)$, are nonisomorphic for $m_1 \neq m_2$ and for $m_1 = m_2$ and $n_1 \neq n_2$.

Claim 2 *For any integers $m \geq 4$ and $n \geq 0$, there are at least $\frac{1}{24m-4)}\binom{n+12m-3}{12m-3}$ nonisomorphic MN-graphs $S_m(\lambda)$, where $\lambda \in \Phi_m(n)$.*

Proof. The automorphism group of S_m is the dihedral group D_{12m-2} of order $24m-4$. Now the claim follows by recalling a well-known fact that $|\Phi_m(n)| = \binom{n+12m-3}{12m-3}$. $\square$

5 Testing 1-immersibility is hard

In this section we prove that testing 1-immersibility is NP-hard. This shows that it is extremely unlikely that there exists a nice classification of MN-graphs.

Theorem 5 *It is NP-complete to decide if a given input graph is 1-immersible.*

Since 1-immersions can be represented combinatorially, it is clear that 1-immersability is in NP. To prove its completeness, we shall make a reduction from a known NP-complete problem, that of 3-colorability of planar graphs of maximum degree at most four [10].

The rest of this section is devoted to the proof of Theorem 5. Only at the very end we shall explain how to modify our reduction in order to obtain a geometric proof of the crossing number minimization for cubic graphs.

Let G be a given plane graph of maximum degree 4 whose 3-colorability is to be tested. We shall show how to construct, in polynomial time, a related graph $\overline{G}$ such that $\overline{G}$ is 1-immersible if and only if G is 3-colorable. We may assume that G has no vertices of degree less than three.

To construct $\overline{G}$, we will use as building blocks graphs which have a unique 1-immersion. These building blocks are connected with each other by edges to form a graph which also has a unique 1-immersion. Then we add some additional paths to obtain $\overline{G}$.

We say that a 1-planar graph G has *unique* 1*-immersion* if, whenever two edges e and f cross each other in some 1-immersion, then they cross each other in every 1-immersion of G, and secondly, if $G^\bullet$ is the planar graph obtained from G by replacing each pair of crossing edges $e = ab$ and $f = cd$ by a new vertex of degree four joined to a, b, c, d, then $G^\bullet$ is 3-connected (and thus has combinatorially unique embedding in the plane – the one obtained from 1-immersions of G.

It was proved in [12] that for every $n \geq 6$, the graph with $4n$ vertices and $13n$ edges shown in Fig. 23(a) has a unique 1-immersion. (To be precise, the paper [12] considers the graph for even values of $n \geq 6$ only, but one can check that the proof does not depend on whether $n \geq 6$ is even or odd.) We call the graph a *U-graph*. Fig. 23(b) shows a designation of the U-graph used in what follows. In the 1-immersion of the U-graph shown in Fig. 23, the vertices $1, 2, 3, \ldots, n-1, n$ which lie on the boundary of the outer face of the spanning embedding (the boundary is called the *outer boundary cycle* of the 1-immersed U-graph) are called the *boundary vertices* of the U-graph in the 1-immersion. If a graph has a U-graph as a subgraph, then the U-graph is called the *U-subgraph* of the graph.

Take two 1-immersed U-graphs U_1 and U_2 of order at least $13 \cdot 7$ and construct the 1-immersed graphs shown in Figs. 24(a) and (b), respectively, where by $1, 2, \ldots, 7$ we denote seven consecutive vertices on the outer boundary cycle of each of the 1-immersed graphs. We say that in Fig. 24(a) (resp. (b)) the U-graphs U_1 and U_2 are connected by a (1)-grid (resp. (2)-grid). The vertices labeled $1, 2, \ldots, 7$ are the *basic vertices* of the grid and for $i = 1, 2, \ldots, 7$, the h-path connecting the vertices labeled i of the (h)-grid, $h \in \{1, 2\}$, is called the *basic path* of the grid connecting these vertices. Let us denote the ith basic path by P_i. The paths P_{i-1} and P_i, $i = 2, 3, \ldots, 7$, are

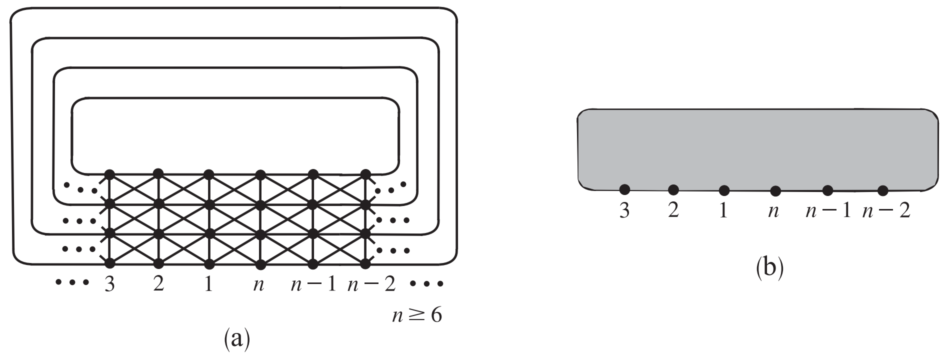


Figure 23: The U-graph.

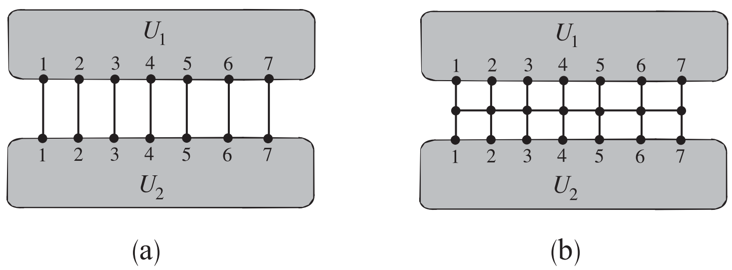


Figure 24: Two U-graphs connected by a grid.

neighboring basic paths of the grid. For two basic paths $P = P_i$ and $P' = P_j$, $1 \leq i < j \leq 7$, denote by $C(P, P')$ the cycle of the graph in Fig. 24 consisting of the two paths and of the edges $(i, i+1), (i+1, i+2), \ldots, (j-1, j)$ of the two graphs U_1 and U_2.

By a *U-supergraph* we mean every graph obtained in the following way. Consider a plane connected graph H. Now, for every vertex $v \in V(H)$, take a 1-immersed U-graph $U(v)$ of order at least $13 \cdot 7 \cdot \deg(v)$ and for any two adjacent vertices u and w of the graph, connect $U(u)$ and $U(w)$ by a (1)- or (2)-grid as shown in Fig. 25 such that any two distinct grids have no basic vertices in common. We obtain a 1-immersed U-supergraph.

Theorem 6 *Every U-supergraph M has a unique 1-immersion.*

Proof. It suffices to show the following:

(a) The graph consisting of two U-graphs connected by an (h)-grid, $h = 1, 2$, has a unique 1-immersion.

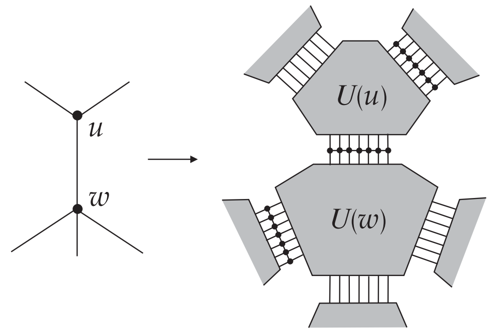


Figure 25: Constructing a U-supergraph.

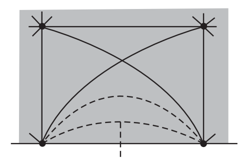


Figure 26: A 1-immersion of a subgraph.

(b) In every 1-immersion φ of M, the edges of distinct grids do not intersect.

Note that M contains no subgraph which can be 1-immersed inside the boundary cycle of a 1-immersed U-subgraph of M in a 1-immersion of M as shown in Fig. 26 in dashed line. Hence, in every 1-immersion of M, the boundary edges of the U-subgraphs of M are not crossed.

We prove (a) and (b) in the following way. We consider a 1-immersed subgraph W of M (cf. Fig. 24) consisting of two U-graphs U_1 and U_2 connected by an (h)-grid Γ, $h \in \{1, 2\}$, and we show that in every 1-immersion φ of M, the graph W has the same 1-immersion and the edges of Γ are not crossed by edges of other grids.

Suppose, for a contradiction, that U_1 and U_2 are 1-immersed under φ as shown in Fig. 27(a). Clearly, there are two basic paths P_i and P_j of Γ, $1 \le i < j \le 7$, which do not intersect. Then the cycle shown in Fig. 27(a) in thick line is embedded in the plane, a contradiction, since the cycle is crossed by 5 other basic paths of Γ, but the cycle has only $2h \le 4$ edges that can be crossed by other edges. Hence, U_1 and U_2 are 1-immersed as shown in Fig. 24.

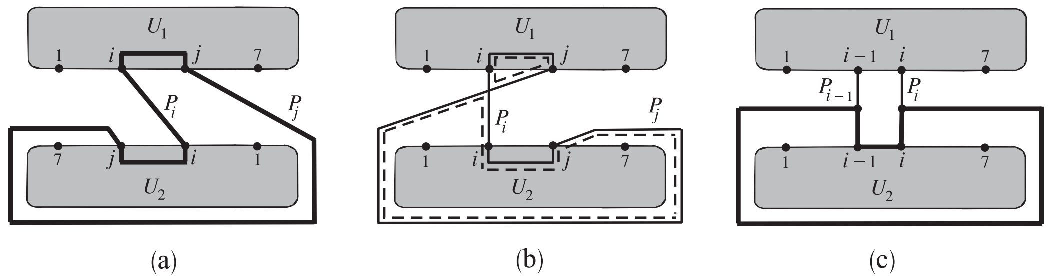


Figure 27: Cycles of two adjacent 1-immersed U-subgraphs.

Suppose that in φ, a basic path P_i of Γ crosses a basic path Q of some grid of M exactly once. If $Q = P_j$ is a basic path of Γ, $j \neq i$ (see Fig. 27(b)), then the closed curves C_1 and C_2 shown in Fig. 27(b) by dashed cycles, are embedded in the plane and each of the other five basic paths of Γ crosses an edge of C_1 or C_2, a contradiction, since C_1 and C_2 have $2h-2 \leq 2$ edges in total which can be crossed by other edges. If Q is a basic path of a grid Γ' different from Γ, then there is a basic path P_j, $j \neq i$, of Γ such that P_j is not crossed by P_i and Q. Hence, the cycle $C(P_i, P_j)$ is embedded and Q crosses the edges of the cycle exactly once. Then for every other basic path Q' of Γ', the cycle $C(Q, Q')$ crosses $C(P_i, P_j)$ at least twice and Q' crosses P_i or P_j (the edges of different U-subgraphs do not intersect). We have that the 7 basic paths of Γ' cross P_i and P_j, a contradiction. Hence, in φ, if two basic paths intersect, then they intersect twice. In particular, only basic paths of (2)-grids can intersect.

Now we claim the following:

(a) If in φ two neighboring basic paths P_{i-1} and P_i of the (2)-grid Γ do not intersect, then the edge e joining the middle vertices of P_{i-1} and P_i lies inside the embedded cycle $C(P_{i-1}, P_i)$.

Indeed, if e lies inside $C(P_{i-1}, P_i)$, then the 4-cycle shown in Fig. 27(c) in thick line is crossed by 5 basic paths P_r, $r \neq i-1, i$, a contradiction.

Suppose that in φ, a basic path of Γ crosses a basic path of a grid Γ' twice (that is, Γ and Γ' are (2)-grids). Since the number of basic paths of Γ is odd (namely, 7), it can not be that every basic path of Γ crosses some other basic path of Γ twice. Then there is a basic path of Γ which is not crossed by other basic paths of Γ. Hence, if $\Gamma = \Gamma'$ (resp. $\Gamma \neq \Gamma'$), then there are two neighboring basic paths P_{i-1} and P_i of Γ such that one of them, say, P_{i-1}, is crossed twice by some basic path Q of Γ (resp. Γ'), and the other basic path $P_i \neq Q$ is not crossed by Q. Then the cycle $C(P_{i-1}, P_i)$ is embedded. By (a), the edge e joining the middle vertices of P_{i-1} and P_i lies inside $C(P_{i-1}, P_i)$. Denote by C_1 and C_2 the two embedded adjacent 4-cycles each of which consists of e and edges of the 6-cycle $C(P_{i-1}, P_i)$. The middle vertex of Q lies outside $C(P_{i-1}, P_i)$ and the two end vertices of Q lie inside C_1 and C_2, respectively. The end vertices of Q belong to two U-subgraphs connected by Γ'. Since the edges of the two U-subgraphs do not cross the edges of C_1 and C_2, we obtain that one of the U-subgraphs lies inside C_1 and the other lies inside C_2, a contradiction, since the two U-subgraphs are connected by at least four basic paths different from Q, P_{i-1}, and P_i. Hence, no basic path of Γ crosses some other basic path twice.

We conclude that the basic paths of the grids connecting U-subgraphs do not intersect.

Now it remains to show that if Γ is a (2)-grid, then the edges joining the middle vertices of the basic paths of Γ are not crossed. Consider any two neighboring basic paths P_{i-1} and P_i of Γ. The cycle $C(P_{i-1}, P_i)$ is embedded and, by (a), the edge e joining the middle vertices of P_{i-1} and P_i lies inside $C(P_{i-1}, P_i)$. It is easy to see that for every edge e' of G not belonging to U-subgraphs and different from e and the edges of $C(P_{i-1}, P_i)$, in the graph $G - e'$ the end vertices of e' are connected by a path which consists of edges of U-subgraphs and basic paths of grids and which does not pass through the vertices of $C(P_{i-1}, P_i)$. Now, if the edge e is crossed by some other edge e', then e' is not an edge of a U-subgraph, the end vertices of e' lie inside the cycles C_1 and C_2, respectively (where C_1 and C_2 are defined as in the preceding paragraph) whose edges are not crossed by edges of U-subgraphs and basic paths, a contradiction.

Therefore, the edges of M do not intersect and that the graph W has a unique 1-immersion. This completes the proof of the theorem. $\square$

Now, given a plane graph G every vertex of which has degree 3 or 4, we construct a graph $\overline{G}$ such that G is 3-colorable if and only if $\overline{G}$ is 1-immersible. To obtain $\overline{G}$, we proceed as follows. First we construct a subgraph $G^{(1)}$ od $\overline{G}$ such that $G^{(1)}$ has a unique 1-immersion. The graph $G^{(1)}$ is obtained from a U-supergraph W by adding some additional vertices and edges. By inspection of the subsequent figures which illustrate the construction of $G^{(1)}$ and its 1-immersion, the reader will easily identify the additional vertices and edges: they do not belong to U-subgraphs and grids. Then one can easily check that given the 1-immersion of W, the additional vertices and

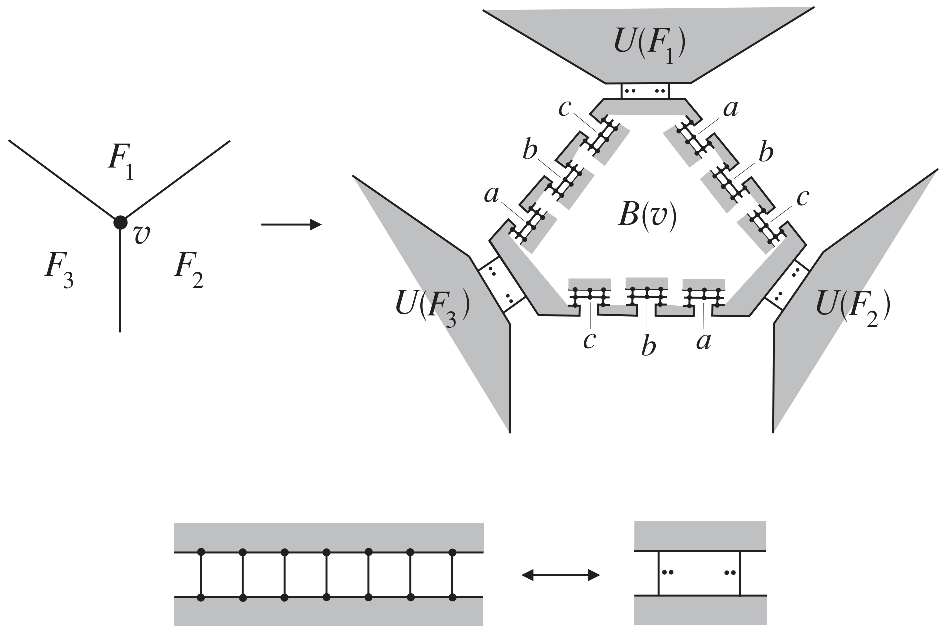


Figure 28: Constructing the graph $G^{(1)}$.

edges can be placed in the plane in a unique way to obtain a 1-immersion of $G^{(1)}$, hence $G^{(1)}$ has a unique 1-immersion also. Now, given the unique 1-immersion of $G^{(1)}$, to construct $\overline{G}$ we place some new additional paths "between" 1-immersed U-subgraphs of $G^{(1)}$.

The graph $G^{(1)}$ is obtained from the plane graph G if we replace every face F of the embedding of G by a U-graph $U(F)$ and replace every vertex v by a *vertex-block* $B(v)$ as shown at the top of Fig. 28. At the bottom of Fig. 28 we show the designation of a (1)-grid used at the top of the figure and at what follows. The vertex-block $B(v)$ has a unique 1-immersion and is obtained from a U-supergraph by adding some additional vertices and edges. Fig. 28 shows schematically the boundary of $B(v)$ and Fig. 31 shows $B(v)$ in detail. For a k-valent vertex v of G, $3 \leq k \leq 4$, the vertex-block $B(v)$ has $3k$ *boundary vertices* labeled clockwise as $a, b, c, a, b, c, \ldots, a, b, c$; these vertices do not belong to U-subgraphs of $B(v)$. In Figs. 28 and 31 we only show the case of a 3-valent vertex v; for a 4-valent vertex the construction is analogous – there are three more boundary vertices labeled a, b, c, respectively.

We say that vertex-blocks $B(v)$ and $B(w)$ are adjacent if v and w are adjacent vertices of G.

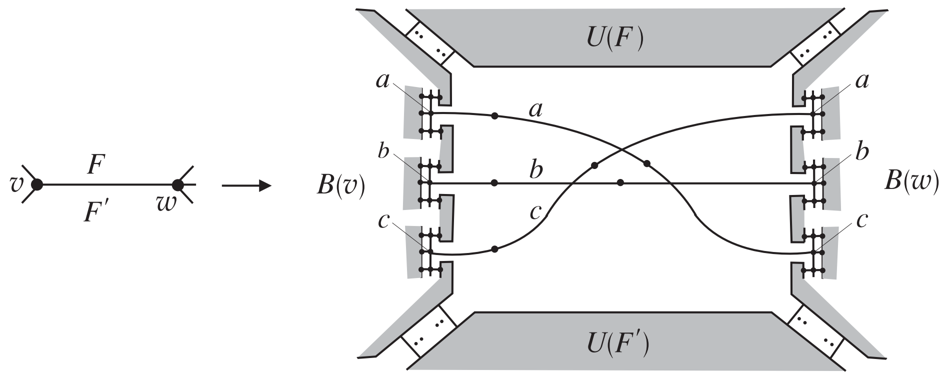


Figure 29: The pending paths connecting adjacent vertex-blocks.

The graph $\overline{G}$ is obtained from $G^{(1)}$ if we take a collection of additional disjoint paths of length ≥ 1 (they are called the *pending paths*) and identify the end vertices of every path with two vertices, respectively, of $G^{(1)}$. The graph $G^{(1)}$ has a unique 1-immersion and the edges of the U-subgraphs of $G^{(1)}$ can not be crossed by the pending paths, hence the 1-immersed $G^{(1)}$ restricts the ways in which the pending paths can be placed in the plane to obtain a 1-immersion of $\overline{G}$. Every pending path connects either boundary vertices of adjacent vertex-blocks or vertices of the same vertex-block.

For any two adjacent vertex-blocks, there are exactly three pending paths connecting the vertices of the vertex-blocks. The paths have length 3 and are shown in Fig. 29; we say that these pending paths are incident with the two vertex-blocks. Each of the three pending paths connects the boundary vertices labeled by the same letter: a, b, or c. For $h \in \{a, b, c\}$, the pending path connecting vertices labeled h is called the (h)-*path* connecting the two vertex-blocks. In Fig. 29 the (h)-path, $h \in \{a, b, c\}$, is labeled by the letter h.

Denote by $G^{(2)}$ the graph obtained from $G^{(1)}$ if we add all triples of pending paths connecting vertex-blocks $B(v), B(w)$, for all edges $vw \in E(G)$.

The pending paths connecting vertices of the same vertex-block $B(v)$ are divided into three families called, respectively, the a-, b-, and c-*families* of $B(v)$. Given the 1-immersion of $G^{(1)}$, for every $h \in \{a, b, c\}$, the h-family of $B(v)$ has the following properties:

(i) Every path P of the h-family admits exactly two embeddings in the plane such that we obtain a 1-immersion of $G^{(2)} \cup P$.

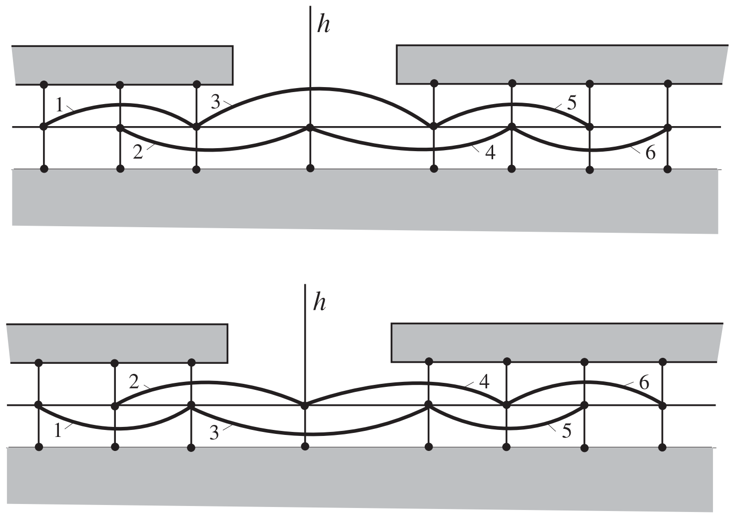


Figure 30: Pending **paths of an h-family.**

(ii) The h-family consists of paths $P_1, P_2, \dots, P_n$ such that the graph $G^{(2)} \cup P_1 \cup P_2 \cup \cdots \cup P_n$ has exactly two 1-immersions. In the two 1-immersions, every path P_i uses its two embeddings. In one of the 1-immersions, paths of the h-family cross all (h)-paths incident with $B(v)$. In the other 1-immersion, the paths of the h-family do not cross any (h)-path incident with $B(v)$.

Fig. 30 shows fragments of the two 1-immersions of the union of $G^{(2)}$ and the pending paths of an h-family. In the figure, each of the depicted (in thick line) six paths of the family, they are labeled by $1, 2, \dots, 6$, respectively, uses its two embeddings in the two 1-immersions.

If in a 1-immersion of $\overline{G}$, paths of an h-family of $B(v)$, $h \in \{a, b, c\}$, cross (h)-paths incident with $B(v)$, then we say that the h-family of $B(v)$ is *activated* in the 1-immersion of $\overline{G}$.

Figs. 31 and 32 show a vertex-block $B(v)$ and the h-families of the vertex-block ($h = a, b, c$) in the case where v is 3-valent (the generalization for a 4-valent vertex v is straightforward). The pending paths of the three h-families are shown by thick lines and the three families are activated. To avoid clattering a figure, Fig. 31 contains a fragment denoted by R which is given in detail in Fig. 32.

In Figs. 31 and 32 we use designations of some fragments of $B(v)$; the designations are given at the left of Fig. 33 and the corresponding fragments

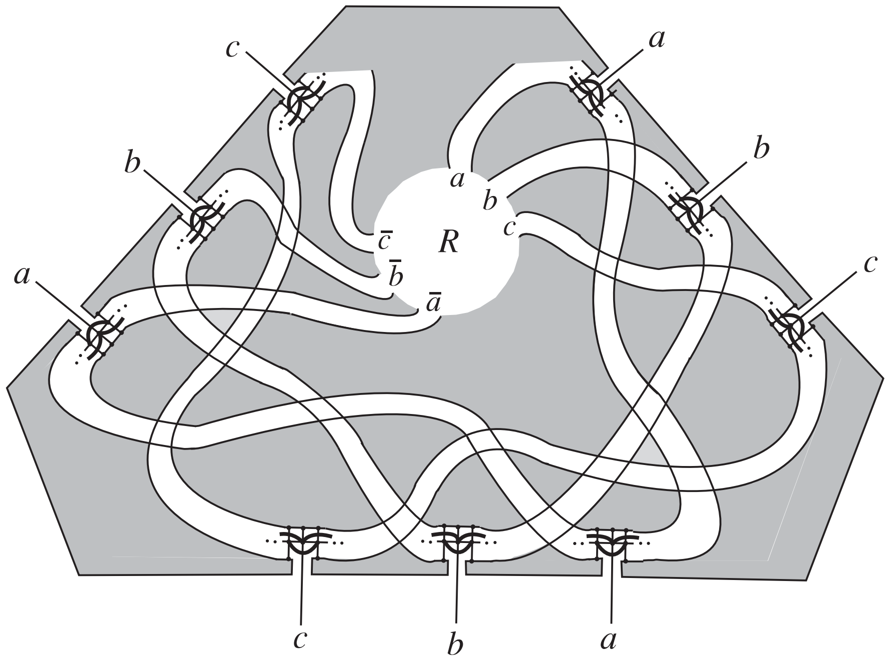


Figure 31: A vertex-block and the activated h-families.

are given at the right of Fig. 33. The reader can easily check that for every pending path P of the three families, there are exactly two ways to embed the path so that we obtain a 1-immersion of $G^{(2)} \cup P$. The vertex-block $B(v)$ contains two 2-paths (in Fig. 32 one of them connects vertices labeled 0, the other one connects the vertices labeled 1; we call the paths the (0)- and (1)-blocking paths, respectively). For every $h \in \{a, b, c\}$, exactly one pending path of the h-family of $B(v)$ crosses a blocking path: the pending path has length 32, crosses the (1)-blocking (resp. (0)-blocking) path when the h-family is activated (resp. not activated), and the pending path in each of its two embeddings crosses exactly one pending path of each of the other two families. Fig. 32 shows the two embeddings of the pending 32-path of the b-family (one of them is in thick line, the other, when the family is not activated, is in dashed line).

Denote by $G_v^{(2)}$ the union of $G^{(2)}$ and the paths of all three h-families of $B(v)$. Now the reader can check that $B(v)$ and the h-families of $B(v)$ are constructed in such a way that the following holds:

(1) For every $h \in \{a, b, c\}$, there is a 1-immersion of $G_v^{(2)}$ such that only

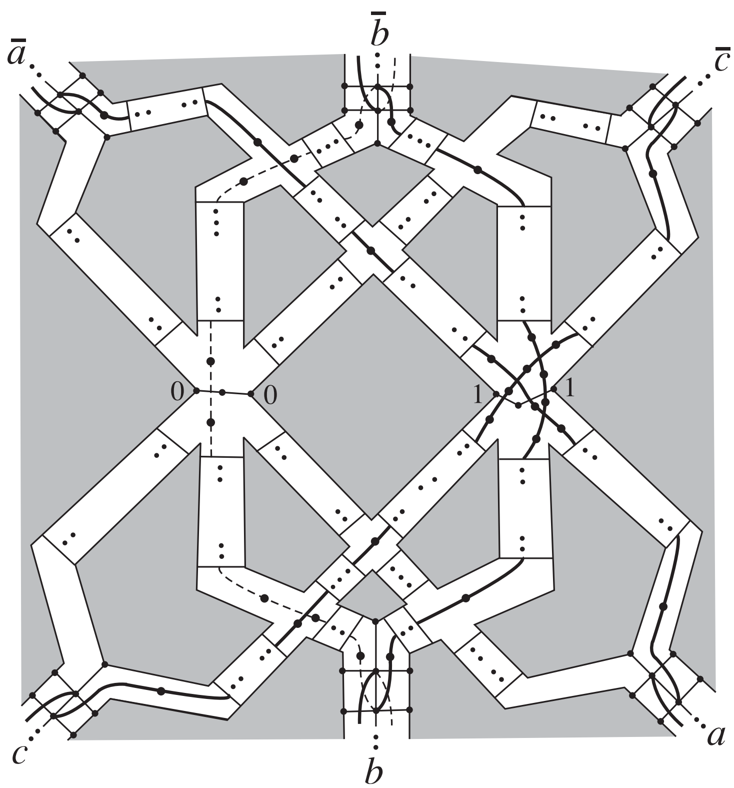


Figure 32: The fragment R of the vertex-block in Fig. 31.

the h-family of $B(v)$ is activated.

(2) For any two $h, h' \in \{a, b, c\}$, there is a 1-immersion of $G_v^{(2)}$ such that only the h- and h'-families of $B(v)$ are activated.

(3) There are no 1-immersions of $G_v^{(2)}$ such that the three h-families of $B(v)$ either are all activated or none is activated.

By construction of $\overline{G}$, if $\overline{G}$ has a 1-immersion, then in the 1-immersion for every vertex v of G, for some $h \in \{a, b, c\}$ the h-family of $B(v)$ is activated and the h-families of the vertex-blocks adjacent to $B(v)$ are not activated.

Now take a 1-immersion of $\overline{G}$ (if it exists) and assign every vertex v of G a color $h \in \{a, b, c\}$ such that the h-family of $B(v)$ is activated in the 1-immersion of $\overline{G}$. We obtain a proper 3-coloring of G with colors $\{a, b, c\}$.

Take a proper 3-coloring of G (if it exists) with colors $\{a, b, c\}$ and for every vertex v of G, if $h(v)$ is the color of v, take the $h(v)$-family of $B(v)$ to be activated and the other two families not to be activated. By the construction of $\overline{G}$, and by the mentioned properties of 1-immersions of its subgraphs $B_v^{(2)}$, it follows that we obtain a 1-immersion of $\overline{G}$.

When constructing $\overline{G}$, we choose the order of every U-subgraph such that every boundary vertex of the U-subgraph is incident with an edge not belonging to the U-subgraph. This implies that for every face F of size k of the plane embedding of G, the number of edges in the U-graph $U(F)$ is bounded by a constant multiple of k. Similarly, for each $v \in V(G)$, the union of $B(v)$ and its three h-families has constant size. Therefore, the whole construction of $\overline{G}$ can be carried over in linear time. This completes the proof of Theorem 5.

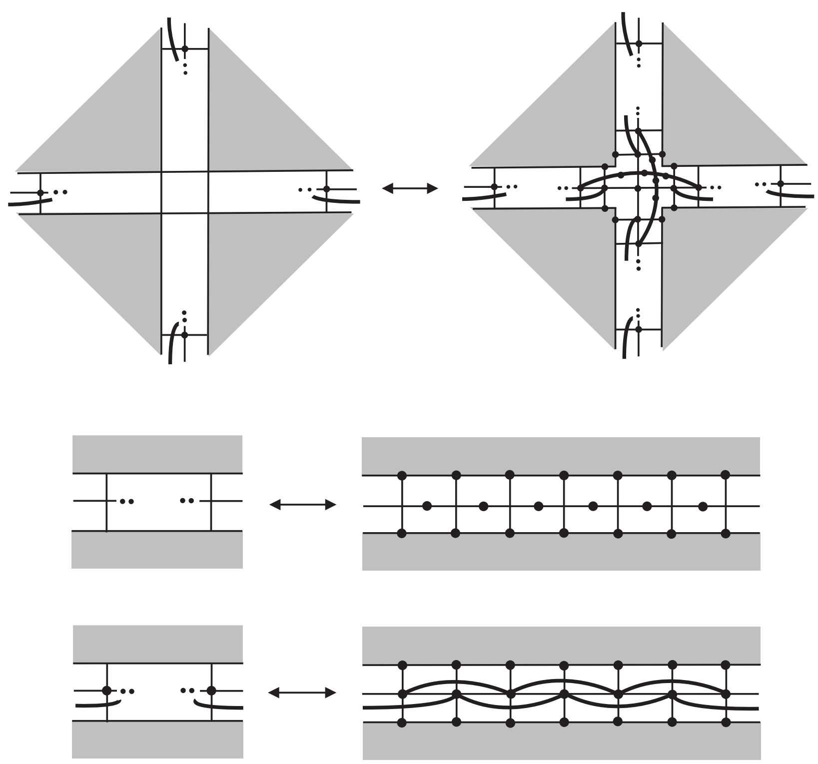


Figure 33: The designations of fragments of the vertex-block $B(v)$.

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