University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Department of Mathematics
Jadranska 19, 1000 Ljubljana, Slovenia

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SPECTRAL RADIUS OF FINITE AND INFINITE PLANAR GRAPHS AND OF GRAPHS OF BOUNDED GENUS

Zdeněk Dvořák Bojan Mohar

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Spectral radius of finite and infinite planar graphs and of graphs of bounded genus

Zdeněk Dvořák[*][†]
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
email: rakdver@kam.mff.cuni.cz

Bojan Mohar[‡][§]
Department of Mathematics
Simon Fraser University
Burnaby, B.C. V5A 1S6
email: mohar@sfu.ca

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Abstract

It is well known that the spectral radius of a tree whose maximum degree is D cannot exceed $2\sqrt{D-1}$. In this paper we derive similar bounds for arbitrary planar graphs and for graphs of bounded genus. It is proved that a the spectral radius $\rho(G)$ of a planar graph G of maximum vertex degree $D \geq 4$ satisfies $\sqrt{D} \leq \rho(G) \leq \sqrt{8D-16}+7.75$. This result is best possible up to the additive constant—we construct an (infinite) planar graph of maximum degree D, whose spectral radius is $\sqrt{8D-16}$. This generalizes and improves several previous results and solves an open problem proposed by Tom Hayes. Similar bounds are derived for graphs of bounded genus. For every k, these bounds can be improved by excluding $K_{2,k}$ as a subgraph. In particular, the upper bound is strengthened for 5-connected graphs. All our results hold for finite as well as for infinite graphs.

At the end we enhance the graph decomposition method introduced in the first part of the paper and apply it to tessellations of the hyperbolic

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[†]On leave from: Institute of Theoretical Informatics, Charles University, Prague, Czech Republic.

[‡]Supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant P1–0297 of ARRS (Slovenia).

[§]On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

plane. We derive bounds on the spectral radius that are close to the true value, and even in the simplest case of regular tessellations of type $\{p, q\}$ we derive an essential improvement over known results, obtaining exact estimates in the first order term and non-trivial estimates for the second order asymptotics.

1 Introduction

Every tree of maximum degree D is a subgraph of the infinite D-regular tree. This observation immediately implies that the spectral radius of every such tree is at most $2\sqrt{D-1}$. In this paper we derive similar bounds for arbitrary planar graphs and for graphs of bounded genus. This generalizes and improves several previous results and solves an open problem proposed by Hayes. Usually higher connectivity of graphs allows more edges in the graph and thus gives rise to graphs with larger spectral radius. However, an interesting outcome of our proof is that higher connectivity has converse effect in the case of planar graphs. The extremal examples for the largest spectral radius need many 4-separations, and hence 5-connected graphs, in particular, allow better upper bounds on the spectral radius.

All graphs in this paper are *simple*, i.e. no loops or multiple edges are allowed. They can be finite or infinite, but we request that they are locally finite. In fact, we shall always have a (finite) upper bound on the maximum degree.

It is well known that the edges of every planar graph G can be partitioned into three acyclic subgraphs. By compactness, this extends to all (locally finite) planar graphs and implies that $\rho(G) \leq 6\sqrt{\Delta - 1}$, where Δ is the maximum degree of G. This bound has been improved by Hayes [10]. He use the following theorem.

Theorem 1.1 (Hayes [10])**.** *Any graph G that has an orientation with maximum indegree k (hence also any k-degenerate graph) and $\Delta = \Delta(G) \geq 2k$ satisfies $\rho(G) \leq 2\sqrt{k(\Delta - k)}$.*

Since each planar graph G has an orientation with maximum indegree 3, this gives $\rho(G) \leq \sqrt{12(\Delta - 3)}$. At the 1st CanaDAM conference (Banff, Alberta, 2007), Tom Hayes asked to what extent the constant factor in his upper bound can be improved. We answer Hayes' question by proving that $\rho(G) \leq \sqrt{8\Delta} + O(1)$ (see Theorem 5.2) and by showing that this bound is essentially best possible. Our bound cannot be improved even when G is bipartite and "tree-like" (i.e. with lots of 2-separations). To some surprise, if the connectivity is increased, the upper bound can be strengthened further. Actually, it suffices to exclude $K_{2,k}$ subgraph, where $k = o(\Delta)$. These results also apply for all graphs of bounded genus, cf. Theorem 5.1.

In the last section we enhance the graph decomposition method used in this paper and apply it to tessellations of the hyperbolic plane, whose graph is p-regular. We derive lower and upper bounds on the spectral radius that are close to each other and asymptotically coincide. Even in the simplest case of regular tessellations of type $\{p, q\}$, previously known bounds were not of the right magnitude asymptotically. Our estimates are exact in the first order term and also give a non-trivial terms for the second order asymptotics. See further discussion about known results in the next section. It is worth pointing out that p-regular graphs of planar tessellations are p-connected (as proved in [20]). It turns out that with q tending to infinity, the spectral radius tends to the same value as the spectral radius of the p-regular tree.

We use standard terminology and notation. For a graph G and $v \in V(G)$, $e \in E(G)$, we denote by $G - v$ and $G - e$ the subgraph of G obtained by deleting v and the subgraph obtained by removing e, respectively. If $e = uv$ is not an edge of G, then we denote by $G + e$ the graph obtained from G by adding the edge e. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and the minimum degree of G, respectively. A graph is said to be *d-degenerate* if every subgraph H of G has $\delta(H) \le d$. This condition is equivalent to the requirement that G can be reduced to the empty graph by successively removing vertices whose degree is at most d. If H is a subgraph of G, we write $H \subseteq G$.

2 Motivation and overview of known results

Our motivation for the study of the spectral radius of planar graphs comes from various directions.

(1) Harmonic analysis. The spectral radius of infinite planar graphs, in particular for tesselations of the hyperbolic plane, is of great interest in harmonic analysis. We refer to [21] and to [24, 25] for an overview.

Tessellations, whose graphs are regular of degree d, may have the spectral radius as large as d. However, this happens precisely when the graph is *amenable* (cf., e.g., [18, 25]). This is also equivalent to the condition that the random walk on the graph is recurrent. This case is well understood. However, in the case of the tessellations of the hyperbolic plane (or more general *Cantor spheres*, see [20]) the random walk is transient (Dodziuk [7]), the isoperimetric number (or the Cheeger constant) is positive [19], and the spectral radius is strictly smaller than d. It can be as small as $2\sqrt{d-1}$ (in the case of the d-regular tree). Quantitative relationship between these notions is provided via Cheeger inequality (see, e.g., [2] or [25]). It is thus surprising that the exact values for the spectral radius of regular tessellations of the hyperbolic plane are not known. Earlier best results are by Žuk [27] and Higuchi and Shirai [12]. They will be reviewed in the last section, where we present improved bounds.

(2) Mixing times of Markov chains. Bounds on the spectral radius of planar graphs can be used in the design and analysis of certain Monte Carlo algorithms and have applications not only in the theory of algorithms but also in theoretical physics. In particular, Hayes [10] and Hayes, Vera, and Vigoda [11] used these to prove $O(n \log n)$ mixing time for the Glauber dynamics for the spin systems on planar graphs. These applications include the Ising model, hard-core lattice gas model, and graph colorings that are important in theoretical physics.

(3) An application in geography. Boots and Royle [3] investigated the spectral radius of planar graphs motivated by an application in geography networks. They conjectured that for every planar graph, $\rho(G) \leq O(\sqrt{n})$, where $n = |G|$, and their computational experiments suggested that the complete join of K_2 and the path P_{n-2} gives the extremal case. Cao and Vince [4] made similar conjecture and proved that $\rho(G) \leq 4 + \sqrt{3(n-3)}$. Yuan [26] and Ellingham and Zha [8] found extensions to graphs of a fixed genus g. It is interesting that all these results are close to best bounds when there is a vertex whose degree is close to n. The setting in this paper provides the same type of the results but the bounds depend on the maximum degree and not the number of vertices.

(4) Structural graph theory. In the study of graph minors, three basic structures appear when one excludes a fixed graph H as a minor. The first one is topological—one gets graphs embeddable in surfaces in which the excluded graph H cannot be embedded. The second structure are extensions of other structures by adding a bounded number of new vertices or adding so-called "vortices" to the surface structure. This is somewhat technical and we will not consider it at this point. The last structure is related to "tree-like decompositions" and, in particular, gives rise to the family of graphs of bounded tree-width. These graphs are degenerate in the sense that they can be reduced to the empty graph by successively removing vertices of small degree. One can prove similar bounds on the spectral radius as presented in this paper, but the detailed analysis requires additional work and we leave details for future work. We refer to [16] for references concerning graph minors theory, and to [5] for some important relations between spectral theory and graph minors.

3 Spectral radius of finite and infinite graphs

If V is a set, we define $\ell^2(V)$ as the set of all functions $f : V \to \mathbb{R}$ such that $||f||^2 = \sum_{v \in V} f(v)^2 < \infty$. For a graph G with vertex set V and edge set E, we define the *adjacency operator* $A = A(G)$ as the linear operator that acts on $\ell^2(V)$ in the same way as the adjacency matrix by the rule of the matrix-vector multiplication:

$$(Af)(v) = \sum_{\{u,v\} \in E} f(u) \,.$$

If the degrees of all vertices in G are bounded above by a finite constant D, then this defines a bounded self-adjoint linear operator, whose spectrum is contained in the interval $[-D, D]$. The supremum of the spectrum is called the *spectral radius* of G and is denoted by $\rho(G)$. We refer to [21] for more details about the spectrum of infinite graphs, and refer to [1, 6, 9] for results about the spectra of finite graphs.

The following basic result [17] enables us to restrict our attention to finite graphs if desired.

Theorem 3.1. *If G is an infinite (locally finite) graph, then its spectral radius $\rho(G)$ is the supremum of spectral radii $\rho(H)$ taken over all finite subgraphs H*

of G, and it is equal to $\sup\{\rho(H_i) \mid i = 1, 2, \dots\}$*, where* $H_1 \subseteq H_2 \subseteq \cdots$ *is any seqence of subgraphs of G such that* $\bigcup_{i\geq 1} H_i = G$*.*

The spectral radius is monotone and subadditive. Formally this is stated in the following lemma.

Lemma 3.2. (a) *If $H \subseteq G$, then $\rho(H) \leq \rho(G)$.*
(b) *If $G = K \cup L$, then $\rho(G) \leq \rho(K) + \rho(L)$.*

Application of Lemma 3.2(a) to the subgraph of G consisting of a vertex of degree $\Delta(G)$ together with all its incident edges gives a lower bound on the spectral radius in terms of the maximum degree. Also, the spectral radius is bounded from above by the maximum degree, so we have the following result:

Lemma 3.3. $\sqrt{\Delta(G)} \leq \rho(G) \leq \Delta(G)$.

4 Partitioning the edges of an embedded graph

The *weight* $w(e)$ of an edge $e = uv$ is $\deg(u)+\deg(v)$. We shall use the following results regarding existence of edges of small weight (also called *light edges*) in graphs on surfaces. If Σ is a surface with Euler characteristic of $\chi(\Sigma)$, then the non-negative integer $g = 2 - \chi(\Sigma)$ is called the *Euler genus* of Σ.

Theorem 4.1 (Ivančo [14])**.** *Let G be a finite graph with minimum degree at least three, embedded in an orientable surface of Euler genus g. Then G contains an edge e with*

$$w(e) \leq \begin{cases} g + 13 & \textit{if } g < 6 \\ 2g + 7 & \textit{if } g \geq 6. \end{cases}$$

Theorem 4.2 (Jendrol' and Tuhársky [15])**.** *Let G be a finite graph with minimum degree at least three, embedded in a non-orientable surface of Euler genus g. Then G contains an edge e with*

$$w(e) \leq \begin{cases} 2g + 11 & \textit{if } 1 \leq g \leq 2 \\ 2g + 9 & \textit{if } 3 \leq g \leq 5 \\ 2g + 7 & \textit{if } g \geq 6. \end{cases}$$

Let us define

$$d(g) = \begin{cases} 10 & \text{if } g \leq 1 \\ 12 & \text{if } 2 \leq g \leq 3 \\ 2g + 6 & \text{if } 4 \leq g \leq 5 \\ 2g + 4 & \text{if } g \geq 6. \end{cases}$$

We conclude the following:

Corollary 4.3. *Let G be a finite graph with minimum degree at least three, embedded in a surface of Euler genus g. Then G contains an edge uv such that $\deg(u) + \deg(v) \leq d(g) + 3$, and hence both u and v have degree at most $d(g)$.*

We show the following decomposition result for the graphs embedded in a fixed surface:

Theorem 4.4. *Let G be a finite graph embedded in a surface of Euler genus g. Let $s = d(g)$ and for each vertex $v \in V(G)$, let $\hat{\delta}(v) = \min\{\deg(v), s\}$. Then G can be decomposed as follows:*

(a) *$G = T \cup L$, where T is a 2-degenerate graph and $\hat{\delta}(v) - 2 \leq \deg_L(v) \leq \hat{\delta}(v)$ for each vertex $v \in V(G)$.*

(b) *$G = T \cup L$, where T is a 2-degenerate graph and $\deg_L(v) \leq \hat{\delta}(v) - 2$ for each vertex $v \in V(G)$ with $\deg(v) \geq 2$, and $\deg_L(v) = 0$ if $\deg(v) \leq 1$.*

(c) *If G does not contain $K_{2,k}$ ($k \geq 2$) as a subgraph, then $G = T \cup T_1 \cup L$, such that T and T_1 are forests, $\Delta(T_1) \leq (k-1)(s-1)+2$, and $\hat{\delta}(v) - 2 \leq \deg_L(v) \leq \hat{\delta}(v)$ for each vertex $v \in V(G)$.*

Proof. Let G be a counterexample with the smallest number of edges. We may assume that G has no isolated vertices. Then G is connected. Let us call a vertex v *small* if $\deg(v) \leq s$. Let S be the set of all small vertices of G, and $S_2 \subseteq S$ the set of all vertices of G of degree at most two. No two vertices in $S \setminus S_2$ are adjacent, as otherwise we can express $G - e$ as $T \cup L'$ or $T \cup T_1 \cup L'$ and set $L = L' + e$, obtaining a decomposition of G. In the cases (a) and (c), the same reduction works for any small vertices, i.e., no two vertices in S are adjacent to each other.

Next, we claim that $\delta(G) \geq 2$. Otherwise, let v be a vertex of degree one, and let w be its neighbor. As G is the smallest counterexample, there exists a decomposition $G - v = T' \cup L$ or $G - v = T' \cup T_1 \cup L$. In the cases (a) and (c), $w \notin S$, hence $\deg_L(w) \geq s - 2$. We let $T = T' + vw$ and obtain a contradiction, as G is supposed to be a counterexample.

In the cases (a) and (b), we similarly conclude that G has minimum degree at least three (by adding both edges incident with a vertex of degree 2 into T). Since G does not contain two adjacent small vertices, this contradicts Corollary 4.3.

It remains to consider the case (c). Suppose that G contains an edge uv with $\deg(u) \leq k(s-1)+1$ and $\deg(v) = 2$, and let w be the neighbor of v distinct from u. By the minimality of G, there exists a decomposition $G - v = T' \cup T_1' \cup L$. We set $T = T' + vw$ and $T_1 = T_1' + uv$. As G does not contain two adjacent small vertices, $\deg(u) > s$ and $\deg_L(u) \geq s - 2$. It follows that $\deg_{T_1}(u) \leq k(s-1)+1-(s-2) = (k-1)(s-1)+2$, hence $\Delta(T_1) \leq (k-1)(s-1)+2$. This is a contradiction, thus each neigbor of a degree-2 vertex has degree at least $k(s-1)+2$.

Let H be the simple graph obtained from G by suppressing the degree-2 vertices and eliminating the arising parallel edges (note that the multiplicity of each such edge is at most k, as otherwise G would contain $K_{2,k}$ as a subgraph). If $v \in V(H)$ is not adjacent to a 2-vertex in G (in particular, if $v \in S \setminus S_2$), then $\deg_H(v) = \deg_G(v) \geq 3$. On the other hand, if v is adjacent to a 2-vertex, then we conclude that $\deg_H(v) \geq \frac{\deg_G(v)}{k} \geq \frac{k(s-1)+2}{k} \geq 3$. It follows that the minimum degree of H is at least three, and by Corollary 4.3, H contains an edge uv with $\deg_H(u) + \deg_H(v) \leq s+3$. We may assume that $\deg_H(u) \leq \deg_H(v)$, and thus $\deg_G(u) \leq k \deg_H(u) \leq k\frac{s+3}{2} \leq k(s-1)+1$. We conclude that u is not adjacent to a degree-2 vertex in G, and hence $\deg_G(u) = \deg_H(u) \leq s$ and u is small. It follows that $uv \in E(G)$ and v is not small, thus $\deg_G(v) > \deg_H(v)$ and v is adjacent to a degree-2 vertex in G, and $\deg_G(v) \geq k(s-1)+2$. However, using the fact that u and v do not have a common neighbor of degree 2, we get $\deg_H(v) \geq 1 + \lceil (\deg_G(v) - 1)/k \rceil \geq 1 + \lceil (k(s-1)+1)/k \rceil = s+1$, which is a contradiction. □

Consider a decomposition of the graph $K_{3,n}$ into a 2-degenerate graph T and a graph L of maximum degree s. Let a_1, a_2 and a_3 be the three vertices of degree n and let B be the set of n vertices of degree three. Let $B' \subseteq B$ be the set of vertices that are not incident with an edge of L. Since the maximum degree of L is s, we obtain $|B'| \geq n - 3s$. As $K_{3,3}$ is not 2-degenerate, $|B'| \leq 2$. Therefore, $n - 3s \leq 2$, and $n \leq 3s+2$. As $K_{3,2g+2}$ can be embedded in a surface of Euler genus g (Ringel [23]), it is not possible to improve the bound on the maximum degree of L in such a decomposition below $\frac{2}{3}g$, i.e., $\Delta(L) = \Omega(g)$.

5 Spectral radius of embedded graphs

We now use the decomposition theorem to obtain a bound on the spectral radius of graphs of bounded genus. In all proofs we assume that the graph G is finite. However, the proof given for the finite case extends to infinite graphs by applying Theorem 3.1 and taking the limit over larger and larger finite subgraphs.

Theorem 5.1. *Let G be a graph embedded in a surface of Euler genus g.*

(a) *If $\Delta(G) \geq d(g) + 2$, then $\rho(G) \leq \sqrt{8(\Delta(G) - d(g))} + d(g)$.*

(b) *If G does not contain $K_{2,k}$ ($k \geq 2$) as a subgraph and $\Delta(G) \geq d(g)$, then*

$$\rho(G) \leq 2\sqrt{\Delta(G) - d(g) + 1} + 2\sqrt{(k-1)(d(g)-1)+1} + d(g).$$

Proof. Let $G = T \cup L$ be a decomposition as guaranteed by Theorem 4.4(a). Note that every vertex of degree $\geq d(g)$ satisfies $\deg_T(v) = \deg_G(v) - \deg_L(v) \leq \deg_G(v) - d(g) + 2$ and every vertex of degree $< d(g)$ in G satisfies $\deg_T(v) \leq 2$. Thus $\Delta(T) - 2 \leq \Delta(G) - d(g)$. By Theorem 1.1, $\rho(T) \leq 2\sqrt{2\Delta(T) - 4} \leq 2\sqrt{2(\Delta(G) - d(g))}$. Furthermore, $\rho(L) \leq \Delta(L) \leq d(g)$. The bound on $\rho(G)$ in (a) follows therefrom by the subadditivity of the spectral radius (Lemma 3.2(b)). Part (b) follows similarly from Theorem 4.4(c). □

The bound of Theorem 5.1(a) can be improved when $\Delta(G)$ is large by using the decomposition of Theorem 4.4(b) instead of (a). We present this improvement only in the special case of planar graphs, where another slight improvement is possible.

Theorem 5.2. *A planar graph G of maximum degree $\Delta = \Delta(G) \geq 10$ has*

$$\rho(G) \leq \sqrt{8\Delta - 80} + 2\sqrt{21} < \sqrt{8\Delta - 80} + 9.17$$

and

$$\rho(G) \leq \sqrt{8\Delta - 16} + 2\sqrt{15} < \sqrt{8\Delta - 16} + 7.75$$

Furthermore, if G does not contain a separating 4-cycle, then

$$\rho(G) \leq 2\sqrt{\Delta - 9} + 2\sqrt{19} + 2\sqrt{21} < 2\sqrt{\Delta - 9} + 17.883.$$

Proof. We proceed as in the proof of Theorem 5.1, considering the decomposition $G = T \cup L$. We may assume that $\Delta \geq 12$ since the bounds follow easily for $\Delta \leq 11$ by using Theorem 1.1. We estimate the contribution of T in the same way. However, we use Theorem 1.1 to bound the spectral radius of L. Every planar graph has an orientation with maximum indegree 3, hence $\rho(L) \leq 2\sqrt{3(\Delta(L) - 3)} \leq 2\sqrt{21} < 9.17$. For the second inequality we apply Theorem 4.4(b) instead of (a).

Consider now the case that G does not contain separating 4-cycles. If G does not contain $K_{2,3}$ as a subgraph, then the bound follows as in Theorem 5.1, using the fact that $\rho(L) \leq 2\sqrt{21}$. So suppose that $K_{2,3} \subseteq G$. As G has no separating 4-cycles, it is easy to see that $V(G) = V(K_{2,3})$, and thus $\Delta(G) \leq 4$. ∎

In the estimates of Theorem 5.1, we can improve the dependency on the genus using the following lemma:

Lemma 5.3. *Let $\varepsilon > 0$ be a real number and let G be a finite graph embedded in a surface of Euler genus g, with $\Delta(G) = O(g)$. Then $\rho(G) = O(\varepsilon^{-1} g^{\frac{1+\varepsilon}{2}})$.*

Proof. Let $k = \lceil \varepsilon^{-1} \rceil$. We construct a decomposition $G = G_1 \cup \cdots \cup G_k$, such that for $i = 1, \ldots, k$, $\Delta(G_i) = O(g^{1-\varepsilon(i-1)})$ and G_i is $O(g^{\varepsilon i})$-degenerate. By Theorem 1.1, $\rho(G_i) = O(g^{(1+\varepsilon)/2})$, and by the subadditivity of the spectral radius, $\rho(G) = O(\varepsilon^{-1} g^{(1+\varepsilon)/2})$.

Suppose that we have already constructed graphs $G_1, G_2, \ldots, G_i$. If $i = 0$, then let $H_0 = G$, otherwise let H_i be the complement of $G_1 \cup \cdots \cup G_i$ in G, i.e., the subgraph of G consisting of the edges that do not belong to $G_1 \cup \cdots \cup G_i$ and the vertices incident with these edges. Let S_{i+1} be the set of vertices obtained in the following way: we take a vertex of degree less than $g^{\varepsilon(i+1)} + 6$ in H_i, add it to S_{i+1}, and remove it from H_i. We repeat this process as long as the graph contains vertices of degree less than $g^{\varepsilon(i+1)} + 6$. We let G_{i+1} consist of the edges incident with at least one vertex of S_{i+1}. This ensures that G_{i+1} is $(g^{\varepsilon(i+1)} + 6)$-degenerate. Note that $H_{i+1} = H_i - S_{i+1}$.

The construction also ensures that for $i \geq 1$, the minimum degree of H_i is at least $g^{\varepsilon i} + 6$ (if $H_i \neq \emptyset$, which we may assume), hence $2|E(H_i)| \geq (6 + g^{\varepsilon i})|V(H_i)|$. On the other hand, as H_i is embedded in a surface of Euler genus g, $2|E(H_i)| \leq 6|V(H_i)| - 12 + 6g$, hence $\Delta(H_i) \leq |V(H_i)| = O(g^{1-\varepsilon i})$. Since $G_{i+1} \subseteq H_i$, this implies the claimed upper bound on the maximum degree of G_{i+1} and completes the proof. ∎

The exponent in the bound of Lemma 5.3 cannot be improved below 1/2, as the complete graph on $\Omega(\sqrt{g})$ vertices can be embedded in a surface of Euler genus g. Together with the decompositions given by Theorem 4.4, Lemma 5.3 gives:

Theorem 5.4. *If a graph G has Euler genus g, then*

$$\rho(G) \leq \sqrt{8\Delta(G)} + O(g^{\frac{1}{2}} \log g).$$

If k is a positive integer and G does not contain $K_{2,k}$ as a subgraph, then

$$\rho(G) \leq 2\sqrt{\Delta(G)} + O(g^{\frac{1}{2}} \log g).$$

Proof. We apply the decompositions given by Theorem 4.4. For the small degree subgraph L we use Lemma 5.3 with $\varepsilon = (\log g)^{-1}$ to conclude that $\rho(L) = O(\varepsilon^{-1} g^{\frac{1+\varepsilon}{2}}) = O(g^{\frac{1}{2}} \log g)$. □

6 Lower bounds

In this section we show that the bounds given by Theorem 5.1 are tight up to the additive term. As the spectral radius of an infinite d-regular tree is $2\sqrt{d-1}$, for any $\varepsilon > 0$ there exists a finite tree T with $\rho(T) > 2\sqrt{\Delta(T)-1} - \varepsilon$, matching the upper bound $2\sqrt{\Delta(G)} + O(1)$ for planar graphs excluding $K_{2,k}$.

Let k and d, $k < d$, be integers such that d is divisible by k. Consider now the following sequence of graphs $H_i^{k,d}$. Let $H_0^{k,d} = K_{k,d-k}$. The graph $H_i^{k,d}$ contains $k\left(\frac{d-k}{k}\right)^{i+1}$ vertices of degree k, let S_{i+1} be the set of these vertices. The graph $H_{i+1}^{k,d}$ is obtained from $H_i^{k,d}$ by partitioning S_{i+1} into k-tuples in some canonical way, then for each such k-tuple C adding $d-k$ new vertices adjacent to each vertex of C (the newly added vertices form the set S_{i+2}). The infinite graph $H^{k,d}$ is the limit of the sequence of the graphs $H_i^{k,d}$. See Figure 1 for an example with $k = 2$ and $d = 8$. The following properties are easy to prove:

- $\Delta(H^{k,d}) = d$.
- $\rho(H^{k,d}) = \sup_i \rho(H_i^{k,d})$.
- The graphs $H_i^{2,d}$ are planar (assuming the natural partitionings of the sets S_i).
- The graphs $H_i^{k,d}$ are k-degenerate.

Lemma 6.1. *The spectral radius of $H^{k,d}$ is $2\sqrt{k(d-k)}$.*

Proof. Due to Theorem 1.1 and the second observation in the previous paragraph, it suffices to show that $\rho(H^{k,d}) \geq 2\sqrt{k(d-k)}$. Let A be the adjacency operator associated with $H = H^{k,d}$. In addition to the sets S_i defined during

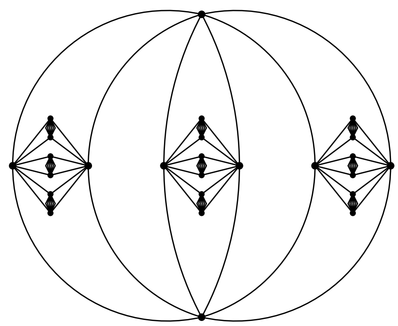

Figure 1: The graph $H^{2,8}$

the construction of H, let S_0 be the set of k vertices of H of degree $d-k$. Let us recall that $|S_i| = k\left(\frac{d-k}{k}\right)^i$. Furthermore, all the edges of H are between the vertices of S_i and S_{i+1}, for $i = 0, 1, 2, \ldots$. Observe that there are exactly $k|S_{i+1}| = (d-k)|S_i|$ edges between S_i and S_{i+1}.

Let f be the function defined by $f(v) = q^i$ for any $v \in S_i$, where $0 < q < \sqrt{\frac{k}{d-k}}$. Note that

$$||f||^2 = \sum_{v \in V(H)} f^2(v) = \sum_{i=0}^{\infty} |S_i| q^{2i} = \sum_{i=0}^{\infty} k \left(q^2 \, \frac{d-k}{k}\right)^i < \infty.$$

Also,

$$\begin{aligned}
\langle f|Af\rangle &= 2 \sum_{uv \in E(H)} f(u)f(v) \\
&= 2\sum_{i=0}^{\infty} (d-k)|S_i|q^{2i+1} \\
&= 2q(d-k)\sum_{i=0}^{\infty} |S_i|q^{2i}.
\end{aligned}$$

It follows from the above calculations that $\frac{\langle f|Af\rangle}{||f||^2} = 2q(d-k)$ can be arbitrarily close to $2(d-k)\sqrt{\frac{k}{d-k}} = 2\sqrt{k(d-k)}$. Therefore, $\rho(H^{k,d}) \geq 2\sqrt{k(d-k)}$. $\square$

We conclude that

- the upper bound in Theorem 1.1 is best possible for graphs that have an orientation with maximum indegree k (i.e., the graphs with maximum average density at most k) and for k-degenerate graphs, and
- as the graph $H^{2,d}$ is planar, the bound $\sqrt{8\Delta} + O(1)$ for the spectral radius of a planar graph as given in Theorem 5.1 is best possible up to the additive term.

7 Hyperbolic tessellations

In this section, we show how to apply a refined decomposition technique to bound the spectral radius of a special kind of infinite planar graphs. For two integers $p, q \geq 3$, where $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$, we call a connected infinite simple plane graph G a $(p, \geq q)$-*tessellation* if it is p-regular and each of its faces has size at least q, and every compact subset of the plane contains only a finite number of its vertices. If all faces have finite size, then this condition implies that G is a one-ended graph, but in the presence of faces of infinite length, G may have more than one end. We will assume that $p \geq 4$ and $q \geq 4$. The cases $p = 3$ and $q = 3$ could be dealt with (assuming that $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$), but the decomposition results would require slight modificiations.

Lemma 7.1. *Let C be a cycle in a $(p, \geq q)$-tessellation G, and let H be the subgraph of G contained in the closed disk bounded by C. Let $d = \sum_{u \in V(C)} \deg_H(u)$ and $k = |V(C)|$. Then $d < 2(k-1)\frac{q-1}{q-2}$.*

Proof. By the definition of $(p, \geq q)$-tessellations, H is finite. Let $n = |V(H)|$, $e = |E(H)|$ and let s be the number of faces of H. By Euler's formula, $n + s = e + 2$. Furthermore, observe that $2e = pn - pk + d$ and $2e \geq qs - q + k$. Combining them, we obtain the following inequality:

$$\frac{d}{p} + 2e\left(\frac{1}{2} - \frac{1}{p} - \frac{1}{q}\right) \leq k - \frac{k}{q} - 1.$$

As $\frac{1}{2} - \frac{1}{p} - \frac{1}{q} \geq 0$ and $2e \geq d$, we get

$$d\,\frac{q-2}{2q} \leq k - \frac{k}{q} - 1,$$

and hence

$$d \leq 2k\frac{q-1}{q-2} - \frac{2q}{q-2} < (2k-2)\frac{q-1}{q-2}.$$

□

Note that Lemma 7.1 implies that G is triangle-free if $q \geq 4$. Let v be a vertex of a $(p, \geq q)$-tessellation G. We define a partition $V_0 \cup V_1 \cup V_2 \cup \cdots$ of vertices of G and a partition $F_0 \cup F_1 \cup F_2 \cup \cdots$ of faces of G in the following way: let $V_0 = \{v\}$ and let F_0 consist of faces incident with v. For each $i > 0$, let V_i consist of the vertices incident with the faces in F_{i-1}, excluding those in V_{i-1}, and let F_i consist of all faces incident with the vertices of V_i, excluding those in F_{i-1}. Let G_i be the subgraph of G induced by V_i. We call the graphs $G_1, G_2, \ldots$ the *layers* of G with respect to v.

Lemma 7.2. *For every $(p, \geq q)$-tessellation G with $p \geq 4$ and $q \geq 4$ and a vertex $v \in V(G)$, the partition $V_0 \cup V_1 \cup V_2 \cup \cdots$ has the following properties, for each $i > 0$:*

(a) *The subgraph G_i is either a union of infinite paths, or a cycle. The face of G_i that contains v is equal to $F_0 \cup F_1 \cup \cdots \cup F_{i-1}$; the boundary of every other face of G_i is bounded by a connected component of G_i.*

(b) *Each vertex of V_i has at most one neighbor in V_{i-1}.*

(c) *A face belonging to F_{i-1} is incident with at most two vertices in V_{i-1}, and if it is incident with two such vertices, then they are adjacent in G_{i-1}.*

Proof. For a contradiction, assume that i is the smallest positive integer such that one of the conditions (a), (b) or (c) is violated. Let us first consider the possibility that condition (b) is false, and let $u \in V_i$ be a vertex with at least two neighbors $w_1, w_2 \in V_{i-1}$. Obviously, $i \geq 2$, and thus G_{i-1} satisfies condition (a). It follows that w_1 and w_2 belong to the same component of G_{i-1}. Let C be the unique cycle in $G_{i-1} + uw_1 + uw_2$ such that the disk bounded by C does not contain v, and let H be the subgraph of G drawn in the closed disk bounded by C. By the conditions (a) and (b) applied for $i-1$, we conclude that $\deg_H(w) \geq p-1 \geq 3$ for each vertex $w \in V(C)$, except for u, w_1 and w_2. Let $k = |V(C)|$ and $d = \sum_{w \in V(C)} \deg_H(w)$. By the above,

$$d \geq 3(k-3) + \deg_H(u) + \deg_H(w_1) + \deg_H(w_2) \geq 3(k-1).$$

However, since $q \geq 4$, Lemma 7.1 implies that

$$d < 2(k-1)\frac{q-1}{q-2} \leq 3(k-1),$$

a contradiction.

Now, consider the possibility that condition (b) holds, but condition (c) fails. As G is triangle-free, we conclude that there exists a face $f \in F_{i-1}$ incident with two non-adjacent vertices w_1 and w_2 in V_{i-1}. Note that $i \geq 2$. We consider a cycle C contained in the union of G_{i-1} and the boundary of f, such that the disk bounded by C contains neither v nor f, and let H be the subgraph of G contained in the closed disk bounded by C. Note that $\deg_H(w) = p$ for any vertex $w \in V(C) \setminus V_{i-1}$, and $\deg_H(w) \geq p-1$ for $w \in V(C) \setminus V(f)$, i.e., all but at most two vertices $w \in V(C)$ satisfy $\deg_H(w) \geq 3$. This again contradicts Lemma 7.1.

Finally, suppose that (b) and (c) hold. Consider a vertex $u \in V_i$. Similarly as in the case (b), we conclude that u is incident with at most two faces in F_{i-1} and that if it is incident with two such faces, then they share an edge uw with $w \in V_{i-1}$. Also, any edge of G_i is incident with a face in F_{i-1}. It follows that G_i is 2-regular, and thus it is a union of cycles and infinite paths. By Lemma 7.1 and the property (b) of G_i, each disk bounded by a cycle in G_i contains v. The claim (a) follows, as $F_0 \cup F_1 \cup \cdots \cup F_{i-1}$ is a connected subset of the plane. □

We also need the following fractional version of Lemma 3.2:

Lemma 7.3. *Let G_1, G_2, …, G_m be subgraphs of a graph G such that each edge of G appears in at least p of the subgraphs. Then $\rho(G) \leq \frac{1}{p}\sum_{i=1}^{m} \rho(G_i)$.*

Proof. By the monotonicity, we may assume that each edge of G appears in exactly p of the subgraphs. Let A be the adjacency operator of G and A_i the adjacency operator of G_i for $1 \le i \le m$, and observe that $A = \frac{1}{p}\sum_{i=1}^{m} A_i$.

Let $\varepsilon > 0$. There exists a function f such that $||f|| = 1$ and $\langle f|Af\rangle \ge \rho(G) - \varepsilon$. By linearity, $\langle f|Af\rangle = \frac{1}{p}\sum_{i=1}^{m}\langle f|A_i f\rangle \le \frac{1}{p}\sum_{i=1}^{m}\rho(G_i)$. Therefore, $\rho(G) - \varepsilon \le \frac{1}{p}\sum_{i=1}^{m}\rho(G_i)$. Since this inequality holds for any $\varepsilon > 0$, the claim of the lemma follows. $\square$

We are now ready to estimate the spectral radius of tessellations:

Theorem 7.4. *If G is a $(p, \ge q)$-tessellation with $p \ge 4$ and $q \ge 4$, then*

$$\rho(G) \le 2\sqrt{p-1} + \frac{2}{q-3}\,.$$

Proof. Choose a vertex $v \in V(G)$ arbitrarily, and consider the layers G_1, G_2, ... with respect to v. Let us color a vertex $u \in V_i$ black if u has a neighbor in V_{i-1}, and white otherwise. Let an *earthworm* be a maximal subgraph H of $G_1 \cup G_2 \cup \cdots$ such that every two vertices of H are joined by a path whose inner vertices are white. By Lemma 7.2(a) and (c), all earthworms are paths of length at least $q-3$. Let M_1, M_2, ..., M_{q-3} be edge-disjoint matchings such that each of them intersects every earthworm in exactly one edge. For $1 \le i \le q-3$, consider the graph $T_i = G - M_i$. We claim that T_i is a forest. Suppose for a contradiction that T_i contains a cycle C. Let j be the greatest index such that $V(C) \cap V_j \ne \emptyset$. As M_i contains at least one edge from each component of G_j, $C \not\subseteq G_j$. Let P be a maximal subpath of $C \cap G_j$. Since each vertex of G_j has at most one neighbor in V_{j-1}, P is not a single vertex. We conclude that P joins two black vertices of G_j and thus it is a supergraph of at least one earthworm. Therefore, $M_i \cap P \ne \emptyset$, which is a contradiction. This proves our claim.

Let $T_{q-2} = G_1 \cup G_2 \cup \cdots$. Observe that each edge of G belongs to at least $q-3$ of the graphs T_1, T_2, ..., T_{q-2}, $\rho(T_i) \le 2\sqrt{p-1}$ for $1 \le i \le q-3$ and $\rho(T_{q-2}) = 2$. By Lemma 7.3, we get $\rho(G) \le 2\sqrt{p-1} + \frac{2}{q-3}$. $\square$

As q goes to infinity, the bound of Theorem 7.4 aproaches $2\sqrt{p-1}$, which is the spectral radius of the p-regular infinite tree. This considerably improves known upper bounds, including the previously best bound of Higuchi and Shirai [12], who proved that

$$\rho(G) \le 2\sqrt{(p-2)(1+\tfrac{1}{q-2})}.$$

A non-trivial lower bound on the spectral radius of p-regular graphs has been obtained only for vertex-transitive graphs. Paschke [22] showed that a vertex transitive p-regular graph containing a q-cycle has spectral radius at least

$$\min_{s>0}\,(p-2)\,\phi\!\left(\frac{1+\cosh sq}{\sinh sq \sinh s}\right) + 2\cosh s,$$

where $\phi(t) = \frac{\sqrt{1+t^2}-1}{t}$. This gives a lower bound of the form

$$2\sqrt{p-1} + \frac{2(p-2)}{(p-1)^{(q+1)/2}}\, h(p,q),$$

where h is a function such that such that $\lim_{p\to\infty} h(p,q) = 1$ and $\lim_{q\to\infty} h(p,q) = 1$. The asymptotics (when p or q is large) of this lower bound is different from our upper bound in Theorem 7.4 in the "second order term" when $(p, \geq q)$-tessellations are considered. It would be of interest to determine the exact behavior.

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