Homotopic Spanners*

Sergio Cabello† Bojan Mohar‡ Arjana Žitnik§

Abstract

We introduce the concept of homotopic spanners in the plane with obstacles and show lower bounds on the number of edges that they require. We also provide a construction based on Θ-graphs for constructing homotopic spanners.

1 Introduction

Spanners have become a basic tool for the design of networks: they are graphs connecting a given set of sites with the property that the distances between sites along the graph is similar to the straight-line distance between the sites. As a basic requirement, spanners have to be sparse, that is, they need to have few edges. Typically, we are interested on spanners that have additional properties, such as bounded degree, small total length, small spanning diameter, etc.

In applications like robot motion planning, we often deal with the scenario where the sites are in the plane and we also have a set of obstacles to be avoided. This naturally leads to the problem of computing spanners under the influence of polyhedral obstacles, already considered by Clarkson [6] and Das [7].

We consider here the construction of spanners in the plane with point-obstacles, but with the additional condition that between each pair of sites there is a short path in the spanner which is homotopically equivalent to the straight-line segment that joins the sites. Although much work has been done on spanners with additional properties, we are not aware of any research on constructing spanners with topological properties.

In the next section we introduce the basic notation and topological background; we also define precisely the concept of homotopic spanners. In Section 3 we show a modification of Θ-graphs that can be used to construct homotopic spanners. In Section 4 we discuss the computational issues related to the construction. In Section 5 we present lower bounds on the number of edges that any homotopic spanner needs.

2 Notions and problem statement

Topological background. A finite set of points $\mathcal{K} \subset \mathbb{R}^2$ will be called *point-obstacles*. If $x, y \in \mathbb{R}^2 \setminus \mathcal{K}$, a *path* from x to y is a continuous mapping $\alpha : [0,1] \to \mathbb{R}^2 \setminus \mathcal{K}$ such that $\alpha(0) = x$ and $\alpha(1) = y$. If β is a path from y to z, then the concatenation $\alpha + \beta$ of paths α and β is a path from x to z defined as $(\alpha+\beta)(u) = \alpha(2u)$ if $0 \le u \le \frac{1}{2}$ and $(\alpha+\beta)(u) = \beta(2u-1)$ if $\frac{1}{2} \le u \le 1$. Two paths α, β joining the same pair of points in $\mathbb{R}^2 \setminus \mathcal{K}$ are said to be *homotopic*, denoted $\alpha \sim_{\mathbb{R}^2\setminus\mathcal{K}} \beta$ if the loop $\alpha - \beta$ (α concatenated with the reverse of β) is a contractible curve in $\mathbb{R}^2 \setminus \mathcal{K}$. The reader is referred to [8], where also the following standard results can be found:

Lemma 1 *Homotopy of paths has the following properties:*

1. *The relation $\alpha \sim_{\mathbb{R}^2\setminus\mathcal{K}} \beta$ is an equivalence relation.*
2. *If $\alpha \sim_{\mathbb{R}^2\setminus\mathcal{K}} \beta$, $\alpha' \sim_{\mathbb{R}^2\setminus\mathcal{K}} \beta'$, and $\alpha(1) = \beta(1) = \alpha'(0) = \beta'(0)$, then $(\alpha + \alpha') \sim_{\mathbb{R}^2\setminus\mathcal{K}} (\beta + \beta')$.*
3. *If the paths α, β share endpoints and are contained in a convex subset of $\mathbb{R}^2\setminus\mathcal{K}$, then $\alpha \sim_{\mathbb{R}^2\setminus\mathcal{K}} \beta$.*

Homotopic Spanners. Let S be a point set in $\mathbb{R}^2$, and let $G = (S, E)$ be a graph on S. The graph is represented in the plane with each vertex represented by the point itself and with straight-line edges. We use ss' to denote both, the edge of G and the straight-line segment joining s and s'. We associate with each edge $ss' \in E$ the length $|ss'|$ of the straight-line segment joining its vertices. The length of a path α in G is the sum of the lengths of its edges; we denote it by $|\alpha|_G$.

For $t \in \mathbb{R}, t \ge 1$, a path in G from $s \in S$ to $s' \in S$ is a *t-path* if its length, is at most $t\,|ss'|$. A graph G is a *t-spanner* if, for each pair of points $s, s' \in S$, there exists a t-path in G from s to s'. We consider the following generalization.

Definition 1 *Given a set of points $S \subset \mathbb{R}^2$ and a set of point-obstacles $\mathcal{K}$, a* $\mathcal{K}$-homotopic t-spanner *of S is a graph $G = (S, E)$ such that, for any $s, s' \in S$, there*

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†Department of Mathematics, IMFM, Ljubljana, Slovenia. sergio.cabello@imfm.uni-lj.si

‡Department of Mathematics, FMF, University of Ljubljana, Slovenia. bojan.mohar@fmf.uni-lj.si

§Department of Mathematics, FMF, University of Ljubljana, Slovenia. arjana.zitnik@fmf.uni-lj.si

is a t-path α in G such that α and the segment ss' are homotopic in $\mathbb{R}^2 \setminus \mathcal{K}$.

We consider the following problem: given a fixed number $t > 1$, construct homotopic t-spanners as a function of S and $\mathcal{K}$ such that the number of edges is the spanner is not too large. We let $n = |S|$ and $k = |\mathcal{K}|$. We assume that no obstacle in $\mathcal{K}$ is aligned with two points of S, as otherwise it may be that the desired spanner does not exist.

3 Construction of homotopic spanners

The idea is to modify the construction of Θ-spanners introduced by Keil and Gutwin [9]. We use a notation similar to Arya, Mount, and Smid [2] and Bose, Gudmundsson, and Morin [3]. Consider an angle $\theta = \frac{2\pi}{T}$ for some integer $T > 8$ such that it holds $t_\theta = \frac{1}{\cos\theta - \sin\theta} \leq t$. For a point s in S, consider the set of rays $\mathcal{R}_{s,\theta} = \{ray_j(s) \mid j \in \{0, \ldots, T-1\}\}$, where $ray_j(s)$ is the straight ray from s with angle $j\theta$ with a horizontal line, and $\mathcal{R}_{s,\mathcal{K}} = \{ray(s, o) \mid o \in \mathcal{K}\}$, where $ray(s, o)$ is the straight ray starting at s with direction towards o. Let $\mathcal{R}_s = \mathcal{R}_{s,\theta} \cup \mathcal{R}_{s,\mathcal{K}}$.

All the rays in $\mathcal{R}_s$ have s as starting point, and therefore they divide the plane into a set of cones, which we denote by $\mathcal{C}_s$. Since t is a fixed constant, also T is a constant. Hence, $\mathcal{C}_s$ consists of $O(1 + k)$ cones. Any cone $C \in \mathcal{C}_s$ has angle at most θ and it contains no obstacles in its interior. For a cone $C \in \mathcal{C}_s$, consider any ray r from s contained in C and let j be the largest value such that $j\theta$ is smaller than the angle of r; we use $ray(C)$ for the ray $ray_j(s) \in \mathcal{R}_{s,\theta}$. Observe that the angle between r and $ray(C)$ is at most θ.

Let the graph $\Theta(S, \mathcal{K}, T)$ be defined as follows:

- The set of vertices of $\Theta(S, \mathcal{K}, T)$ is S;
- For each point $s \in S$, for each cone $C \in \mathcal{C}_s$ such that $C \cap (S \setminus \{s\}) \neq \emptyset$, we put an edge connecting s and a point s_C in $C \cap S \setminus \{s\}$ that has the orthogonal projection onto $ray(C)$ closest to s. If there are more than one candidate for s_C, we select one which is closest to $ray(C)$.

Observe that $\Theta(S, \mathcal{K}, T)$ has $O(nk)$ edges.

Theorem 2 *The graph $\Theta(S, \mathcal{K}, T)$ is a $\mathcal{K}$-homotopic t_θ-spanner of S with $O(nk)$ edges.*

Proof. Consider two points $s, s' \in S$, and let C be the cone of $\mathcal{C}_s$ that contains s'. By construction, we know that there is a point $s_c \in C$ such that ss_c is an edge in $\Theta(S, \mathcal{K}, T)$. Using that $ray(s, s')$ and $ray(s, s_c)$ form an angle at most θ, the same argument that is used for the standard Θ-graph [2] implies

$$t_\theta\,|s_c s'| \leq t_\theta\,|ss'| - |ss_c|. \tag{1}$$

We show by induction on the rank of the interpoint distances that for any pair of points $s, s' \in S$ there is a t_θ-path in $\Theta(S, \mathcal{K}, T)$ that is homotopic to ss'. If the pair s, s' is a closest pair, then it holds that $s_c = s'$ and therefore the segment ss' is in $\Theta(S, \mathcal{K}, T)$.

Consider a pair of points $s, s' \in S$. If $s' = s_c$, then the segment ss' is in $\Theta(S, \mathcal{K}, T)$ and there is nothing to show. Otherwise, $s' \neq s_c$. Because of (1), we have $|s_c s'| < |ss'|$, and by induction hypothesis there is a t_θ-path α in $\Theta(S, \mathcal{K}, T)$ from s_c to s' that is homotopic to the segment $s_c s'$ in $\mathbb{R}^2 \setminus \mathcal{K}$, that is $\alpha \sim_{\mathbb{R}^2\setminus\mathcal{K}} s_c s'$. Let $\beta = ss_c + \alpha$. We have

$$|\beta|_G = |ss_c| + |\alpha|_G \leq |ss_c| + t_\theta\,|s_c s'| \leq t_\theta\,|ss'|,$$

where the last inequality follows from equation (1). This means that β is a t_θ-path from s to s'.

We next show that $\beta \sim_{\mathbb{R}^2\setminus\mathcal{K}} ss'$, which finishes the proof. Since $\beta = ss_c + \alpha$ and $\alpha \sim_{\mathbb{R}^2\setminus\mathcal{K}} s_c s$, we have $\beta \sim_{\mathbb{R}^2\setminus\mathcal{K}} ss_c + s_c s$ because of property 2 in Lemma 1. Because the triangle $\triangle ss's_c$ is contained in the cone $C \in \mathcal{C}_s$ we have $\mathcal{K} \cap \triangle ss's_c = \emptyset$, and by property 3 in Lemma 1 we conclude that $ss_c + s_c s' \sim_{\mathbb{R}^2\setminus\mathcal{K}} ss'$. Since $\sim$ is an equivalence relation we get $\beta \sim_{\mathbb{R}^2\setminus\mathcal{K}} ss'$. $\square$

For any value $t > 1$ we can take a constant $T \in \mathbb{N}$ large enough such that $t \geq \frac{1}{\cos(2\pi/T) - \sin(2\pi/T)}$, and we conclude that for any fixed t we can construct a $\mathcal{K}$-homotopic t-spanner with $O(nk)$ edges.

4 Efficient construction

Consider a set of n sites S and k obstacles $\mathcal{K}$. We assume that $k \leq n$, as otherwise we can just consider the complete graph as a spanner and we are within the bound of $O(nk)$ edges for a spanner that we are aiming to. For a fixed value T, the graph $\Theta(S, \mathcal{K}, T)$ can be constructed in $O(n^2 \log k)$ time as follows:

1. for each site $s \in S$
   (a) split the sites $S \setminus \{s\}$ into the cones of $\mathcal{C}_s$. This can be done by making a tree-like structure for the boundary rays $\mathcal{R}_s$ of $\mathcal{C}_s$ in $O(k \log k)$ and locating each point of $S \setminus \{s\}$ in the appropriate cone in $O(\log k)$ time per point. This takes $O(k \log k + n \log k) = O(n \log k)$ time.
   (b) for each cone $C \in \mathcal{C}_s$, scan the points and choose the one that s gets connected to, according to the criteria in Section 3. This takes $O(n)$ time overall because each point appears at most in two cones of $\mathcal{C}_s$.

We discuss how the graph $\Theta(S, \mathcal{K}, T)$ can be constructed in a more efficient way. The idea is to consider all the cones as range spaces and use the standard trade-offs for simplex range queries; see Matoušek [10] or the survey by Agarwal and Erickson [1].

The main result to be used is the following (we use the notation $\tilde{O}(f(n)) = O(f(n)n^{\varepsilon})$ for any $\varepsilon > 0$, where the constant in $\tilde{O}(f(n))$ may depend on ε).

Lemma 3 *For any set S of n points in the plane and any value $n \le m \le n^2$ there is a family $\mathcal{F}(S) = \{F_1, \ldots, F_p\}$ of subsets of S and a data structure $\mathcal{D}(S)$ such that*

- *$p = O(m)$, that is, $\mathcal{F}(S)$ has $O(m)$ members;*
- *$\sum_{i=1}^{p} |F_i| = \tilde{O}(m)$;*
- *for any triangle Δ in the plane, there is a group $\mathcal{F}(\Delta)$ of $\tilde{O}(n/\sqrt{m})$ elements of $\mathcal{F}(S)$ such that $\Delta \cap S = \bigcup_{F\in\mathcal{F}(\Delta)} F$;*
- *$\mathcal{D}(S)$ has size $O(m)$ and can be constructed in $\tilde{O}(m)$ time;*
- *for a query triangle Δ, the data structure $\mathcal{D}$ provides $\mathcal{F}(\Delta) \subset \mathcal{F}$ in $\tilde{O}(n/\sqrt{m})$ time.*

For a point set S and an angle α let $Point(S, \alpha)$ denote a point in S such that the line passing through it with angle $\alpha + \pi/2$ has all the points of S to its right; that is, $Point(S, \alpha)$ is a point with minimum x-coordinate after rotating S with angle $-\alpha$.

We extend the data structure of the previous lemma as follows: for each set $F \in \mathcal{F}$ and each value $j = 0, \ldots, T-1$ we store $Point(F, j\frac{2\pi}{T})$. For any triangle Δ we have

$$Point(S \cap \Delta, j\tfrac{2\pi}{T}) \in \{Point(F, j\tfrac{2\pi}{T}) \mid F \in \mathcal{F}(\Delta)\},$$

and using the previous lemma we conclude that we can find the point $Point(S \cap \Delta, j\frac{2\pi}{T})$ in $\tilde{O}(n/\sqrt{m})$ time per triangle Δ.

The augmented data structure can be constructed by considering each $j = 0, \ldots, T-1$ and scanning each $F \in \mathcal{F}$. Since we regard T as a constant, and each $F \in \mathcal{F}$ is considered T times, we need $O(T\sum_{F\in\mathcal{F}} |F|) = \tilde{O}(m)$ time to construct the augmented data structure.

Consider the construction of $\Theta(S, \mathcal{K}, T)$ given in Section 3. For a cone C with apex s, we have to find a point in C with the orthogonal projection onto $ray(C)$ closest to s. Since $ray(C)$ has an angle of the form $j_C\frac{2\pi}{T}$ for some j_C, it follows that this point is $Point(C \cap S, j_C\frac{2\pi}{T})$. Since a cone is a special case of a triangle, we can use the previous discussion to conclude that we can find the edge that the cone C contributes to $\Theta(S, \mathcal{K}, T)$ in $\tilde{O}(n/\sqrt{m})$ time.

By setting $m = n^{4/3}k^{2/3}$ we can find the edge corresponding to a cone in $\tilde{O}(n/\sqrt{n^{4/3}k^{2/3}}) = \tilde{O}(n^{1/3}k^{-1/3})$ time. This makes sense since we are assuming $k \le n$. The preprocessing of the data structure for this case takes $\tilde{O}(n^{4/3}k^{2/3})$ time. Since we have to consider $O(nk)$ cones for the construction of $\Theta(S, \mathcal{K}, T)$, we can find all the edges in time $O(nk) \cdot \tilde{O}(n^{1/3}k^{-1/3}) = \tilde{O}(n^{4/3}k^{2/3})$ time. We summarize.

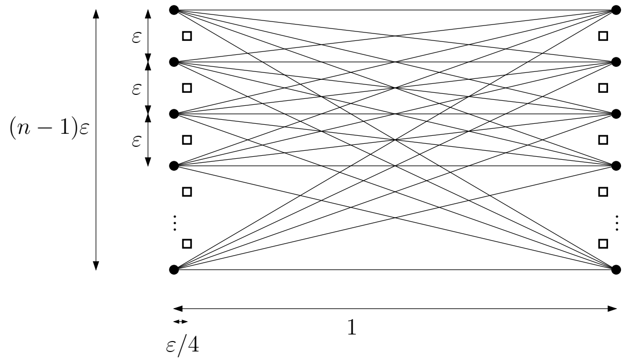

Figure 1: Lower bound for homotopic spanners when $k = \Theta(n)$. The dots are sites and the squares are obstacles.

Theorem 4 *If $k \le n$, we can construct $\Theta(S, \mathcal{K}, T)$ in $O(n^{4/3+\varepsilon}k^{2/3})$ time, for any fixed $\varepsilon > 0$.*

This result improves the $O(n^2 \log k)$ time construction given above whenever $k = O(n^{1-\varepsilon})$ for any fixed $\varepsilon > 0$.

5 Lower bounds

The homotopic spanner that we have constructed above has $O(nk)$ edges, where n is the number of points and k is the number of point-obstacles. In contrast, the standard spanners have only $O(n)$ edges. It is natural to wonder if $\Omega(nk)$ edges are indeed necessary for constructing a homotopic spanner. We have the following construction for the case $k = \Theta(n)$.

Lemma 5 *For any value of t, $1 < t < 3$, and any value of n, there is a set S of $O(n)$ points and a set $\mathcal{K}$ of $O(n)$ point obstacles such that any $\mathcal{K}$-homotopic t-spanner of S needs $\Omega(n^2)$ edges.*

Proof. Take $\varepsilon = \frac{3-t}{3n}$ and consider the configuration in Figure 1; we have sites

$$L = \{(0, j\varepsilon) \mid j \in [n]\}, \quad R = \{(1, j\varepsilon) \mid j \in [n]\},$$

and obstacles

$$\mathcal{K}_L = \{(\tfrac{\varepsilon}{4}, \tfrac{1}{2} + j\varepsilon) \mid j \in [n-1]\}, \tag{2}$$

$$\mathcal{K}_R = \{(1 - \tfrac{\varepsilon}{4}, \tfrac{1}{2} + j\varepsilon) \mid j \in [n-1]\}, \tag{3}$$

where we use the notation $[n] = \{1, \ldots, n\}$. This configuration has $2n$ points and $2n-2$ obstacles. It remains to argue that any homotopic t-spanner has $\Omega(n^2)$ edges.

The key observation is that any homotopic t-path from a site $l \in L$ to a site $r \in R$ has to use the segment

lr. Note that if a path α from l to r is homotopic to lr in $\mathbb{R}^2 \setminus (\mathcal{K}_L \cup \mathcal{K}_R)$ and only "crosses" from L to R once, then the segment lr has to be part of α.

Assume for the contrary that there is a $(\mathcal{K}_L \cup \mathcal{K}_R)$-homotopic t-spanner G of $L \cup R$ that does not contain the edge lr for some $l \in L, r \in R$. Let α be the t-path in G from l to r that is homotopic to lr in $\mathbb{R}^2 \setminus (\mathcal{K}_L \cup \mathcal{K}_R)$. Since α does not contain the segment lr, it has to "cross" from L to R (or vice versa) at least three times. We conclude that $|\alpha|_G \geq 3$. Using that $|lr| \leq 1 + n\varepsilon$ it follows that $|\alpha|_G / |lr| > t$, and G cannot be a $(\mathcal{K}_L \cup \mathcal{K}_R)$-homotopic t-spanner.

Since each segment from L to R has to be in a homotopic t-spanner, then any homotopic t-spanner has at least $n^2 = \Omega(n^2)$ edges. □

The above construction for the lower bound generalizes for general values of n, k as $\Omega(n + \min\{n^2, k^2\})$. Given n and k, if $k \geq n$ then we can take the construction of the previous result and add $k - n$ extra obstacles; we need $\Omega(n^2) = \Omega(n + \min\{n^2, k^2\})$ edges in any homotopic t-spanner. If $k < n$ then take the construction above with $n = k$ and add the extra $n - k$ sites far enough not to influence the construction; we need $\Omega(k^2)$ edges to make a t-spanner of the first part, and we need $n - k - 1$ edges to connect all the sites added afterwards, which adds to $\Omega(k^2 + n - k) = \Omega(k^2 + n) = \Omega(n + \min\{n^2, k^2\})$ because $k < n$. We summarize:

Theorem 6 *For any value t, $1 < t < 3$, and any values of n, k, there is a set S of $O(n)$ points and a set $\mathcal{K}$ of $O(k)$ point obstacles such that any $\mathcal{K}$-homotopic t-spanner of S needs $\Omega(n + \min\{n^2, k^2\})$ edges.*

6 Discussion

We have introduced the concept of homotopic spanners in the plane with point-obstacles. It is not clear how this concept generalizes to higher dimensions, where all paths are homotopic with respect to point-obstacles, neither how it generalizes to polyhedral obstacles, where a straight-line segment connecting two sites may intersect obstacles.

For n sites and k point-obstacles, we have presented a construction for homotopic spanners that uses $O(nk)$ edges. However, we can only provide an example showing that a homotopic spanner may need $\Omega(n + \min\{n^2, k^2\})$ edges. Our construction is based on Θ-graphs. The most natural alternative to consider is the Well Separated Pairs Decomposition of Callahan and Kosaraju [5, 4], but it does not seems easy to handle the homotopy classes induced by the obstacles in a better way than with Θ-graphs.

As with normal spanners, we can also be interested on homotopic spanners with additional properties, such as small maximum degree, small spanner diameter, small total weight, etc. As for the maximum degree D, the construction given above shows that in the worst case $D = \Omega(k)$, and so we cannot aim to get bounded degree. Adapting the ordered Θ-spanners of Bose, Gudmundsson, and Morin [3] to handle point-obstacles, it is possible to construct spanners with $O(nk)$ edges and maximum degree $O(k \log n)$. As for the spanner diameter, a randomized construction similar to Arya, Mount, and Smid [2], where we keep all the obstacles at each stage, will lead to randomized algorithms for constructing homotopic spanners with $O(nk)$ edges and $O(\log n)$ spanner diameter.

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