Third-Order Inference for the Weibull Distribution

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Abstract

Recent third-order likelihood analysis theory is applied to obtain improved confidence intervals for scalar component interest parameters of the Weibull model. This theory involves implicit but appropriate conditioning and marginalization. Both the censored and complete Weibull models are examined where primary interest is focussed on assessing the shape, scale or median parameter. The simulation results indicate the increased accuracy of this method relative to the conventional first-order likelihood ratio and Wald departure procedures. Applications to strike duration data are presented.

Keywords: Likelihood; Conditioning; Weibull distribution

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1 Introduction

The Weibull distribution has been widely used in the empirical modelling of economic models. Applications include the modelling of unemployment spells, strike durations, income distributions, the length of a firm's innovation period, and the size of research and development budgets. Depending on the particular problem, the variable under consideration may not be fully observed, requiring censoring procedures for estimation.

Confidence intervals for testing the parameters of the Weibull model are typically constructed using maximum likelihood estimates which are asymptotically normally distributed. These confidence intervals have first-order accuracy and are computationally simple and readily accessible from statistical packages. However, these intervals have been found to perform quite poorly in terms of coverage probabilities. As an alternative, first-order confidence intervals based on likelihood ratio tests have been examined, and while computationally more costly, they have been found to perform relatively well in certain cases. Considerable scope for improvement nonetheless remains.

In this paper, recent likelihood analysis theory is used to obtain highly accurate (third-order) inferences for scalar interest parameters of the Weibull distribution. The asymptotic theory developed by Fraser and Reid (1995) is based on the likelihood structure of the data and involves conditioning and marginalization for intrinsic measures of departure to produce third-order accurate tail approximations. The methodology provides an improvement to the results obtained using large sample normal theory and those based on the likelihood ratio test. Confidence intervals and simulations are provided for both the censored and complete cases that show the improved accuracy of this method. An example using the benchmark strike duration data provided in Kennan (1985) is examined.

The paper is structured as follows. Section 2 presents the Weibull distribution. The third-order likelihood theory is described and applied to both the censored and complete cases in Section 3. Applications and simulations are contained in Section 4. Section 5 concludes and gives directions for future research.

2 The Weibull Distribution

The Weibull probability density function for the random variable X is given by

$$f(x; \beta, \eta) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1} \exp \left\{ - \left(\frac{x}{\eta}\right)^{\beta} \right\} \quad x > 0, \quad (1)$$

where $\beta > 0$ and $\eta > 0$ are the shape and scale parameters, respectively. Writing

$$W = \mu + \sigma Y,$$

where Y is a random variable with standardized extreme value density,

$$f(y) = \exp\{y - e^y\} \quad -\infty < y < \infty,$$

then the density for W can be obtained as

$$f(w; \mu, \sigma) = \frac{1}{\sigma} \exp \left\{ \frac{w - \mu}{\sigma} - \exp \left(\frac{w - \mu}{\sigma} \right) \right\} \quad -\infty < w < \infty. \quad (2)$$

The distribution of W is known as the Smallest Extreme Value Distribution (SEV). If $W = \log X$, so that, $X = e^W$, then

$$f(x; \mu, \sigma) = \frac{1}{x} \frac{1}{\sigma} (xe^{-\mu})^{\frac{1}{\sigma}} \exp\{-(xe^{-\mu})^{\frac{1}{\sigma}}\}.$$

With $\sigma = 1/\beta$ and $\mu = \log \eta$, X is distributed as Weibull with shape parameter β and scale parameter η . Given this, for analytical and computational convenience, this paper works in the $W = \log X$ scale, the results, however, are reported directly for the Weibull parameters. In addition to the shape and scale parameters, there is typically an interest in inference regarding the α^{th} percentile. This quantity is calculated as

$$w_\alpha = \log\{\eta[-\log(1 - \alpha)]^{1/\beta}\}.$$

Third-order likelihood analysis results for model (2) are provided below assuming a pre-specified censoring point. The complete data model is easily handled as a special case.

3 Third-Order Likelihood Analysis

3.1 Censored Case

Let $w = (w_1, \dots, w_n)$ be a random sample from the SEV distribution given in (2). Here, r observations with ranked data are observed, $w_1 \leq w_2 \leq \dots \leq w_r \leq w_c$, where w_c represents the censoring point, that is, $(n - r)$ observations survive beyond w_c . See Lawless (1982) for details. The log-likelihood function of $\theta = (\mu, \sigma)$ is given by

$$l(\theta) = -r \log \sigma + \sum_{i=1}^r \left\{ \frac{w_i - \mu}{\sigma} - \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} - (n - r) \exp \left\{ \frac{w_c - \mu}{\sigma} \right\}, \quad (3)$$

with derivatives

$$\begin{aligned} l_\mu(\theta) &= \sum_{i=1}^r \left\{ -\frac{1}{\sigma} + \frac{1}{\sigma} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} + (n - r) \frac{1}{\sigma} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} \\ l_\sigma(\theta) &= -\frac{r}{\sigma} + \sum_{i=1}^r \left\{ -\frac{w_i - \mu}{\sigma^2} + \frac{w_i - \mu}{\sigma^2} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} + (n - r) \frac{w_c - \mu}{\sigma^2} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\}. \end{aligned}$$

The normal equations are solved to obtain the maximum likelihood estimate $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$. The information matrix denoted, $j_{\theta\theta}(\theta)$, is calculated as

$$j_{\theta\theta}(\theta) = \begin{bmatrix} -l_{\mu\mu}(\theta) & -l_{\mu\sigma}(\theta) \\ -l_{\mu\sigma}(\theta) & -l_{\sigma\sigma}(\theta) \end{bmatrix} = \begin{bmatrix} j_{\mu\mu}(\theta) & j_{\mu\sigma}(\theta) \\ j_{\mu\sigma}(\theta) & j_{\sigma\sigma}(\theta) \end{bmatrix},$$

where

$$\begin{aligned} l_{\mu\mu}(\theta) &= \sum_{i=1}^r \left\{ -\frac{1}{\sigma^2} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} - (n-r) \frac{1}{\sigma^2} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} \\ l_{\mu\sigma}(\theta) &= \sum_{i=1}^r \left\{ \frac{1}{\sigma^2} - \frac{1}{\sigma^2} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} - \frac{w_i - \mu}{\sigma^3} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} - \\ &\quad (n-r) \frac{1}{\sigma^2} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} - (n-r) \frac{w_c - \mu}{\sigma^3} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} \\ l_{\sigma\sigma}(\theta) &= \frac{r}{\sigma^2} + \sum_{i=1}^r \left\{ \frac{2(w_i - \mu)}{\sigma^3} - \frac{2(w_i - \mu)}{\sigma^3} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} - \frac{(w_i - \mu)^2}{\sigma^4} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} - \\ &\quad (n-r) \frac{2(w_c - \mu)}{\sigma^3} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} - (n-r) \frac{(w_c - \mu)^2}{\sigma^4} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\}. \end{aligned}$$

The observed information matrix is obtained as

$$j_{\theta\theta}(\hat{\theta}) = \begin{bmatrix} j_{\mu\mu}(\hat{\theta}) & j_{\mu\sigma}(\hat{\theta}) \\ j_{\mu\sigma}(\hat{\theta}) & j_{\sigma\sigma}(\hat{\theta}) \end{bmatrix},$$

where $j_{\mu\mu}(\hat{\theta}) = -l_{\mu\mu}(\hat{\theta})$, $j_{\mu\sigma}(\hat{\theta}) = -l_{\mu\sigma}(\hat{\theta})$, and $j_{\sigma\sigma}(\hat{\theta}) = -l_{\sigma\sigma}(\hat{\theta})$. Familiar first-order asymptotic methods can be used to test the null hypothesis of $\mu = \mu_0$. The standardized maximum likelihood departure and signed square root of the log likelihood ratio statistic are given respectively by

$$q = (\hat{\mu} - \mu_0) \{j^{\mu\mu}(\hat{\theta})\}^{-1/2} \quad (4)$$

$$r = \text{sgn}(\hat{\mu} - \mu_0) \{2[l(\hat{\theta}) - l(\hat{\theta}_{\mu_0})]\}^{1/2}, \quad (5)$$

where $\hat{\theta}_{\mu_0} = (\mu_0, \hat{\sigma}_{\mu_0})$ is the constrained maximum likelihood estimate of θ at μ_0 and $j^{\mu\mu}(\hat{\theta})$ is the appropriate entry from the estimated asymptotic variance matrix of $\hat{\theta}$

$$j^{\theta\theta}(\hat{\theta}) = \begin{bmatrix} j^{\mu\mu}(\hat{\theta}) & j^{\mu\sigma}(\hat{\theta}) \\ j^{\mu\sigma}(\hat{\theta}) & j^{\sigma\sigma}(\hat{\theta}) \end{bmatrix}.$$

The quantities q and r have the standard normal as their limiting distribution and thus tail probabilities for testing the null can be approximated by $\Phi(q)$ and $\Phi(r)$ respectively, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Recent third-order likelihood methods developed by Fraser and Reid (1995) are applied to obtain highly accurate inferences for any scalar component parameter of interest of the full-dimensional

model. The theory involves the construction of a particular maximum likelihood departure, Q , that is used together with the signed square root of the log-likelihood ratio (5) to produce third-order approximations to the left tail probabilities for testing scalar interest parameters. Explicit nuisance parameterization is not required for this method. Details are given in Fraser et al. (1999).

There are two fundamental steps involved in this likelihood analysis theory. The first involves a dimension reduction from the dimension of the variable to the dimension of the parameter. This dimension reduction is achieved by conditioning on an implicit ancillary statistic. In fact, only tangent directions, V , to this ancillary statistic are necessary. The resulting model is then approximated by a tangent exponential model with the canonical parameter involving the conditioning vectors as well as derivatives of the log-likelihood function with respect to the data. The second step involves a further dimension reduction through marginalization which reduces the dimension of the exponential model to the desired dimension of one for testing the null hypothesis of $\psi(\theta) = \psi_0$. A review of the necessary technical components resulting from this theory and the third-order tail approximations used in the analysis of the Weibull model are given below.

Maximization of the Lagrangean, $H(\theta)$, is used to obtain the constrained maximum likelihood estimate $\hat{\theta}_{\psi_0}$

$$\max_{\theta, \lambda} H(\theta) = l(\theta) + \lambda(\psi(\theta) - \psi_0). \quad (6)$$

Setting the first-order conditions

$$\begin{aligned} H_\mu(\theta) &= l_\mu(\theta) + \lambda\psi_\mu(\theta) \\ H_\sigma(\theta) &= l_\sigma(\theta) + \lambda\psi_\sigma(\theta) \\ H_\lambda(\theta) &= \psi(\theta) - \psi_0 \end{aligned}$$

equal to zero and solving the system of equations yields the constrained maximum likelihood solution for the parameter θ and the Lagrange multiplier λ . Note that $l_\mu(\theta) = \partial l(\theta)/\partial \mu$ and $\psi_\mu(\theta) = \partial \psi(\theta)/\partial \mu$. The tilted log-likelihood is defined as the quantity

$$\tilde{l}(\theta) = l(\theta) + \tilde{\lambda}(\psi(\theta) - \psi_0), \quad (7)$$

where $\tilde{\lambda}$, the Lagrange multiplier solution, is substituted into the Lagrangean, $H(\theta)$. A nominal information matrix is given by

$$\tilde{j}_{\theta\theta}(\hat{\theta}_{\psi_0}) = -\tilde{l}_{\theta\theta}(\hat{\theta}_{\psi_0}).$$

Fraser and Reid (1995) show that the tangent vectors, V , to the ancillary statistic can be obtained by using a pivotal quantity, k , and differentiating the data, w , with respect to the full parameter, θ ,

while holding the pivotal quantity fixed. The vectors are calculated as follows:

$$V = \frac{\partial w}{\partial \theta} \Big|_{\hat{\theta}} = \left\{ \frac{\partial k(w, \theta)}{\partial w} \right\}^{-1} \left\{ \frac{\partial k(w, \theta)}{\partial \theta} \right\} \Big|_{\hat{\theta}}. \quad (8)$$

These vectors are used to calculate the local canonical parameter

$$\varphi(\theta) = \frac{\partial l(\theta)}{\partial w} V. \quad (9)$$

This locally defined canonical parameter then becomes the relevant parameter space. Given this new parameterization, a marginalization procedure gives the recalibrated interest parameter

$$\chi(\theta) = \psi_{\theta}(\hat{\theta}_{\psi_0}) \varphi_{\theta}^{-1}(\hat{\theta}_{\psi_0}) \varphi(\theta), \quad (10)$$

where $\chi(\theta) = \psi(\theta)$ in the $\varphi(\theta)$ scale and $\varphi_{\theta}(\theta) = \partial \varphi(\theta) / \partial \theta$. Additionally, the estimate of the variance of $\chi(\theta)$ is given by

$$\hat{\sigma}_{\chi}^2 = \psi_{\theta}(\hat{\theta}_{\psi_0}) \tilde{J}^{\theta\theta}(\hat{\theta}_{\psi_0}) \psi'_{\theta}(\hat{\theta}_{\psi_0}).$$

The standardized maximum likelihood departure in the $\varphi(\theta)$ scale is

$$Q = \text{sgn}(\hat{\psi} - \psi_0) |\chi(\hat{\theta}) - \chi(\hat{\theta}_{\psi_0})| \left\{ \frac{|j_{(\theta\theta)}(\hat{\theta})|}{\hat{\sigma}_{\chi}^2 |\tilde{j}_{(\theta\theta)}(\hat{\theta}_{\psi_0})|} \right\}, \quad (11)$$

where the recalibrated determinants are calculated as

$$|j_{(\theta\theta)}(\hat{\theta})| = |j_{\theta\theta}(\hat{\theta})| |\varphi_{\theta}(\hat{\theta})|^{-2} \quad (12)$$

$$|\tilde{j}_{(\theta\theta)}(\hat{\theta}_{\psi_0})| = |\tilde{j}_{\theta\theta}(\hat{\theta}_{\psi_0})| |\varphi_{\theta}(\hat{\theta}_{\psi_0})|^{-2}. \quad (13)$$

The determinants in the above expressions involving the $\varphi_{\theta}(\hat{\theta})$ and $\varphi_{\theta}(\hat{\theta}_{\psi_0})$ terms can be thought of as Jacobian terms. Two asymptotically equivalent third-order approximations to tail probabilities are given by Lugannani and Rice (1980)

$$\Phi(R) + \phi(R) \left\{ \frac{1}{R} - \frac{1}{Q} \right\} \quad (14)$$

and Barndorff-Nielsen (1991)

$$\Phi(R^*), \quad (15)$$

where $\phi(\cdot)$ is the standard normal density function, R is the modified signed log-likelihood ratio statistic given by

$$R = \text{sgn}(\hat{\psi} - \psi_0) \left\{ 2[l(\hat{\theta}) - l(\hat{\theta}_{\psi_0})] \right\}^{1/2} \quad (16)$$

$$= \text{sgn}(\hat{\psi} - \psi_0) \left\{ 2[l(\hat{\theta}) - \tilde{l}(\hat{\theta}_{\psi_0})] \right\}^{1/2} \quad (17)$$

and R^* is the modified signed log-likelihood ratio statistic given by

$$R^* = R - R^{-1} \log \left(\frac{R}{Q} \right). \quad (18)$$

The equality from (16) to (17) follows directly from the optimality conditions.

The complexity of the third-order method lies in the calculation of the maximum likelihood departure measure, Q , given by (11). Once this quantity has been obtained, with a large portion of its components likewise required for the computation of the likelihood ratio and *mle*-based intervals, the complexity of the computations does not increase with sample size.

3.1.1 Interest Parameters

Three scalar component interest parameters for the censored model given in (3) are considered: the shape parameter, $\beta = 1/\sigma$, the scale parameter, $\eta = e^\mu$, and the α^{th} percentile, $x_\alpha = e^\mu [-\log(1 - \alpha)]^\sigma \Leftrightarrow w_\alpha = \log\{\eta [-\log(1 - \alpha)]^{1/\beta}\}$. Making use of the 1-1 relationships, the following three cases are considered.

Case 1: $\psi = \sigma$

The constrained maximum likelihood estimate for this case is $\psi_0 = \sigma_0$. The Lagrangean is

$$H(\theta) = l(\theta) + \lambda(\sigma - \sigma_0)$$

and differentiation yields

$$\begin{aligned} H_\mu(\theta) &= l_\mu(\theta) \\ H_\sigma(\theta) &= l_\sigma(\theta) + \lambda \\ H_\lambda(\theta) &= \sigma - \sigma_0. \end{aligned}$$

The constrained maximum likelihood estimates satisfy

$$\begin{aligned} l_\mu(\tilde{\theta}) &= 0 \\ \tilde{\lambda} &= -l_\sigma(\tilde{\theta}) \\ \tilde{\sigma} &= \sigma_0. \end{aligned}$$

The tilted likelihood then is

$$\tilde{l}(\theta) = l(\theta) + \tilde{\lambda}(\sigma - \sigma_0),$$

with

$$\begin{aligned}
\tilde{l}_\mu(\theta) &= l_\mu(\theta) \\
\tilde{l}_\sigma(\theta) &= l_\sigma(\theta) + \tilde{\lambda} \\
\tilde{l}_{\mu\mu}(\theta) &= l_{\mu\mu}(\theta) \\
\tilde{l}_{\mu\sigma}(\theta) &= l_{\mu\sigma}(\theta) \\
\tilde{l}_{\sigma\sigma}(\theta) &= l_{\sigma\sigma}(\theta).
\end{aligned}$$

Notice the above implies

$$\tilde{j}_{\theta\theta}(\tilde{\theta}) = j_{\theta\theta}(\tilde{\theta}).$$

The ancillary direction vectors, V , are

$$V = \begin{bmatrix} 1 & \frac{w_1 - \hat{\mu}}{\hat{\sigma}} \\ \vdots & \vdots \\ 1 & \frac{w_r - \hat{\mu}}{\hat{\sigma}} \\ 1 & \frac{w_c - \hat{\mu}}{\hat{\sigma}} \end{bmatrix}.$$

These tangent directions are used to obtain the canonical parameter

$$\varphi(\theta) = [\varphi_1(\theta) \quad \varphi_2(\theta)],$$

where

$$\begin{aligned}
\varphi_1(\theta) &= \sum_{i=1}^r \left\{ \frac{1}{\sigma} - \frac{1}{\sigma} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} - (n-r) \frac{1}{\sigma} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} = -l_\mu(\theta) \\
\varphi_2(\theta) &= \sum_{i=1}^r \left\{ \left\{ \frac{1}{\sigma} - \frac{1}{\sigma} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} \frac{w_i - \hat{\mu}}{\hat{\sigma}} \right\} - (n-r) \frac{1}{\sigma} \exp \left\{ \frac{w_c - \mu}{\sigma} \right\} \frac{w_c - \hat{\mu}}{\hat{\sigma}}.
\end{aligned}$$

The parameter $\chi(\theta)$ is then obtained as

$$\chi(\theta) = \psi_\theta(\tilde{\theta}) \varphi_\theta^{-1}(\tilde{\theta}) \varphi(\theta),$$

with $\psi_\theta(\theta) = [\psi_\mu(\theta) \quad \psi_\sigma(\theta)] = [0 \quad 1]$ and $\varphi_\theta^{-1}(\theta)$ is calculated from

$$\varphi_\theta(\theta) = \begin{bmatrix} \varphi_{1\mu}(\theta) & \varphi_{1\sigma}(\theta) \\ \varphi_{2\mu}(\theta) & \varphi_{2\sigma}(\theta) \end{bmatrix},$$

where

$$\varphi_{1\mu}(\theta) = -l_{\mu\mu}(\theta) = j_{\mu\mu}(\theta)$$

$$\begin{aligned}
\varphi_{1\sigma}(\theta) &= -l_{\mu\sigma}(\theta) = j_{\mu\sigma}(\theta) \\
\varphi_{2\mu}(\theta) &= \sum_{i=1}^r \frac{1}{\sigma^2} \exp\left\{\frac{w_i - \mu}{\sigma}\right\} \frac{w_i - \hat{\mu}}{\hat{\sigma}} + (n-r) \frac{1}{\sigma^2} \exp\left\{\frac{w_c - \mu}{\sigma}\right\} \frac{w_c - \hat{\mu}}{\hat{\sigma}} \\
\varphi_{2\sigma}(\theta) &= \sum_{i=1}^r \left[-\frac{1}{\sigma^2} + \frac{1}{\sigma^2} \exp\left\{\frac{w_i - \mu}{\sigma}\right\} + \frac{w_i - \mu}{\sigma^3} \exp\left\{\frac{w_i - \mu}{\sigma}\right\} \right] \frac{w_i - \hat{\mu}}{\hat{\sigma}} + \\
&\quad (n-r) \frac{1}{\sigma^2} \exp\left\{\frac{w_c - \mu}{\sigma}\right\} \frac{w_c - \hat{\mu}}{\hat{\sigma}} + (n-r) \frac{w_c - \mu}{\sigma^3} \exp\left\{\frac{w_c - \mu}{\sigma}\right\} \frac{w_i - \hat{\mu}}{\hat{\sigma}}.
\end{aligned}$$

As before

$$\hat{\sigma}_\chi^2 = \psi_\theta(\tilde{\theta}) \tilde{j}^{\theta\theta}(\tilde{\theta}) \psi_\theta(\tilde{\theta}),$$

where $\tilde{j}^{\theta\theta}(\tilde{\theta})$ is the estimated asymptotic covariance matrix of $\tilde{\theta}$. The signed likelihood ratio departure is

$$R = \text{sgn}(\hat{\psi} - \psi_0) \{2[l(\hat{\theta}) - l(\tilde{\theta})]\}^{1/2}$$

and the maximum likelihood departure is

$$Q = \text{sgn}(\hat{\psi} - \psi_0) |\chi(\hat{\theta}) - \chi(\tilde{\theta})| \left\{ \frac{|\hat{j}_{\theta\theta}(\hat{\theta})| |\varphi_\theta(\hat{\theta})|^{-2}}{\hat{\sigma}_\chi^2 |\tilde{j}_{\theta\theta}(\tilde{\theta})| |\varphi_\theta(\tilde{\theta})|^{-2}} \right\}^{1/2},$$

with the recalibrated determinants calculated as in (12) and (13). Tail probabilities can be calculated using either the Lugannani and Rice or Barndorff-Nielsen formulas.

Case 2: $\psi = \mu$

The Lagrangean for this case is

$$H(\theta) = l(\theta) + \lambda(\mu - \mu_0).$$

Differentiation with respect to the parameters and the Lagrange multiplier leads to

$$H_\mu(\theta) = l_\mu(\theta) + \lambda$$

$$H_\sigma(\theta) = l_\sigma(\theta)$$

$$H_\lambda(\theta) = \mu - \mu_0.$$

The constrained maximum likelihood estimates satisfy

$$\tilde{\lambda} = -l_\mu(\tilde{\theta})$$

$$l_\sigma(\tilde{\theta}) = 0$$

$$\tilde{\mu} = \mu_0.$$

The tilted log-likelihood is

$$\tilde{l} = l(\theta) + \tilde{\lambda}(\mu - \mu_0),$$

with

$$\begin{aligned}\tilde{l}_\mu(\theta) &= l_\mu(\theta) + \tilde{\lambda} \\ \tilde{l}_\sigma(\theta) &= l_\sigma(\theta) \\ \tilde{l}_{\mu\mu}(\theta) &= l_{\mu\mu}(\theta) \\ \tilde{l}_{\mu\sigma}(\theta) &= l_{\mu\sigma}(\theta) \\ \tilde{l}_{\sigma\sigma}(\theta) &= l_{\sigma\sigma}(\theta).\end{aligned}$$

The rest of the quantities are the same as in *Case 1* with $\psi_\theta(\theta) = [1 \ 0]$.

Case 3: $\psi = w_\alpha$

The interest parameter ψ can be expressed as $\psi = \mu + \sigma A$ where $A = \log[-\log(1 - \alpha)]$ and w_α is the α^{th} percentile. The Lagrangean for this case is

$$H(\theta) = l(\theta) + \lambda(\mu + \sigma A - \psi_0).$$

Differentiation leads to

$$\begin{aligned}H_\mu(\theta) &= l_\mu(\theta) + \lambda \\ H_\sigma(\theta) &= l_\sigma(\theta) + \lambda A \\ H_\lambda(\theta) &= \mu + \sigma A - \psi_0.\end{aligned}$$

The constrained maximum likelihood estimates satisfy

$$\begin{aligned}\tilde{\lambda} &= -l_\mu(\tilde{\theta}) \\ l_\sigma(\tilde{\theta}) &= A l_\mu(\tilde{\theta}) \\ \tilde{\mu} &= \psi_0 - A\tilde{\sigma}.\end{aligned}$$

The tilted log-likelihood is

$$\tilde{l} = l(\theta) + \tilde{\lambda}(\mu + \sigma A - \psi_0),$$

with

$$\tilde{l}_\mu(\theta) = l_\mu(\theta) + \tilde{\lambda}$$

$$\begin{aligned}
\tilde{l}_\sigma(\theta) &= l_\sigma(\theta) + \tilde{\lambda}A \\
\tilde{l}_{\mu\mu}(\theta) &= l_{\mu\mu}(\theta) \\
\tilde{l}_{\mu\sigma}(\theta) &= l_{\mu\sigma}(\theta) \\
\tilde{l}_{\sigma\sigma}(\theta) &= l_{\sigma\sigma}(\theta).
\end{aligned}$$

The rest of the quantities are likewise the same as in *Case 1* with $\psi_\theta(\theta) = [1 \quad A]$.

3.2 Complete Case

In the absence of censoring, the special complete data case ($r = n$) results. For the random sample $w = (w_1, \dots, w_n)$ from the SEV distribution, the log-likelihood function for $\theta = (\mu, \sigma)$ is adjusted as follows

$$l(\theta) = -n \log \sigma + \sum_{i=1}^n \left\{ \frac{w_i - \mu}{\sigma} - \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\}, \quad (19)$$

with derivatives

$$\begin{aligned}
l_\mu(\theta) &= \sum \left\{ -\frac{1}{\sigma} + \frac{1}{\sigma} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} \\
l_\sigma(\theta) &= -\frac{n}{\sigma} + \sum \left\{ -\frac{w_i - \mu}{\sigma^2} + \frac{w_i - \mu}{\sigma^2} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\}.
\end{aligned}$$

The normal equations are solved to obtain the maximum likelihood estimate $\hat{\theta} = (\hat{\mu}, \hat{\sigma})$. In particular, $\hat{\theta}$ simultaneously satisfies

$$\begin{aligned}
\sum \exp \left\{ \frac{w_i - \hat{\mu}}{\hat{\sigma}} \right\} &= n \\
\sum \left\{ \frac{w_i - \hat{\mu}}{\hat{\sigma}} \exp \left\{ \frac{w_i - \hat{\mu}}{\hat{\sigma}} \right\} - \frac{w_i - \hat{\mu}}{\hat{\sigma}} \right\} &= n.
\end{aligned}$$

The information matrix denoted, $j_{\theta\theta}(\theta)$, is calculated as

$$j_{\theta\theta}(\theta) = \begin{bmatrix} -l_{\mu\mu}(\theta) & -l_{\mu\sigma}(\theta) \\ -l_{\mu\sigma}(\theta) & -l_{\sigma\sigma}(\theta) \end{bmatrix} = \begin{bmatrix} j_{\mu\mu}(\theta) & j_{\mu\sigma}(\theta) \\ j_{\mu\sigma}(\theta) & j_{\sigma\sigma}(\theta) \end{bmatrix},$$

where

$$\begin{aligned}
l_{\mu\mu}(\theta) &= \sum \left\{ -\frac{1}{\sigma^2} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} \\
l_{\mu\sigma}(\theta) &= \sum \left\{ \frac{1}{\sigma^2} - \frac{1}{\sigma^2} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} - \frac{w_i - \mu}{\sigma^3} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\} \\
l_{\sigma\sigma}(\theta) &= \frac{n}{\sigma^2} + \sum \left\{ \frac{2(w_i - \mu)}{\sigma^3} - \frac{2(w_i - \mu)}{\sigma^3} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} - \frac{(w_i - \mu)^2}{\sigma^4} \exp \left\{ \frac{w_i - \mu}{\sigma} \right\} \right\}.
\end{aligned}$$

The observed information matrix is obtained as

$$j_{\theta\theta}(\hat{\theta}) = \begin{bmatrix} j_{\mu\mu}(\hat{\theta}) & j_{\mu\sigma}(\hat{\theta}) \\ j_{\mu\sigma}(\hat{\theta}) & j_{\sigma\sigma}(\hat{\theta}) \end{bmatrix},$$

where

$$\begin{aligned} j_{\mu\mu}(\hat{\theta}) &= \frac{n}{\hat{\sigma}^2} \\ j_{\mu\sigma}(\hat{\theta}) &= \frac{1}{\hat{\sigma}^2} \sum \frac{w_i - \hat{\mu}}{\hat{\sigma}} \exp \left\{ \frac{w_i - \hat{\mu}}{\hat{\sigma}} \right\} \\ j_{\sigma\sigma}(\hat{\theta}) &= \frac{n}{\hat{\sigma}^2} + \frac{1}{\hat{\sigma}^2} \sum \left(\frac{w_i - \hat{\mu}}{\hat{\sigma}} \right)^2 \exp \left\{ \frac{w_i - \hat{\mu}}{\hat{\sigma}} \right\}. \end{aligned}$$

The ancillary directions are

$$V = \begin{bmatrix} 1 & \frac{w - \hat{\mu}}{\hat{\sigma}} \end{bmatrix},$$

where the 1 represents the n -dimensional unit vector and the $(w - \hat{\mu})/\hat{\sigma}$ is the n -dimensional standardized residual vector

$$\left[\frac{w_1 - \hat{\mu}}{\hat{\sigma}}, \dots, \frac{w_n - \hat{\mu}}{\hat{\sigma}} \right].$$

Given these quantities the third-order computations are conceptually similar to those for the uncensored case, and as such, the details are not provided here.

3.2.1 Interest Parameters

The same three scalar component interest parameters for the complete Weibull model are considered: the shape parameter, $\beta = 1/\sigma$, the scale parameter, $\eta = e^\mu$, and the α^{th} percentile, $x_\alpha = e^\mu [-\log(1 - \alpha)]^\sigma \Leftrightarrow w_\alpha = \log\{\eta[-\log(1 - \alpha)]^{1/\beta}\}$.

Case 1: $\psi = \sigma$

The constrained maximum likelihood estimate for this case is then $\psi_0 = \sigma_0$. The Lagrangean is

$$H(\theta) = l(\theta) + \lambda(\sigma - \sigma_0)$$

and differentiation yields

$$\begin{aligned} H_\mu(\theta) &= l_\mu(\theta) \\ H_\sigma(\theta) &= l_\sigma(\theta) + \lambda \\ H_\lambda(\theta) &= \sigma - \sigma_0. \end{aligned}$$

The constrained maximum likelihood estimates are solved as

$$\begin{aligned} \tilde{\sigma} &= \sigma_0 \\ \tilde{\mu} &= \tilde{\sigma} \log \left\{ \frac{1}{n} \sum \exp \left\{ \frac{w_i}{\tilde{\sigma}} \right\} \right\} \\ \tilde{\lambda} &= \frac{n}{\tilde{\sigma}} + \sum \left\{ \frac{w_i - \tilde{\mu}}{\tilde{\sigma}^2} - \frac{w_i - \tilde{\mu}}{\tilde{\sigma}^2} \exp \left\{ \frac{w_i - \tilde{\mu}}{\tilde{\sigma}} \right\} \right\}. \end{aligned}$$

The tilted likelihood then is

$$\tilde{l}(\theta) = l(\theta) + \tilde{\lambda}(\sigma - \sigma_0),$$

with

$$\begin{aligned}\tilde{l}_\mu(\theta) &= l_\mu(\theta) \\ \tilde{l}_\sigma(\theta) &= l_\sigma(\theta) + \tilde{\lambda} \\ \tilde{l}_{\mu\mu}(\theta) &= l_{\mu\mu}(\theta) \\ \tilde{l}_{\mu\sigma}(\theta) &= l_{\mu\sigma}(\theta) \\ \tilde{l}_{\sigma\sigma}(\theta) &= l_{\sigma\sigma}(\theta).\end{aligned}$$

Notice the above implies

$$\tilde{j}_{\theta\theta}(\tilde{\theta}) = j_{\theta\theta}(\tilde{\theta}).$$

The ancillary direction vectors, V , are

$$V = \begin{bmatrix} 1 & \frac{w - \hat{\mu}}{\hat{\sigma}} \end{bmatrix},$$

where the 1 represents the n -dimensional unit vector and the $w - \hat{\mu}/\hat{\sigma}$ is the n -dimensional standardized residual vector. These tangent directions are used to obtain the canonical parameter

$$\varphi(\theta) = [\varphi_1(\theta) \quad \varphi_2(\theta)],$$

where

$$\begin{aligned}\varphi_1(\theta) &= \sum \left\{ \frac{1}{\sigma} - \frac{1}{\sigma} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} \right\} \\ \varphi_2(\theta) &= \sum \left\{ \left\{ \frac{1}{\sigma} - \frac{1}{\sigma} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} \right\} \frac{w_j - \hat{\mu}}{\hat{\sigma}} \right\}.\end{aligned}$$

The parameter $\chi(\theta)$ is then obtained as

$$\chi(\theta) = \psi_\theta(\tilde{\theta}) \varphi_\theta^{-1}(\tilde{\theta}) \varphi(\theta)$$

with $\psi_\theta(\theta) = [\psi_\mu(\theta) \quad \psi_\sigma(\theta)] = [0 \quad 1]$ and $\varphi_\theta^{-1}(\theta)$ is calculated from

$$\varphi_\theta(\theta) = \begin{bmatrix} \varphi_{1\mu}(\theta) & \varphi_{1\sigma}(\theta) \\ \varphi_{2\mu}(\theta) & \varphi_{2\sigma}(\theta) \end{bmatrix},$$

where

$$\begin{aligned}
\varphi_{1\mu}(\theta) &= \sum \frac{1}{\sigma^2} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} \\
\varphi_{1\sigma}(\theta) &= \sum \left[-\frac{1}{\sigma^2} + \frac{1}{\sigma^2} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} + \frac{w_j - \mu}{\sigma^3} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} \right] \\
\varphi_{2\mu}(\theta) &= \sum \frac{1}{\sigma^2} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} \frac{w_j - \hat{\mu}}{\hat{\sigma}} \\
\varphi_{2\sigma}(\theta) &= \sum \left[-\frac{1}{\sigma^2} + \frac{1}{\sigma^2} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} + \frac{w_j - \mu}{\sigma^3} \exp \left\{ \frac{w_j - \mu}{\sigma} \right\} \right] \frac{w_j - \hat{\mu}}{\hat{\sigma}}.
\end{aligned}$$

As before

$$\hat{\sigma}_\chi^2 = \psi_\theta(\tilde{\theta}) \tilde{j}^{\theta\theta}(\tilde{\theta}) \psi_\theta(\tilde{\theta}),$$

where $\tilde{j}^{\theta\theta}(\tilde{\theta})$ is the estimated asymptotic covariance matrix of $\tilde{\theta}$. The signed likelihood ratio departure is

$$R = \text{sgn}(\hat{\psi} - \psi_0) \{2[l(\hat{\theta}) - l(\tilde{\theta})]\}^{1/2}$$

and the maximum likelihood departure is

$$Q = \text{sgn}(\hat{\psi} - \psi_0) |\chi(\hat{\theta}) - \chi(\tilde{\theta})| \left\{ \frac{|\hat{j}_{\theta\theta}(\hat{\theta})| |\varphi_\theta(\hat{\theta})|^{-2}}{\hat{\sigma}_\chi^2 |\tilde{j}_{\theta\theta}(\tilde{\theta})| |\varphi_\theta(\tilde{\theta})|^{-2}} \right\}^{1/2}.$$

Tail probabilities can be calculated using either the Lugannani and Rice or Barndorff-Nielsen formulas.

Case 2: $\psi = \mu$

The Lagrangean for this case is

$$H(\theta) = l(\theta) + \lambda(\mu - \mu_0).$$

Differentiation with respect to the parameters and the Lagrange multiplier leads to

$$H_\mu(\theta) = l_\mu(\theta) + \lambda$$

$$H_\sigma(\theta) = l_\sigma(\theta)$$

$$H_\lambda(\theta) = \mu - \mu_0.$$

The constrained maximum likelihood estimates are solved as

$$\tilde{\mu} = \mu_0$$

$$\tilde{\lambda} = -l_\mu(\tilde{\theta}),$$

where $\tilde{\sigma}$ satisfies

$$-\frac{n}{\tilde{\sigma}} - \sum \left\{ \frac{w_i - \tilde{\mu}}{\tilde{\sigma}^2} - \frac{w_i - \tilde{\mu}}{\tilde{\sigma}^2} \exp \left\{ \frac{w_i - \tilde{\mu}}{\tilde{\sigma}} \right\} \right\} = 0.$$

The tilted log-likelihood is

$$\tilde{l} = l(\theta) + \tilde{\lambda}(\mu - \mu_0),$$

with

$$\begin{aligned} \tilde{l}_\mu(\theta) &= l_\mu(\theta) + \tilde{\lambda} \\ \tilde{l}_\sigma(\theta) &= l_\sigma(\theta) \\ \tilde{l}_{\mu\mu}(\theta) &= l_{\mu\mu}(\theta) \\ \tilde{l}_{\mu\sigma}(\theta) &= l_{\mu\sigma}(\theta) \\ \tilde{l}_{\sigma\sigma}(\theta) &= l_{\sigma\sigma}(\theta). \end{aligned}$$

The rest of the quantities are the same as in *Case 1* with $\psi_\theta(\theta) = [1 \ 0]$.

Case 3: $\psi = w_\alpha$

The interest parameter ψ can be expressed as $\psi = \mu + \sigma A$ where $A = \log[-\log(1 - \alpha)]$ and w_α is the α^{th} percentile. The Lagrangean for this case is

$$H(\theta) = l(\theta) + \lambda(\mu + \sigma A - \psi_0).$$

Differentiation leads to

$$\begin{aligned} H_\mu(\theta) &= l_\mu(\theta) + \lambda \\ H_\sigma(\theta) &= l_\sigma(\theta) + \lambda A \\ H_\lambda(\theta) &= \mu + \sigma A - \psi_0. \end{aligned}$$

The constrained maximum likelihood estimates are solved as

$$\begin{aligned} \tilde{\mu} &= \psi_0 - A\tilde{\sigma} \\ \tilde{\lambda} &= -l_\mu(\tilde{\theta}), \end{aligned}$$

where $\tilde{\sigma}$ satisfies

$$-\frac{n}{\tilde{\sigma}} - \sum \left\{ \frac{w_i - \psi_0}{\tilde{\sigma}^2} - \frac{w_i - \psi_0}{\tilde{\sigma}^2} \exp \left\{ \frac{w_i - \psi_0}{\tilde{\sigma}} + A \right\} \right\} = 0.$$

The tilted log-likelihood is

$$\tilde{l} = l(\theta) + \tilde{\lambda}(\mu + \sigma A - \psi_0),$$

with

$$\begin{aligned}\tilde{l}_\mu(\theta) &= l_\mu(\theta) + \tilde{\lambda} \\ \tilde{l}_\sigma(\theta) &= l_\sigma(\theta) + \tilde{\lambda}A \\ \tilde{l}_{\mu\mu}(\theta) &= l_{\mu\mu}(\theta) \\ \tilde{l}_{\mu\sigma}(\theta) &= l_{\mu\sigma}(\theta) \\ \tilde{l}_{\sigma\sigma}(\theta) &= l_{\sigma\sigma}(\theta).\end{aligned}$$

The rest of the quantities are likewise the same as in *Case 1* with $\psi_\theta(\theta) = [1 \ A]$.

As the maximum likelihood departure, Q , is explicitly provided in this paper, obtaining the third-order intervals becomes a programming matter. In the next section, results are provided for the 90%, 95%, and 99% first- and third-order confidence intervals for Kennan's (1985) strike duration data. In addition, simulation results are recorded for the interest parameters discussed in the above three cases, specifically for the shape, scale, and median parameters. Results are provided for both the censored and complete data sets.

4 Results

For the empirical illustration, a sample of the strike duration data given in Kennan (1985) is used. The data used by Kennan is from the Bureau of Labor Statistics and reflects strike durations, in days, for the U.S. manufacturing industries for the 1968-1976 period. The complete and censored data sets considered are as in Kiefer (1988); the complete sample is restricted to strikes beginning in June of each year for a total number of 62 observations, the censored sample is formed by censoring durations greater than or equal to 80 days, resulting in 12 censored observations or approximately a 20% censored sample.

The first-order results were computed using Limdep. All third-order computations were done in Splus. In the tables below "lr" refers to the first-order likelihood ratio approach, "mle" refers to the first-order maximum likelihood approach, "LR" and "BN" refer to the third-order Lugannani and Rice and Barndorff-Nielsen approaches, respectively.

4.1 Censored Case

4.1.1 Confidence Intervals

The resulting 90%, 95%, and 99% confidence intervals are provided in Tables 1 to 3 below for the shape, scale, and median parameters. For the shape parameter, the confidence intervals resulting from using either the likelihood ratio or the *mle* are wider than the confidence intervals obtained from using the third-order methods. For the scale parameter, the first-order confidence intervals differ from the third-order ones predominantly on the upper tail, with the *mle*-based intervals noticeably discordant. Examination of the confidence intervals for the median reveals that the intervals based on the likelihood ratio are close to those based on the third-order methods, while the *mle*-based intervals are consistently very different.

Table 1: Confidence Intervals for Shape

Method	90% CI		95% CI		99% CI	
	lower	upper	lower	upper	lower	upper
lr	0.7254	1.0762	0.6959	1.1141	0.6403	1.1902
mle	0.6874	1.0943	0.6484	1.1333	0.5722	1.2095
LR	0.7085	1.0567	0.6793	1.0943	0.6243	1.1700
BN	0.7086	1.0568	0.6794	1.0944	0.6244	1.1701

Table 2: Confidence Intervals for Scale

Method	90% CI		95% CI		99% CI	
	lower	upper	lower	upper	lower	upper
lr	32.2662	54.0480	30.6670	58.2784	27.7084	65.5739
mle	32.4231	54.0187	30.8765	56.7244	28.0621	62.4134
LR	32.4226	55.9020	30.7924	59.2698	27.7728	66.9105
BN	32.4226	55.9010	30.7924	59.2684	27.7728	66.9083

Table 3: Confidence Intervals for the Median

Method	90% CI		95% CI		99% CI	
	lower	upper	lower	upper	lower	upper
lr	20.7006	36.5278	19.4879	38.5315	17.2222	42.8831
mle	20.6559	34.8133	19.3004	36.1688	16.6497	38.8195
LR	20.6409	36.8054	19.4097	38.8670	17.1119	43.3584
BN	20.6409	36.8053	19.4097	38.8668	17.1119	43.3581

4.1.2 Simulation

To examine the accuracy of the censored case, a simulation study was conducted with 10,000 simulated samples. The shape and scale parameters were set equal to 1. Two sample sizes (n) with various

censoring percentages were considered. Results are reported for the following cases:

Simulation Settings: Shape=1, Scale=1

Case	n	Censored %
1	10	10
2	10	20
3	10	30
4	20	10
5	20	20
6	20	30

For each simulated sample, only the 95% confidence intervals are reported. Tables 4 to 6 record the central coverage, coverage error, upper error, lower error, and average bias. The theoretical values for these quantities are 0.95, 0.05, 0.025, 0.025 and 0, respectively. Central coverage is the proportion of intervals for which the true parameter value is within 95% confidence. Coverage error is defined as the proportion of intervals for which the true parameter value is outside 95% confidence. Upper and lower errors correspond to the proportion of intervals that fall outside the upper and lower confidence limits. Average bias is defined as $(|\text{Upper Error} - \alpha/2| + |\text{Lower Error} - \alpha/2|)/2$, where $1 - \alpha$ is the nominal coverage level.

From these tables it is evident that the confidence intervals based on third-order likelihood theory have excellent coverage properties. The conventional intervals based on the maximum likelihood estimate perform very poorly with highly asymmetric errors. For the shape parameter, holding the sample size constant, the average bias of the *mle*-based intervals increases as the censoring percentage increases. The likelihood ratio-based intervals improve on the *mle*-based ones in terms of central coverage, however, they nonetheless yield particularly asymmetric errors for the scale and median parameters. The third-order intervals possess the smallest average bias for all cases and parameters considered.

An additional simulation that matches the parameters of the strike duration data example more closely was considered. The simulation was based on 10,000 samples of 60 observations with a shape parameter set equal to 0.9 and a scale parameter set equal to 43. The censoring percentage was set equal to 20. The results are recorded in Table 7. As can be seen from the table, for a sample size of 60, the third-order method outperforms the other methods, especially for the shape parameter, in terms of average bias.

Table 4: Simulation for Shape - Nominal: 95%

Case	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
1	lr	0.9502	0.0498	0.0264	0.0234	0.0015
	mle	0.9318	0.0682	0.0004	0.0678	0.0337
	LR	0.9511	0.0489	0.0253	0.0236	0.0009
	BN	0.9504	0.0496	0.0244	0.0252	0.0004
2	lr	0.9510	0.0490	0.0242	0.0248	0.0005
	mle	0.9195	0.0805	0.0001	0.0804	0.0402
	LR	0.9498	0.0502	0.0256	0.0246	0.0005
	BN	0.9499	0.0501	0.0248	0.0253	0.0003
3	lr	0.9483	0.0517	0.0261	0.0256	0.0009
	mle	0.9095	0.0905	0.0000	0.0905	0.0453
	LR	0.9507	0.0493	0.0241	0.0252	0.0006
	BN	0.9504	0.0496	0.0245	0.0251	0.0003
4	lr	0.9515	0.0485	0.0268	0.0217	0.0026
	mle	0.9398	0.0602	0.0036	0.0566	0.0265
	LR	0.9498	0.0502	0.0258	0.0244	0.0007
	BN	0.9506	0.0494	0.0248	0.0246	0.0003
5	lr	0.9507	0.0493	0.0257	0.0236	0.0011
	mle	0.9322	0.0678	0.0035	0.0643	0.0304
	LR	0.9501	0.0499	0.0244	0.0255	0.0005
	BN	0.9499	0.0501	0.0242	0.0259	0.0009
6	lr	0.9479	0.0521	0.0254	0.0267	0.0011
	mle	0.9298	0.0702	0.0019	0.0683	0.0332
	LR	0.9492	0.0508	0.0247	0.0261	0.0007
	BN	0.9498	0.0502	0.0249	0.0253	0.0002

Table 5: Simulation for Scale - Nominal: 95%

Case	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
1	lr	0.9522	0.0478	0.0311	0.0167	0.0072
	mle	0.9291	0.0709	0.0455	0.0254	0.0105
	LR	0.9485	0.0515	0.0279	0.0236	0.0022
	BN	0.9497	0.0503	0.0263	0.0240	0.0012
2	lr	0.9522	0.0478	0.0323	0.0155	0.0084
	mle	0.9250	0.0750	0.0435	0.0315	0.0125
	LR	0.9494	0.0506	0.0263	0.0243	0.0010
	BN	0.9491	0.0509	0.0258	0.0251	0.0004
3	lr	0.9487	0.0513	0.0352	0.0161	0.0096
	mle	0.9227	0.0773	0.0368	0.0405	0.0137
	LR	0.9500	0.0500	0.0259	0.0241	0.0009
	BN	0.9492	0.0508	0.0256	0.0252	0.0004
4	lr	0.9534	0.0466	0.0290	0.0176	0.0057
	mle	0.9388	0.0612	0.0369	0.0243	0.0063
	LR	0.9497	0.0503	0.0256	0.0247	0.0005
	BN	0.9498	0.0502	0.0253	0.0249	0.0002
5	lr	0.9528	0.0472	0.0313	0.0159	0.0077
	mle	0.9379	0.0621	0.0344	0.0277	0.0061
	LR	0.9503	0.0497	0.0240	0.0257	0.0009
	BN	0.9505	0.0495	0.0244	0.0251	0.0004
6	lr	0.9503	0.0497	0.0330	0.0167	0.0082
	mle	0.9348	0.0652	0.0294	0.0358	0.0076
	LR	0.9488	0.0512	0.0251	0.0261	0.0006
	BN	0.9509	0.0491	0.0236	0.0255	0.0009

Table 6: Simulation for Median - Nominal: 95%

Case	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
1	lr	0.9473	0.0527	0.0264	0.0234	0.0015
	mle	0.9278	0.0722	0.0556	0.0166	0.0195
	LR	0.9502	0.0498	0.0259	0.0239	0.0010
	BN	0.9515	0.0485	0.0241	0.0244	0.0008
2	lr	0.9565	0.0435	0.0297	0.0138	0.0080
	mle	0.9215	0.0785	0.0585	0.0200	0.0193
	LR	0.9502	0.0498	0.0257	0.0241	0.0008
	BN	0.9502	0.0498	0.0251	0.0247	0.0002
3	lr	0.9549	0.0451	0.0314	0.0137	0.0089
	mle	0.9170	0.0830	0.0576	0.0254	0.0165
	LR	0.9521	0.0479	0.0252	0.0227	0.0013
	BN	0.9528	0.0472	0.0231	0.0241	0.0014
4	lr	0.9542	0.0458	0.0278	0.0180	0.0049
	mle	0.9398	0.0602	0.0446	0.0156	0.0145
	LR	0.9499	0.0501	0.0262	0.0239	0.0012
	BN	0.9510	0.0490	0.0244	0.0246	0.0005
5	lr	0.9546	0.0454	0.0290	0.0164	0.0063
	mle	0.9356	0.0644	0.0470	0.0174	0.0148
	LR	0.9502	0.0498	0.0257	0.0241	0.0008
	BN	0.9496	0.0504	0.0262	0.0242	0.0010
6	lr	0.9536	0.0464	0.0306	0.0158	0.0074
	mle	0.9296	0.0704	0.0453	0.0251	0.0102
	LR	0.9492	0.0508	0.0251	0.0257	0.0004
	BN	0.9504	0.0496	0.0244	0.0252	0.0004

Table 7: Simulation ($n=60$, shape=0.9, scale=43, censored %=20) - Nominal: 95%

Parameter	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
Shape	lr	0.9499	0.0501	0.0229	0.0272	0.0022
	mle	0.9652	0.0348	0.0197	0.0151	0.0076
	LR	0.9502	0.0498	0.0245	0.0253	0.0004
	BN	0.9501	0.0499	0.0245	0.0254	0.0004
Scale	lr	0.9505	0.0495	0.0260	0.0235	0.0013
	mle	0.9438	0.0562	0.0299	0.0263	0.0031
	LR	0.9501	0.0499	0.0258	0.0241	0.0009
	BN	0.9501	0.0499	0.0258	0.0241	0.0009
Median	lr	0.9494	0.0506	0.0259	0.0247	0.0006
	mle	0.9444	0.0556	0.0302	0.0254	0.0028
	LR	0.9494	0.0506	0.0255	0.0251	0.0003
	BN	0.9495	0.0505	0.0255	0.0250	0.0002

4.2 Complete Case

4.2.1 Confidence Intervals

The resulting 90%, 95%, and 99% confidence intervals for the complete (uncensored) strike duration data are provided in Tables 8 to 10 below for the shape, scale, and median parameters. For the shape parameter, the confidence intervals resulting from using either the likelihood ratio or the *mle* are again wider than the confidence intervals obtained from using the third-order methods. For the scale and median parameters, the likelihood ratio-based confidence intervals are close to the third-order ones, the *mle*-based intervals are, however, very discordant especially for the median.

Table 8: Confidence Intervals for Shape

Method	90% CI		95% CI		99% CI	
	lower	upper	lower	upper	lower	upper
lr	0.7804	1.0816	0.7542	1.1131	0.7043	1.1760
mle	0.7407	1.1087	0.7055	1.1439	0.6366	1.2128
LR	0.7675	1.0675	0.7414	1.0989	0.6918	1.1615
BN	0.7675	1.0675	0.7415	1.0989	0.6918	1.1616

Table 9: Confidence Intervals for Scale

Method	90% CI		95% CI		99% CI	
	lower	upper	lower	upper	lower	upper
lr	32.1673	52.1166	30.6225	54.5717	27.7333	59.8118
mle	32.4088	52.1420	30.9663	54.5709	28.3283	59.6526
LR	32.1757	52.4719	30.6111	54.9834	27.6871	60.3551
BN	32.1757	52.4721	30.6111	54.9833	27.6871	60.3548

Table 10: Confidence Intervals for the Median

Method	90% CI		95% CI		99% CI	
	lower	upper	lower	upper	lower	upper
lr	20.8600	35.7189	19.6778	37.4504	17.4700	41.0558
mle	21.0801	34.2316	19.8209	35.4908	17.3585	37.9532
LR	20.7370	35.7720	19.5452	37.5311	17.3212	41.1991
BN	20.7370	35.7720	19.5452	37.5311	17.3213	41.1991

4.2.2 Simulation

In this section simulation results for the complete data case for 10,000 simulated samples are reported. The shape parameter took on the values 1, 0.5 and 5, and the scale parameter took on the values 1 and 10. Sample sizes (n) of size 5 and 10 were considered. Results are reported for the following 12 cases:

Simulation Settings

Case	n	shape	scale
1	5	1	1
2	10	1	1
3	5	1	10
4	10	1	10
5	5	0.5	1
6	10	0.5	1
7	5	0.5	10
8	10	0.5	10
9	5	5	1
10	10	5	1
11	5	5	10
12	10	5	10

For each simulated sample, only the 95% confidence intervals are reported. Tables 11 to 13 record the central coverage, coverage error, upper error, lower error, and average bias. The simulation results for the complete case illustrate the dominance of the third-order approximations. For every case considered, the average bias is again smallest for these third-order intervals compared to the first-order intervals. Further, this bias decreases for any given shape and scale combination as the sample size increases from 5 to 10. The intervals based on the maximum likelihood estimates perform very poorly and produce highly asymmetric errors. The likelihood ratio-based intervals produce highly asymmetric error rates as well, although in terms of central coverage they do outperform the *mle*-based intervals.

An additional simulation similar to that produced in Table 7 was considered. The simulation was based on 10,000 samples of 60 observations with a shape parameter set equal to 0.9 and a scale parameter set equal to 43. The results are recorded in Table 14. The third-order method generally outperforms the other methods in terms of central coverage and average bias even for this sample size of 60. Again, the largest gain, in terms of bias, from using the third-order method arises for the shape parameter.

Table 11: Simulation for Shape - Nominal: 95%

Case	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
1	lr	0.9465	0.0535	0.0323	0.0212	0.0056
	mle	0.9582	0.0418	0.0094	0.0324	0.0115
	LR	0.9490	0.0510	0.0275	0.0235	0.0020
	BN	0.9492	0.0508	0.0289	0.0219	0.0035
2	lr	0.9423	0.0577	0.0309	0.0268	0.0039
	mle	0.9369	0.0631	0.0112	0.0519	0.0204
	LR	0.9500	0.0500	0.0260	0.0240	0.0010
	BN	0.9499	0.0501	0.0263	0.0238	0.0013
3	lr	0.9440	0.0560	0.0328	0.0232	0.0048
	mle	0.9386	0.0614	0.0169	0.0445	0.0138
	LR	0.9490	0.0510	0.0275	0.0235	0.0020
	BN	0.9492	0.0508	0.0289	0.0219	0.0035
4	lr	0.9471	0.0529	0.0288	0.0241	0.0024
	mle	0.9474	0.0526	0.0227	0.0299	0.0036
	LR	0.9500	0.0500	0.0260	0.0240	0.0010
	BN	0.9499	0.0501	0.0263	0.0238	0.0013
5	lr	0.9504	0.0496	0.0298	0.0198	0.0050
	mle	0.9471	0.0529	0.0192	0.0337	0.0073
	LR	0.9488	0.0512	0.0277	0.0235	0.0021
	BN	0.9491	0.0509	0.0290	0.0219	0.0036
6	lr	0.9551	0.0449	0.0186	0.0263	0.0039
	mle	0.9533	0.0467	0.0295	0.0172	0.0062
	LR	0.9502	0.0498	0.0260	0.0238	0.0011
	BN	0.9501	0.0499	0.0263	0.0236	0.0014
7	lr	0.9406	0.0594	0.0324	0.0270	0.0047
	mle	0.9607	0.0393	0.0169	0.0224	0.0054
	LR	0.9488	0.0512	0.0277	0.0235	0.0021
	BN	0.9491	0.0509	0.0290	0.0219	0.0036
8	lr	0.9447	0.0553	0.0244	0.0309	0.0033
	mle	0.9394	0.0606	0.0205	0.0401	0.0098
	LR	0.9502	0.0498	0.0260	0.0238	0.0011
	BN	0.9501	0.0499	0.0263	0.0236	0.0014
9	lr	0.9435	0.0565	0.0362	0.0203	0.0080
	mle	0.9399	0.0601	0.0099	0.0502	0.0202
	LR	0.9490	0.0510	0.0275	0.0235	0.0020
	BN	0.9492	0.0508	0.0289	0.0219	0.0035
10	lr	0.9502	0.0498	0.0326	0.0172	0.0077
	mle	0.9318	0.0682	0.0141	0.0541	0.0200
	LR	0.9499	0.0501	0.0261	0.0240	0.0011
	BN	0.9500	0.0500	0.0264	0.0236	0.0014
11	lr	0.9494	0.0506	0.0315	0.0191	0.0062
	mle	0.9406	0.0594	0.0057	0.0537	0.0240
	LR	0.9490	0.0510	0.0275	0.0235	0.0020
	BN	0.9492	0.0508	0.0289	0.0219	0.0035
12	lr	0.9452	0.0548	0.0237	0.0311	0.0037
	mle	0.9346	0.0654	0.0192	0.0462	0.0135
	LR	0.9499	0.0501	0.0261	0.0240	0.0011
	BN	0.9500	0.0500	0.0264	0.0236	0.0014

Table 12: Simulation for Scale - Nominal: 95%

Case	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
1	lr	0.9443	0.0557	0.0317	0.0240	0.0039
	mle	0.9171	0.0829	0.0530	0.0299	0.0165
	LR	0.9480	0.0520	0.0261	0.0259	0.0010
	BN	0.9468	0.0532	0.0272	0.0260	0.0016
2	lr	0.9500	0.0500	0.0264	0.0236	0.0014
	mle	0.9387	0.0613	0.0330	0.0283	0.0057
	LR	0.9497	0.0503	0.0258	0.0245	0.0007
	BN	0.9495	0.0505	0.0261	0.0244	0.0009
3	lr	0.9439	0.0561	0.0326	0.0235	0.0046
	mle	0.9169	0.0831	0.0472	0.0359	0.0166
	LR	0.9480	0.0520	0.0261	0.0259	0.0010
	BN	0.9468	0.0532	0.0272	0.0260	0.0016
4	lr	0.9491	0.0509	0.0311	0.0198	0.0057
	mle	0.9259	0.0741	0.0413	0.0328	0.0121
	LR	0.9497	0.0503	0.0258	0.0245	0.0007
	BN	0.9495	0.0505	0.0261	0.0244	0.0009
5	lr	0.9440	0.0560	0.0359	0.0201	0.0079
	mle	0.9179	0.0821	0.0507	0.0314	0.0161
	LR	0.9477	0.0523	0.0264	0.0259	0.0012
	BN	0.9466	0.0534	0.0274	0.0260	0.0017
6	lr	0.9439	0.0561	0.0323	0.0238	0.0043
	mle	0.9365	0.0635	0.0362	0.0273	0.0068
	LR	0.9496	0.0504	0.0259	0.0245	0.0007
	BN	0.9494	0.0506	0.0262	0.0244	0.0009
7	lr	0.9470	0.0530	0.0299	0.0231	0.0034
	mle	0.9409	0.0591	0.0343	0.0248	0.0048
	LR	0.9477	0.0523	0.0264	0.0259	0.0012
	BN	0.9466	0.0534	0.0274	0.0260	0.0017
8	lr	0.9502	0.0498	0.0305	0.0193	0.0056
	mle	0.9435	0.0565	0.0370	0.0195	0.0088
	LR	0.9496	0.0504	0.0259	0.0245	0.0007
	BN	0.9494	0.0506	0.0262	0.0244	0.0009
9	lr	0.9514	0.0486	0.0277	0.0209	0.0034
	mle	0.9605	0.0395	0.0297	0.0098	0.0100
	LR	0.9478	0.0522	0.0262	0.0260	0.0011
	BN	0.9467	0.0533	0.0273	0.0260	0.0017
10	lr	0.9495	0.0505	0.0303	0.0202	0.0051
	mle	0.9487	0.0513	0.0314	0.0199	0.0058
	LR	0.9496	0.0504	0.0258	0.0246	0.0006
	BN	0.9494	0.0506	0.0261	0.0245	0.0008
11	lr	0.9396	0.0604	0.0341	0.0263	0.0052
	mle	0.9281	0.0719	0.0401	0.0318	0.0110
	LR	0.9478	0.0522	0.0262	0.0260	0.0011
	BN	0.9467	0.0533	0.0273	0.0260	0.0017
12	lr	0.9479	0.0521	0.0337	0.0184	0.0077
	mle	0.9266	0.0734	0.0428	0.0306	0.0117
	LR	0.9496	0.0504	0.0258	0.0246	0.0006
	BN	0.9494	0.0506	0.0261	0.0245	0.0008

Table 13: Simulation for Median - Nominal: 95%

Case	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
1	lr	0.9562	0.0438	0.0267	0.0171	0.0048
	mle	0.9530	0.0470	0.0299	0.0171	0.0064
	LR	0.9483	0.0517	0.0250	0.0267	0.0009
	BN	0.9482	0.0518	0.0249	0.0269	0.0010
2	lr	0.9534	0.0466	0.0265	0.0201	0.0032
	mle	0.9542	0.0458	0.0277	0.0181	0.0048
	LR	0.9485	0.0515	0.0261	0.0254	0.0007
	BN	0.9485	0.0515	0.0261	0.0254	0.0007
3	lr	0.9500	0.0500	0.0304	0.0196	0.0054
	mle	0.9498	0.0502	0.0313	0.0189	0.0062
	LR	0.9483	0.0517	0.0250	0.0267	0.0009
	BN	0.9482	0.0518	0.0249	0.0269	0.0010
4	lr	0.9535	0.0465	0.0266	0.0199	0.0034
	mle	0.9529	0.0471	0.0279	0.0192	0.0044
	LR	0.9485	0.0515	0.0261	0.0254	0.0007
	BN	0.9485	0.0515	0.0261	0.0254	0.0007
5	lr	0.9581	0.0419	0.0245	0.0174	0.0041
	mle	0.9501	0.0499	0.0212	0.0287	0.0038
	LR	0.9482	0.0518	0.0250	0.0268	0.0009
	BN	0.9481	0.0519	0.0250	0.0269	0.0009
6	lr	0.9521	0.0479	0.0269	0.0210	0.0030
	mle	0.9500	0.0500	0.0302	0.0198	0.0052
	LR	0.9486	0.0514	0.0261	0.0253	0.0007
	BN	0.9485	0.0515	0.0262	0.0253	0.0007
7	lr	0.9526	0.0474	0.0307	0.0167	0.0070
	mle	0.9538	0.0462	0.0322	0.0140	0.0091
	LR	0.9482	0.0518	0.0250	0.0268	0.0009
	BN	0.9481	0.0519	0.0250	0.0269	0.0009
8	lr	0.9489	0.0511	0.0280	0.0231	0.0025
	mle	0.9492	0.0508	0.0287	0.0221	0.0033
	LR	0.9486	0.0514	0.0261	0.0253	0.0007
	BN	0.9485	0.0515	0.0262	0.0253	0.0007
9	lr	0.9496	0.0504	0.0313	0.0191	0.0061
	mle	0.9536	0.0464	0.0345	0.0119	0.0113
	LR	0.9483	0.0517	0.0251	0.0266	0.0008
	BN	0.9481	0.0519	0.0251	0.0268	0.0009
10	lr	0.9509	0.0491	0.0299	0.0192	0.0054
	mle	0.9534	0.0466	0.0313	0.0153	0.0080
	LR	0.9486	0.0514	0.0261	0.0252	0.0007
	BN	0.9485	0.0515	0.0262	0.0251	0.0007
11	lr	0.9466	0.0534	0.0316	0.0218	0.0049
	mle	0.9555	0.0445	0.0347	0.0098	0.0125
	LR	0.9483	0.0517	0.0251	0.0266	0.0008
	BN	0.9481	0.0519	0.0251	0.0268	0.0009
12	lr	0.9489	0.0511	0.0285	0.0226	0.0030
	mle	0.9504	0.0496	0.0294	0.0202	0.0046
	LR	0.9486	0.0514	0.0261	0.0252	0.0007
	BN	0.9485	0.0515	0.0262	0.0251	0.0007

Table 14: Simulation ($n=60$, shape=0.9, scale=43) - Nominal: 95%

Parameter	Method	Coverage	Coverage Error	Upper Error	Lower Error	Average Bias
Shape	lr	0.9496	0.0504	0.0231	0.0273	0.0021
	mle	0.9631	0.0369	0.0202	0.0167	0.0066
	LR	0.9500	0.0500	0.0247	0.0253	0.0003
	BN	0.9501	0.0499	0.0247	0.0252	0.0003
Scale	lr	0.9498	0.0502	0.0264	0.0238	0.0013
	mle	0.9419	0.0581	0.0319	0.0262	0.0041
	LR	0.9507	0.0493	0.0252	0.0241	0.0006
	BN	0.9508	0.0492	0.0252	0.0240	0.0006
Median	lr	0.9480	0.0520	0.0268	0.0252	0.0010
	mle	0.9422	0.0578	0.0321	0.0257	0.0039
	LR	0.9506	0.0494	0.0253	0.0241	0.0006
	BN	0.9508	0.0492	0.0252	0.0240	0.0006

5 Conclusion

Recent third-order likelihood theory was applied to obtain highly accurate confidence intervals for the shape, scale, and median parameters of the Weibull distribution. Both the censored and complete cases were examined and applied to Kennan's (1985) strike duration data. The intervals obtained using the first-order quantities, namely, the likelihood ratio and the maximum likelihood estimate were considerably different from those intervals obtained using the third-order methods. The accuracy of the third-order intervals was assessed through simulations. For both the complete and censored cases, the simulations revealed that a significant improvement could be attained using the third-order-based confidence intervals. As stated in Section 3, the extra computation required for the third-order intervals is due to the calculation of the maximum departure measure, Q . However, the explicit equations for this quantity are provided in the paper. It is further noted that the calculation of Q does not depend on the sample size but on the dimension of the parameter, and in this case the dimension is two, which makes the extra calculations minimal. On the other hand, the calculation of the maximum likelihood estimate and constrained maximum likelihood estimate depends on the sample size. Hence, for simulations, when the sample size is large, even the first-order methods will require considerable computational costs. In terms of the first-order methods, the likelihood ratio intervals performed relatively well and are the preferred method to the *mle*-based intervals. With respect to future research, it will be interesting to pursue these methods to Weibull problems that include regressors and hence more parameters..

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