Homework #5 • Numerical Analysis II (math 416) • IVPs & Midterm Warm-up

- due Wednesday 18 October.
- please indicate any collaborations, or acknowledge any useful e-mails.

A) (2 pages) Chapter Nine in Heath is a pretty readable survey of numerical IVPs. Address the true/false questions (# 9.1-7) in Heath, p294. For each problem, give a clear one-sentence justification and/or a reference/quote from the book. Also discuss questions # 9.29 & 9.34 – also give references, but your answers should show understanding and not just be a parroting of the book.

B) (2 pages) Derive and then solve the difference equation that corresponds to the 3^{rd}-order Adams-Bashforth discretization of the simple ODE

$$y' = q\,y\ ,$$

where q is a complex constant. Download and modify the script *lect16.m* to make a contourplot of the stability region in the complex $q\,\Delta t$-plane. You will need to solve a cubic polynomial, but *help roots* is the way to go. Explain why it is enough to consider only the $max(abs(\cdot))$ of the three roots. (Hint: my new code only needs two new lines; also, smile when you see the new plot.) Make a statement about the relative comparison between the RK2 and AB3 stability regions.

A consequence of the stability region is that your AB3 code from the last homework will go unstable for a sufficiently large Δt – investigate what happens to the numerical solution as Δt approaches (but is below) the stability edge? **Bonus:** try to show that this value of timestep is consistent with the AB3 stability diagram.

C) (3 pages) Investigate numerically, the properties of the Matthieu equation

$$y'' + (\alpha + \beta\,\cos t)\,y = 0\ ,$$

with respect to the parameters α and β, which are positive constants. You may use the Matlab intrinsic ODE solver, but choose your method appropriately. The issue is "for what parameter values do the (true) ODE solutions remain bounded as $t \to \infty$?" One way is to construct a *parameter map*, whose axes are α and β, which maps out regions of bounded and unbounded ODE solutions. Focus on the vicinity of $\alpha \approx 1/4$ and $\alpha \approx 1$ for β not too large.

A related calculation is to compute the eigenvalues of the matrix formed from two independent solutions $y_1(t)$ and $y_2(t)$

$$M = \begin{bmatrix} y_1(2\pi) & y_1'(2\pi) \\ y_2(2\pi) & y_2'(2\pi) \end{bmatrix}$$

where the solutions are distinguished by the initial conditions

$$y_1(0) = 1\ ;\ y_1'(0) = 0 \qquad\qquad y_2(0) = 0\ ;\ y_2'(0) = 1\ .$$

Make careful observations and then, pose a conjecture.

D) (optional) In homework #2, you solved a nonlinear BVP by combining a linear BVP solve with a Newton iteration. Another way to solve a nonlinear BVP is to combine a nonlinear IVP solve with a Newton iteration – the so-called *shooting method.* Use an IVP shooting method to solve numerically the BVP of Heath #10.2, p313.

To get things started, define $y(t;s)$ as the solution to the IVP

$$y_{tt} + (1 + e^y) = 0 \qquad \text{on} \qquad 0 \le t \le 1$$

$$y(0) = 0 \quad ; \quad y_t(0) = s \ ,$$

(subscripts indicate derivatives) and also $f(s) = y(1;s) - 1$. Thus, the solution to the above problem satisfies the BVP when s is exactly a zero of $f(s) = 0$. The Newton iteration for numerically obtaining s is

$$\bar{s}_{k+1} = \bar{s}_k - \frac{f(\bar{s}_k)}{f_s(\bar{s}_k)}$$

which is very straightforward to implement provided one can compute the derivative $f_s(s)$. But, the s-derivative of the ODE solution $u(t,s) = y_s(t;s)$ satisfies the *linearized* ODE

$$u_{tt} + (e^y)\, u = 0 \qquad \text{on} \qquad 0 \le t \le 1$$

$$u(0) = 0 \quad ; \quad u_t(0) = 1 \ ,$$

which is just the s-derivative of the IVP for $y(t;s)$. So by, solving two simultaneous IVPs for y, u – the Newton iteration can be evaluated.

Write a matlab script to compute the solutions to this problem. You are encouraged to use the Matlab intrinsic ODE solver to save time. You may also hand in a one-page pseudocode, if you want the practice for the midterm. (Note: this ODE has a first integral.)