Homework #1 • MATH 462 • The Euler Equations

- please respect page limits.
- submit your write-up by noon, Thursday 26 January.
- remember that webct is an open forum for discussion.
- please acknowledge collaborations & assistance from colleagues.

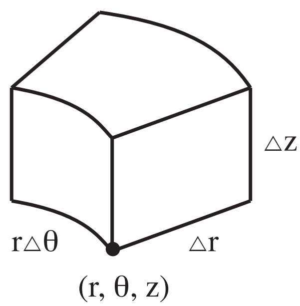
A) Polar Coordinates (4 pages, 20pts) Present a *first principles* derivation of the Euler equations for two-dimensional fluid flow in three-dimensional cylindrical coordinates (r, θ, z). In this context, the two-dimensional flow is the special case with no vertical flow ($w \equiv 0$) and no vertical variations ($\partial/\partial z \equiv 0$). For convenience, define the velocities $U(r, \theta, t)$ and $V(r, \theta, t)$ to be the flow components in the $\hat{r}$ and $\hat{\theta}$-directions. Invoking

(a) conservation of mass, and

(b) Newton's law with a given body force (per unit volume) $\vec{f}(r, \theta, t)$;

should result in three PDEs for the flow velocities, density $\rho(r, \theta, t)$ and pressure $p(r, \theta, t)$.

Quality (clarity & conciseness) of the presentation counts. Please use a few words to explain your reasoning, and include several line diagrams in your derivation.



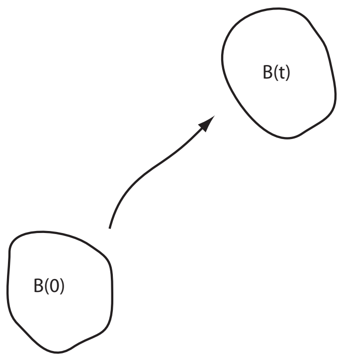
Added calculus note: Remember that there is a θ-dependence of the unit basis vectors $\hat{r}(\theta)$ and $\hat{\theta}(\theta)$.

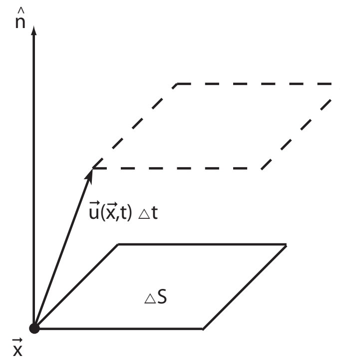
$$\hat{r}(\theta + \triangle\theta) \approx \hat{r}(\theta) + \hat{\theta}(\theta)\triangle\theta \quad ; \quad \hat{\theta}(\theta + \triangle\theta) \approx \hat{\theta}(\theta) - \hat{r}(\theta)\triangle\theta \ .$$

bonus: Use Appendix A.6 to verify that your conservation of mass equation is consistent with the expected result in Cartesian coordinates.

B) Lagrangian Derivation of Continuity (2 pages, 10pts) Another common derivation of the fluid equations relies on the integral identities of multi-variable calculus — instead of the infinitesimal control volume approach shown in the lectures. Present a Lagrangian derivation of the continuity equation based on the ideas and notation sketched out below. Organize your write-up as a numbered list identifying each of the major concepts/steps.

Consider any initial blob of fluid, $\mathcal{B}(0)$, and let $\mathcal{B}(t)$ be the blob of this original fluid as it moves with the flow (as in the figure on the left). Note that we assume the flow velocity is continuous so that we do not have to be concerned with the fluid blob breaking up into multiple bloblets.





Define the total fluid mass inside the blob by the volume integral over the space occupied by the blob

$$M(t) = \iiint_{\mathcal{B}(t)} \rho(\vec{x}, t)\ dx\, dy\, dz,$$

and then explain how the principle of conservation of mass requires

$$\frac{dM}{dt} = 0.$$

Taking the derivative of the integral formula for $M(t)$ involves two contributions due to time variations in the integrand $\rho(\vec{x}, t)$, and the blob $\mathcal{B}(t)$. This differentiation is essentially a three-dimensional version of Leibnitz's rule. In particular, explain explicitly how the second contribution can be expressed as a surface integral. The basic idea is that over a time-interval $\triangle t$ the change to the blob geometry can be accounted for by summing over the displacements of small surface patches $\triangle S$ (see figure on the right for notation). This surface integral can then be converted to a volume integral using the divergence theorem (Appendix A3, Acheson). Finally, complete the derivation by invoking the Dubois-Reymond Lemma which states that if an integral of the form

$$\iiint_{\mathcal{B}(t)} \{\text{integrand}\}\ dx\, dy\, dz = 0$$

for <u>all</u> choices of $\mathcal{B}(t)$, then the integrand must be zero everywhere.