Homework #2 • MATH 462 • Simple Incompressible Flows

- submit your write-up noon, Friday 29 January.
- remember that webct is open for discussion & thursday in AQ5004.
- please acknowledge collaborations & assistance from colleagues.

A) A Spinning Bucket Problem (2 pages, 10pts) Consider the flow velocity for a uniformly rotating fluid $(u, v, w) = (-\Omega y, \Omega x, 0)$. Find the accompanying pressure field $p(x, y, z)$ which produces a flow solution to the *incompressible* Euler equations in the presence of gravity $\vec{F} = -\rho g \hat{z}$. Having done this, now discuss and resolve the *apparent confusion* suggested in the first two paragraphs of Problem 1.2 (Acheson) involving the Bernoulli streamline theorem (Section 1.3).

extra: Why might an astronomer with access to a lot of mercury find this result interesting?

B) Streamlines & Streamfunctions (4 pages + plot, 15pts) A keen-eyed student making the flow plots for problem **B)** of Homework #0 will have noticed that the contours of constant ψ seemed oddly parallel to the flow arrows everywhere! This is no accident, of course, and this behavior is further investigated in the following problem for a different flow example.

In cylindrical coordinates, an *axisymmetric* flow has no dependence on the θ variable. A *Stokes streamfunction*, $\Psi(r, z)$, provides a definition of a steady, axisymmetric flow velocity having the form $\vec{u} = U(r, z)\,\hat{r} + W(r, z)\,\hat{z}$ through the derivative relations:

$$U = -\frac{1}{r}\frac{\partial \Psi}{\partial z} \qquad ; \qquad W = +\frac{1}{r}\frac{\partial \Psi}{\partial r} \ .$$

i) For the specific case of the Stokes streamfunction

$$\Psi(r, z) = \frac{A}{2} r^2 + \frac{m}{4\pi}\left(1 - \frac{z}{\sqrt{r^2 + z^2}}\right)$$

verify that the flow satisfies the incompressibility condition.

ii) Show that the streamfunction Ψ is constant along streamlines by verifying that

$$\frac{D\Psi}{Dt} = 0 \ .$$

iii) Show that the flow direction is tangent to the streamline contours everywhere, and note that the calculation is exactly the same as in **ii)**. (Is this a coincidence, or a conspiracy?)

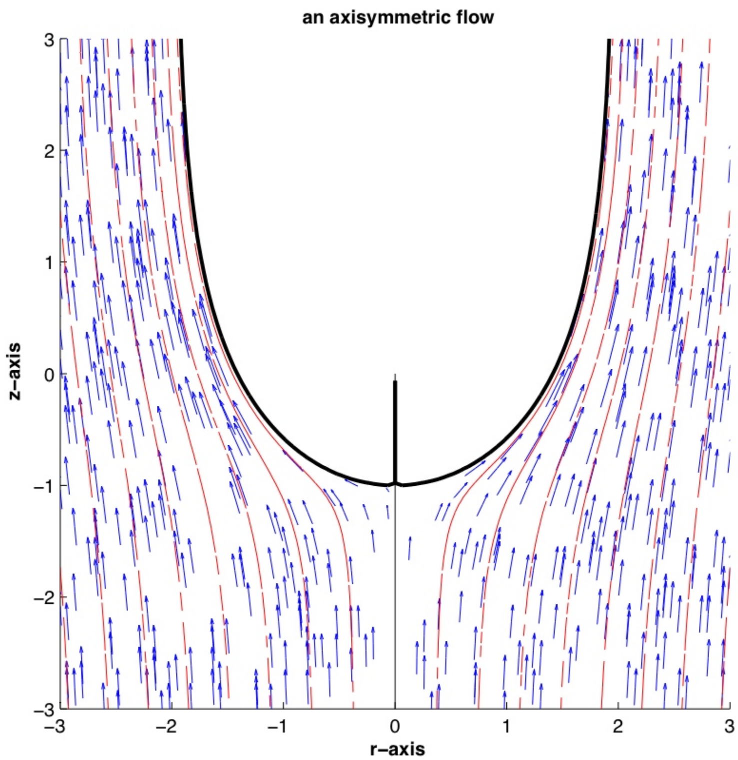
iv) Use the *contour* command in Matlab (or better yet, modify an earlier script) to produce a graphic illustrating the flow by streamlines and flow direction arrows. Annotate with parameter values, and all other relevant comments.

v) Find the exact location of the flow stagnation point (where $\vec{u} = \vec{0}$). What is the value of the streamfunction there? Add this special contour to your plot, and note that this contour can act as a *separating* streamline of the flow (i.e. separating into "inside" and "outside" parts).

vi) Use the Bernoulli streamline theorem to give a formula for the pressure deviation (assume that the pressure far-upstream is a constant, p^{atm}, and no body forces). Label regions of relatively high (H) and/or low (L) pressure regions on your flow graphic (you may add these by hand, or plot the pressure field as an extra Matlab plot). Verify that these regions are consistent with the observed streamline curvatures.

vii) Based on parts **i)-vi)**, is it clear that this combination of velocity and pressure fields is a solution to the incompressible Euler equations? Explain your reasoning.

extra: Is this example illustrative/suggestive of any realistic flow? In particular, consider the outside part of the flow only, with the inside part (containing the origin) replaced by a solid object.



outstanding challenges:

a - **Bernoulli Or Not:** Follow the *bernoulli or not* link & decide whether or not the observation can be explained using the Bernoulli principle.

b - **Bernoulli in the news:** How many errors?