<u>Homework #3 • MATH 462 • Vorticity & Flow</u>

- submit your write-up by noon Friday 05 February (homework box #8).
- remember that webct is open for discussion.
- please acknowledge collaborations & assistance from colleagues.
- it is recommended that you begin problem **B)** well before the due date.

A) A Patch of Vorticity (3 pages + 1 for plots, 15pts) Consider an incompressible 2D fluid whose initial condition is characterized by a circular patch of vorticity

$$\omega(\vec{x},0) = \begin{cases} \dfrac{1}{\pi a^2} & 0 \leq r < a \\ 0 & a < r < \infty \end{cases}$$

where ω is the $\hat{z}$-component of vorticity $\vec{\omega}$ and $r = |\,\vec{x}\,|$.

i) Solve for the initial streamfunction $\psi(\vec{x},0)$, and hence, determine the initial flow velocity. Your task is simplified in this geometry since the Poisson PDE for the streamfunction $\psi(\vec{x},0)$ is really just an ODE. Invoke the BC that the flow is bounded at the origin. It is also necessary to impose continuity on the streamfunction and its associated flow at $r = a$. Choose the constant part of the streamfunction to be zero at $r \to \infty$. Describe the resulting flow pattern.

ii) Using the vorticity equation, deduce the time evolution of this flow for $t > 0$. This result, in turn, makes it easy to determine the pressure field. Invoke the conditions that the pressure approaches a constant value p^∞ as $r \to \infty$ and is continuous at $r = a$. Explain why the presssure field is consistent with the flow pattern.

iii) Make a subplot (in matlab, type *help subplot*) or two showing the important flow quantities as a function of r.

iv) Finally, show that there is a limiting streamfunction as the patch parameter $a \to 0$

$$\Psi(\vec{x},t) = \lim_{a\to 0} \; \psi(\vec{x},t) \; .$$

This limit is commonly known as the *line vortex* in 3D flow.

B) Exact Flow Trajectories (4 pages, 15pts) Despite the need for taking many derivatives, this is really a thought-intensive problem. In the partial derivative notation, you will need to be clear about what is being held constant. There is an old-fashioned type of notation that makes this issue clear. For a function $F(x,y,t)$, the usual partial derivative notation

$$\frac{\partial F}{\partial x} \quad \text{is used to mean} \quad \left(\frac{\partial F}{\partial x}\right)_{y,t}$$

where the subscripts indicate that the derivative is the rate of change in x, but with y and t held constant. In your write-up of this problem, **every partial derivative must indicate the fixed variables**.

In 1802, the Czech-born Franz Joseph von Gerstner discovered a remarkable set of exact 2D flow trajectories consistent with Euler's incompressible flow equations

$$\begin{aligned} x(t,\bar{x},\bar{y}) &= \bar{x} + \frac{1}{k}\, e^{k\bar{y}} \,\sin k(\bar{x}+ct) \\ y(t,\bar{x},\bar{y}) &= \bar{y} - \frac{1}{k}\, e^{k\bar{y}} \,\cos k(\bar{x}+ct) \end{aligned} \tag{1}$$

where k and c are given constants. All trajectories are circles with radius $e^{k\bar{y}}/k$ and the point $(\bar{x}, \bar{y})$ as its centre. Please read the problems below carefully.

i) The u flow velocity is given by a certain partial time-derivative of $x(t, \bar{x}, \bar{y})$ — exactly what needs to be held fixed in this calculation? Give the velocity, $\vec{u}$, of the flow as a function of the variables $(\bar{x}, \bar{y}, t)$, and clearly state where in the (x, y)-plane this flow velocity is found. Explain why you cannot give a formula for the flow velocity, $\vec{u}(x, y, t)$, i.e. explicitly as a function of the usual coordinates.

ii) Despite that we only have an implicit representation of the flow, we can verify that the velocity field has zero divergence. The calculation requires thoughtful application of the chain rule. As a first step, you will need the following partial derivatives

$$\left(\frac{\partial \bar{x}}{\partial x}\right)_{y,t} \quad ; \quad \left(\frac{\partial \bar{y}}{\partial x}\right)_{y,t} = \frac{e^{k\bar{y}} \sin k(\bar{x} + ct)}{e^{2k\bar{y}} - 1} \quad ; \quad \left(\frac{\partial \bar{x}}{\partial y}\right)_{x,t} \quad ; \quad \left(\frac{\partial \bar{y}}{\partial y}\right)_{x,t}$$

which are derivable from the four relations obtained by taking the x and y partials of the trajectories (1).

iii) You should now be well-equipped to evaluate the divergence using the chain rule

$$\nabla \cdot \vec{u} = \left(\frac{\partial u}{\partial x}\right)_{y,t} + \left(\frac{\partial v}{\partial y}\right)_{x,t} .$$

iv) Lastly, calculate the vorticity. What principle of two-dimensional vorticity dynamics is illustrated by your result?

***) extra:** Is this flow steady? How might one design a visualization of this flow? What do you think about fluid dynamical analyses based on particle paths?

Finally, this flow was independently rediscovered by Rankine in 1863. The original publication is posted on webct.

*) **Almost a Homework Problem** (for your personal challenge) Here is a problem that didn't quite make it into this week's homework list. Solve, using PDE techniques, for a one-dimensional (incompressible) flow that consists of a fixed-pressure pump filling a cylinder capped with a frictionless, moving piston. The density of the filling fluid is ρ_0. The mass of the piston is M_p. The cross-sectional area of the cylinder is A.

Denote the location of the piston by $L(t)$. Given that the piston begins from $L(0) = 0$ with initial velocity $L'(0) = 0$, determine a second-order, nonlinear ODE problem that fully determines the piston motion, $L(t)$.

Some of the boundary conditions are tricky. The easiest is $p(0) = p^{pump}$, the other two involve $u(L(t))$ and $p(L(t))$. The inside pressure at the piston end is exactly that required to be consistent with the piston's acceleration (hence the need for the piston mass). The $\hat{x}$-momentum equation eventually becomes the ODE for $L(t)$. (You may neglect the airflow on the outside of the cylinder — assume it is much less dense than ρ_0. No body forces.) Unfortunately, this ODE seems not to have a closed form solution.