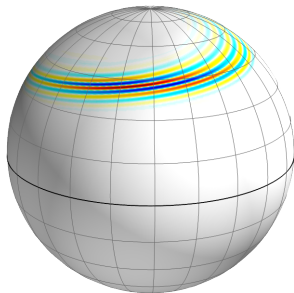
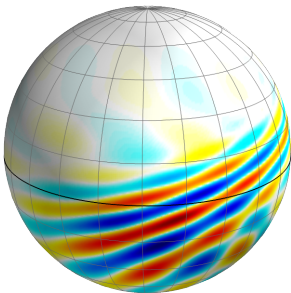


Rossby Wave Modes for Rotating Shallow Water on the Sphere

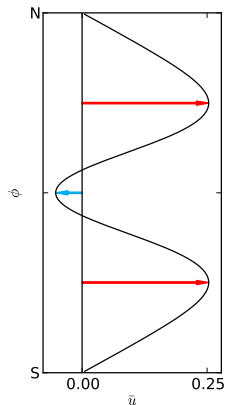
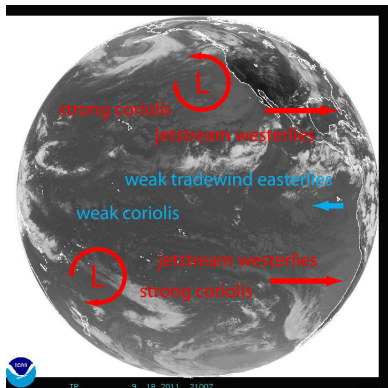
- ▷ **midlatitude** waves & critical latitude exclusion from the tropics
- ▷ equatorial β -plane & **tropical** waveguide
- ▷ finite number of **global** Rossby waves observed in climatology



- ▷ David J Muraki & Kevin Mitchell, Simon Fraser University



Midlatitude & Equatorial Regions



Zonal Winds & Coriolis Rotation

- ▷ midlatitudes: strong **jetstream westerlies** & **strong coriolis** effect
- ▷ equatorial (dry) tropics: weak **tradewind easterlies** & **weak coriolis** effect (sign change)

Three Communities of Rossby Wave Theory

Midlatitudes (meridionally-sheared jet)

- ▷ midlatitude β -plane, slow waves at wavelengths scaling on Rossby radius
- ▷ diminishing of midlatitude jet \rightarrow NO propagation into tropics
 - ▷ critical latitude (zonal phase speed = angular wind speed), Dickinson (1970)

Tropics (negligible shear)

- ▷ equatorial β -plane theory, equatorially-trapped modes
- ▷ infinite number of discrete modes, at all wavelength scales, Matsuno (1966)

Global

- ▷ long-wavelength, planetary-scale waves
- ▷ finite number observed in climatological data, Madden (2007)
 - ▷ projection onto spherical harmonics
 - ▷ link with limited number of computed discrete modes, Kasahara (1980) ?

How are these perspectives reconciled into one framework?

Rotating Shallow Water (rSW) on the Sphere

Laplace Tidal Equations

- ▷ winds, \vec{u} & height field, $H = 1 + \epsilon \eta$
- ▷ spherical longitude-latitude coordinates, (λ, ϕ)

$$\epsilon^2 \frac{D\vec{u}}{Dt} + (\sin \phi \hat{r}) \times \vec{u} = -\epsilon \nabla \eta$$

$$\epsilon \frac{D\eta}{Dt} + (1 + \epsilon \eta) (\nabla \cdot \vec{u}) = 0$$

- ▷ atmospheric scaling: small **Rossby number** asymptotics

$$\epsilon = \left(\frac{4\Omega^2 r^2}{gH_0} \right)^{-1/2} \ll 1$$

A Long History of Linear Waves ...

- ▷ ... Margules (1892/93), Hough (1897/98) ... Longuet-Higgins (1968) ...
- ▷ wave modes: inertia gravity (fast), mixed-Rossby/gravity, Kelvin & Rossby (slow)
- ▷ midlatitude waves: Rossby (1939/40) ...
- ▷ equatorially-trapped waves: Matsuno (1966) ...

Linear Wave Modes

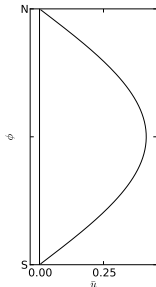
Background Zonal Wind

- ▷ linearize rSW about midlatitude jet & tradewind profile, $\bar{u}(\phi)$

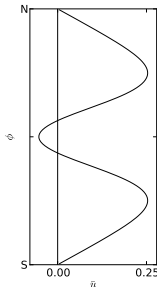
$$\begin{pmatrix} u \\ v \\ h \end{pmatrix} = \begin{pmatrix} \bar{u}(\phi) \\ 0 \\ \bar{h}(\phi) \end{pmatrix} + \begin{pmatrix} \hat{u}(\phi) \\ \hat{v}(\phi) \\ \hat{h}(\phi) \end{pmatrix} e^{i(m\lambda - \omega t)}$$

- ▷ non-constant coefficient eigenvalue problem for $\omega_n(m)$
 - ▷ FFT spectral computing & WKB analysis
- ▷ dispersion relation & wave spectrum comparison:

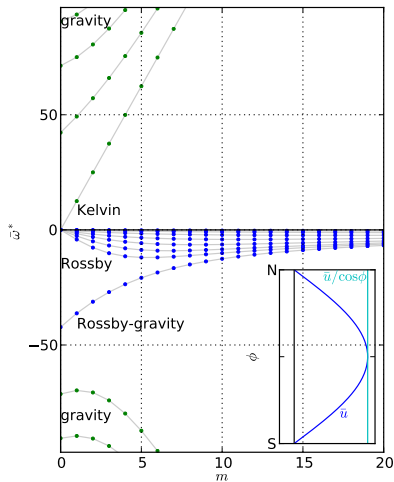
super-rotation (no shear)



tradewind (climatological shear)



Super-Rotation Profile \rightarrow Standard Spectrum



▷ reduced frequency:

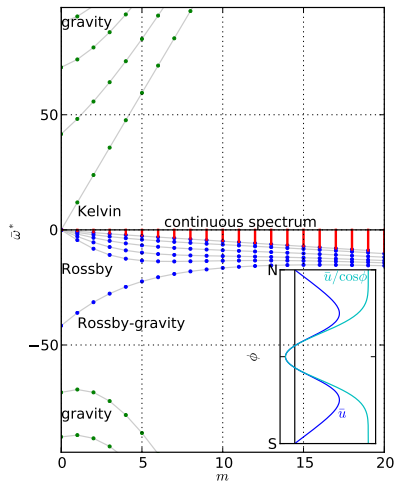
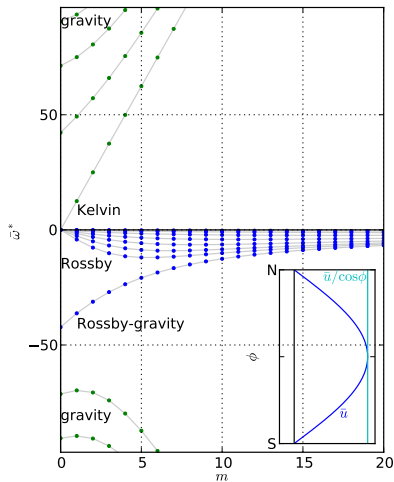
$$\bar{\omega}^* = \omega - m \left(\frac{\bar{u}(\phi)}{\cos \phi} \right)_{\max}$$

▷ slow **Rossby** & fast **gravity** waves

▷ $\epsilon = 0.084 \ll 1$

▷ discrete eigenvalue spectrum, $n = 0 \dots \infty$

Super-Rotation vs Tradewind Profiles



▷ tradewind profile \Rightarrow continuous spectrum, Case (1960) & Farrell (1982)

Tradewind Profile \rightarrow Singular Spectrum

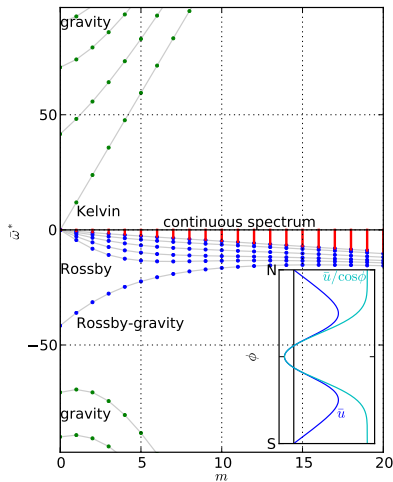
- ▷ reduced frequency:

$$\bar{\omega}^* = \omega - m \left(\frac{\bar{u}(\phi)}{\cos \phi} \right)_{\max}$$

- ▷ $\epsilon = 0.084 \ll 1$

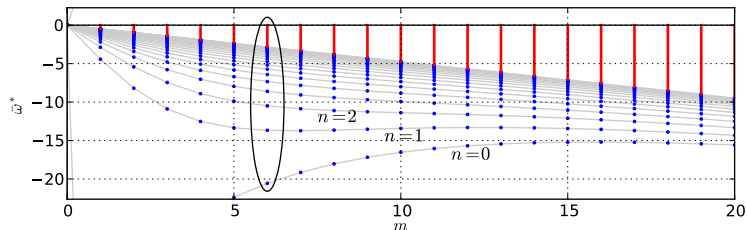
- ▷ **discrete** spectrum, $n = 0 \dots \infty$
- ▷ **continuous** spectrum:

$$\left(\frac{\bar{u}(\phi)}{\cos \phi} \right)_{\min} < \frac{\omega}{m} < \left(\frac{\bar{u}(\phi)}{\cos \phi} \right)_{\max}$$



- ▷ tradewind shear introduces two infinite sets of Rossby wavemodes

Three Flavors of Rossby Wave Modes? ---



Discrete Rossby Spectrum

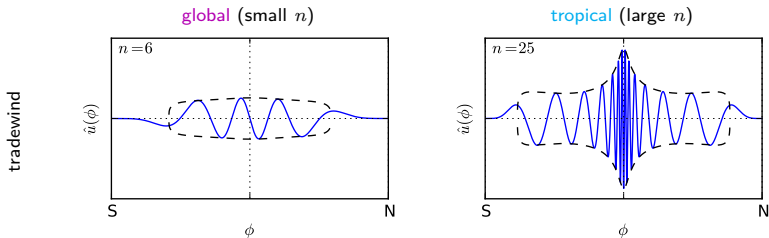
- ▷ global planetary-scale waves, accounted for in Madden (2007)?
- ▷ tropical shortwaves ... equatorial- β , but longwave in midlatitudes

Continuous Rossby Spectrum

- ▷ midlatitude shortwaves, all have critical latitudes
- ▷ zonal phase speed lies in (angular) wind band:

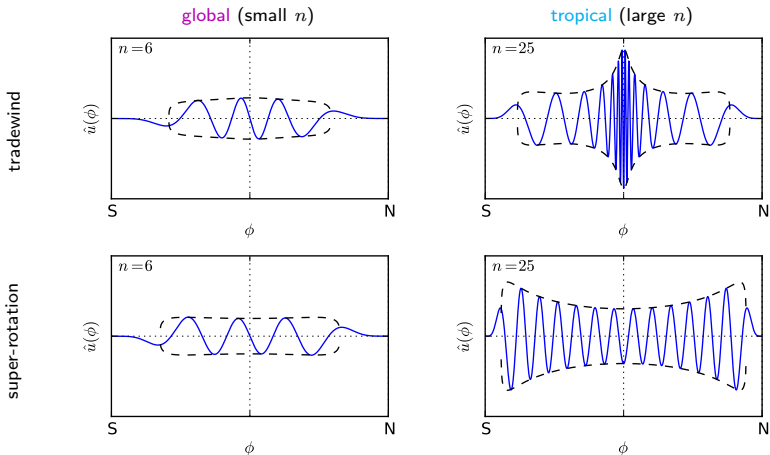
$$\left(\frac{\bar{u}(\phi)}{\cos \phi} \right)_{\min} < \frac{\omega}{m} < \left(\frac{\bar{u}(\phi)}{\cos \phi} \right)_{\max}$$

Discrete Eigenmode, Meridional Rossby Wave ($m = 6$)



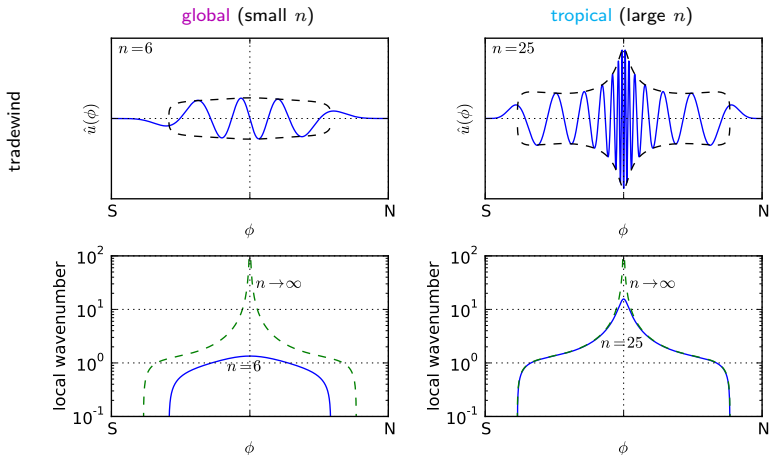
- ▷ global planetary-scale wave, relatively uniform wavelength
- ▷ tropical shortwave, extremely latitude-dependent wavelength

Discrete Eigenmode, Meridional Rossby Wave ($m = 6$)



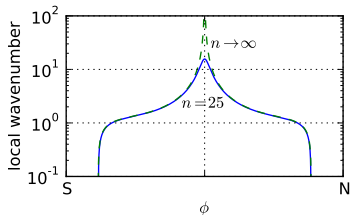
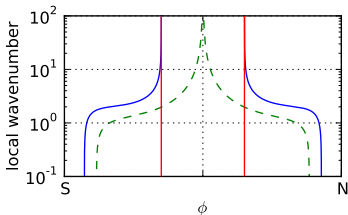
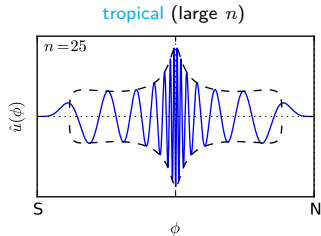
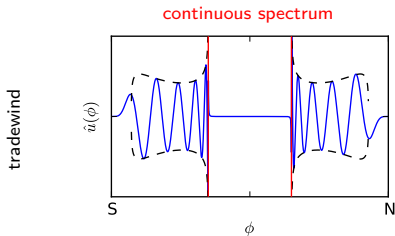
- ▷ **global planetary-scale wave**, similar to spherical harmonic, Hough mode, etc
- ▷ **tropical shortwave** ... but longwave in midlatitudes (not in equatorial- β theory)

Discrete Eigenmode, WKB Analysis ($m = 6$)



- ▷ WKB analysis of tropical shortwave, $n \rightarrow \infty$:
 - ▷ maximum limit of wavenumber in midlatitudes, unbounded at equator

Continuous Singular Mode, WKB Analysis ($m = 6$)



- ▷ **midlatitude shortwave**: wavescales shorter than discrete modes
- ▷ exclusion of waves from tropics by **critical latitude** (meridional group velocity $\rightarrow 0$)

WKB Analysis

Oscillatory Asymptotics

- ▷ introduce fast meridional phase $\rho(\phi)$ with $\epsilon\rho'(\phi) = O(1)$:

$$\begin{pmatrix} u \\ v \\ h \end{pmatrix} = \begin{pmatrix} \bar{u}(\phi) \\ 0 \\ \bar{h}(\phi) \end{pmatrix} + \vec{A}(\phi) e^{i\rho(\phi)} e^{i(m\lambda - \omega t)}$$

- ▷ local dispersion relation $\bar{\omega}(\phi) = \bar{\omega}(m, \rho')$ with generalized wavenumber $\rho'(\phi)$:

$$\frac{\bar{\omega}}{m} \left(\frac{(\epsilon m)^2}{\cos^2 \phi} + (\epsilon \rho')^2 + \frac{\sin^2 \phi}{\bar{h}} \right) + \frac{\bar{h}}{\cos \phi} \left(\frac{\sin \phi}{\bar{h}} \right)_{\phi} = 0$$

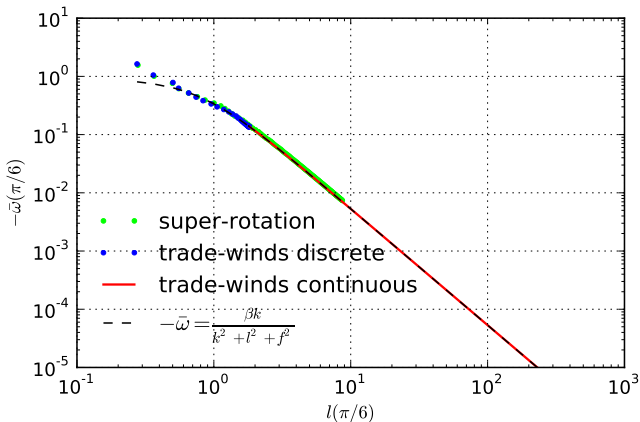
- ▷ contains key features of spectrum:
 - ▷ turning points: equatorial-trapping & infinite number of discrete eigenmodes
 - ▷ critical latitudes: singular point & continuous modes
- ▷ midlatitude scalings ...

$$\bar{\omega} \rightarrow \epsilon \bar{\omega} \quad ; \quad \frac{\epsilon m}{\cos \phi} \rightarrow k \quad ; \quad \epsilon \rho' \rightarrow l \quad ; \quad \frac{\sin^2 \phi}{\bar{h}} \rightarrow f^2$$

- ▷ ... recover midlatitude Rossby wave dynamics, Hoskins/Karoly (1981)

$$\omega \approx Uk - \frac{\beta k}{k^2 + l^2 + f^2}$$

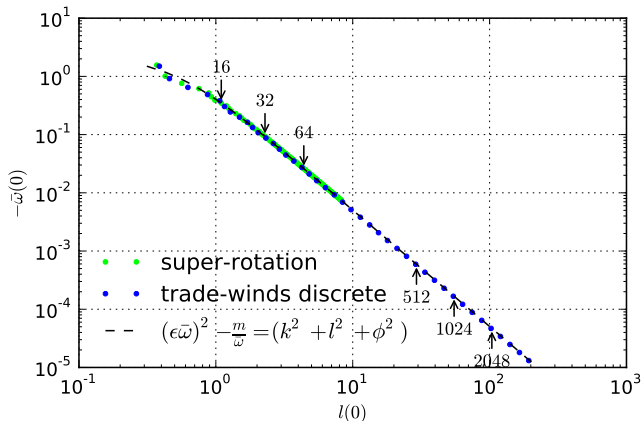
Local Dynamical Equivalence ($m = 6$)



Midlatitude β -Plane Theory ($\phi = \pi/6$)

- ▷ equivalent dispersion relation: $\bar{\omega}_n(m)$ as function of local wavenumber
- ▷ super-rotation & tropical shortwave + midlatitude shortwave dynamics coincide

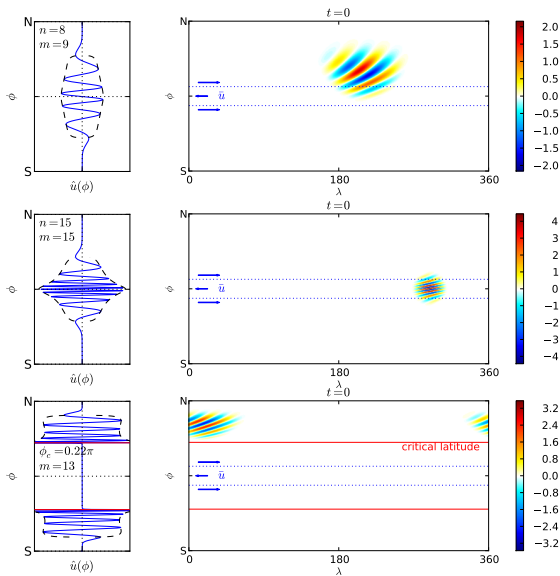
Local Dynamical Equivalence ($m = 6$)



Equatorial β -Plane Theory ($\phi = 0, \epsilon = 0.084 \ll 1$)

- ▷ equivalent dispersion relation: $\bar{\omega}_n(m)$ as function of local wavenumber
- ▷ super-rotation & **tropical shortwave** dynamics coincide

Rossby Wavepacket Propagation



global

planetary scale wave
crosses tropics

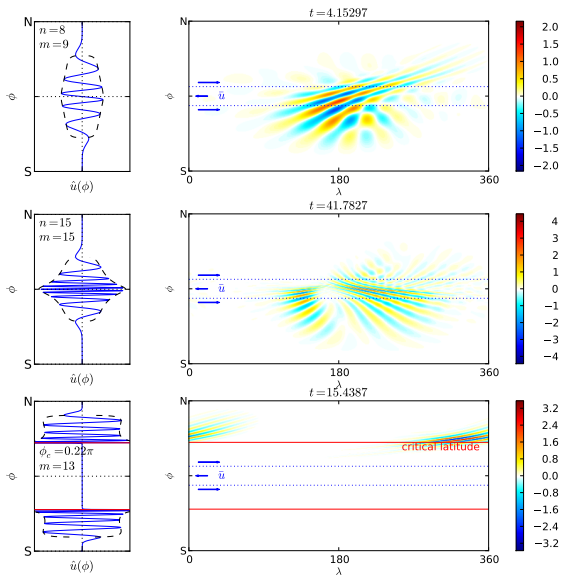
tropical

tropical shortwave
midlatitude longwave

midlatitude

critical latitude
exclusion from tropics

Rossby Wavepacket Propagation



global

planetary scale wave
crosses tropics

tropical

tropical shortwave
midlatitude longwave

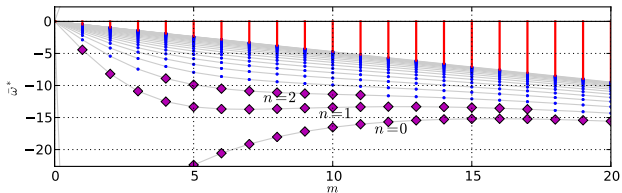
midlatitude

critical latitude
exclusion from tropics

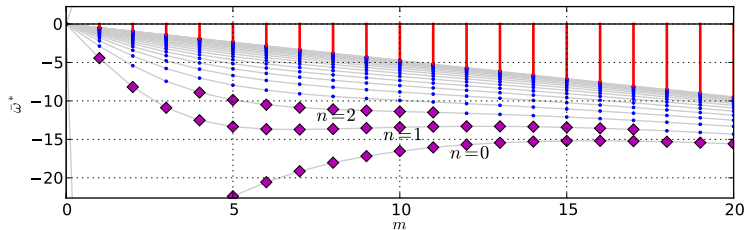
Rossby Wave Modes for rSW on the Sphere

Midlatitude, Tropical & Global Waves

- ▷ dynamics consistent with local β -plane, $\omega(k, l)$
- ▷ discrete spectrum accounts for equatorial-crossing waves (+ numerical quirks)
- ▷ continuous spectrum accounts for critical latitude behavior



Global Modes



Which Global Modes are Identifiable by Spherical Harmonics?

- ▷ global planetary-scale waves, accounted for in Madden (2007)?
- ▷ which discrete modes have a unique index correspondence with a spherical harmonic?
 - ▷ only longest meridional wavelengths due to extreme shortwave behavior
 - ▷ fewer matched modes with increasing ϵ