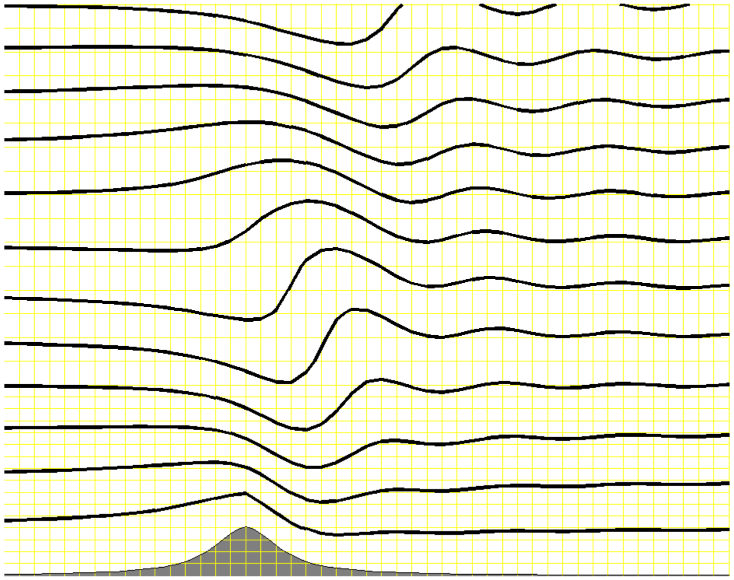
# Waves Generated by Airflow over a Mountain Ridge

- ▷ Long's 1953 theory for steady, density-stratified flow over obstacles
- ▷ integral equation approaches for a Helmholtz problem



http://www.fridgeproductions.pwp.blueyonder.co.uk/

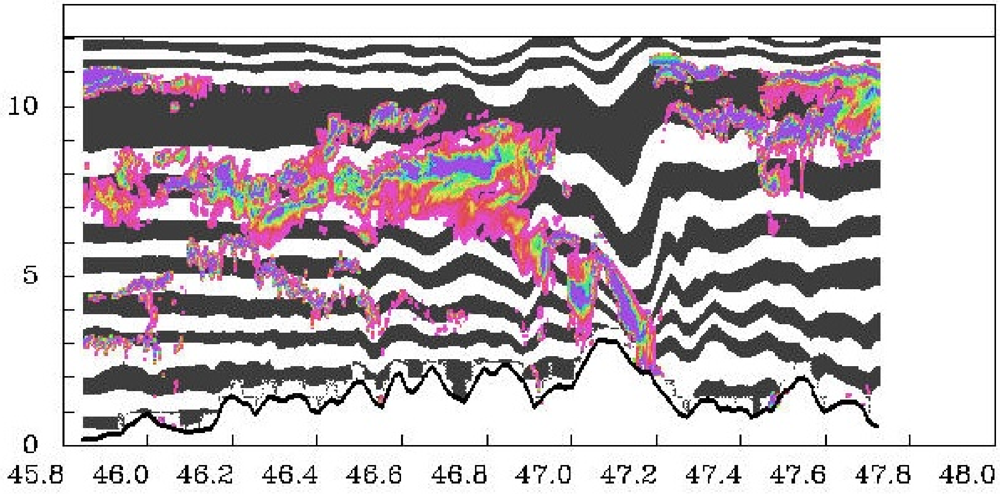

- ▷ Dave Muraki, Simon Fraser University
- ▷ Youngsuk Lee & David Alexander, SFU
- ▷ Craig Epifanio, Texas A&M

# Topographic Gravity Waves

## Atmospheric Concerns


http://www.casa.gov.au/avreg/fsa/download/02jan/ATSB.pdf


http://users.snowcrest.net/weshawk/LayeredLentic.jpeg




Volkert, et.al. 2002

- ▷ mathematical story: idealized steady 2D flows & their stability

# Atmospheric Fluid Dynamics

## Fluid Dynamics & Thermodynamics

- incompressible 2D Euler equations with Boussinesq buoyancy

$$
\begin{aligned}
u_x + w_z &= 0 \\
\frac{Du}{Dt} &= -\phi_x \\
\frac{Dw}{Dt} - \left(\frac{g}{\theta_r}\right)\theta &= -\phi_z \\
\frac{D\theta}{Dt} &= 0
\end{aligned}
$$

- potential temperature, $\theta$ (cold/heavy ; warm/light) & geopotential, $\phi$ (pressure)
- advection: $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\,\dfrac{\partial}{\partial x} + w\,\dfrac{\partial}{\partial z}$

## Stratified Atmosphere

- buoyancy disturbance from constant stratification: $\left(\dfrac{g}{\theta_r}\right)\theta = \mathcal{N}^2 z + b$
  - hydrostatic balance: $\phi = \mathcal{N}^2 z^2/2 + \phi$
  - $\mathcal{N}$, Brunt-Väisälä frequency $\longrightarrow$ gravity waves via buoyancy oscillations

# Stably Stratified Flow

$$
\begin{aligned}
u_x + w_z &= 0 \\
\frac{Du}{Dt} &= -\phi_x \\
\frac{Dw}{Dt} - b &= -\phi_z \\
\left(\frac{g}{\theta_r}\right)\frac{D\theta}{Dt} = \frac{Db}{Dt} + \mathcal{N}^2 w &= 0
\end{aligned}
$$

## Streamfunction & Vorticity

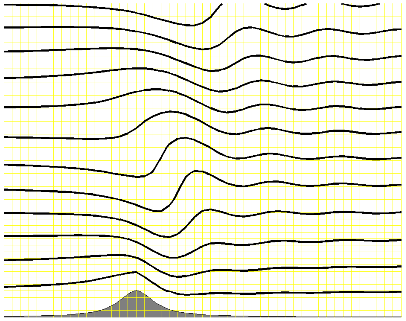
- streamfunction, $\Psi \quad \longrightarrow \quad u = \Psi_z \quad ; \quad w = -\Psi_x$
- vorticity, $\eta \quad \longrightarrow \quad \eta = u_z - w_x = \nabla^2 \Psi$

- vorticity/buoyancy formulation

$$
\begin{aligned}
\frac{D\eta}{Dt} + b_x &= \eta_t + J(\eta, \Psi) + b_x = 0 \\
\frac{Db}{Dt} - \mathcal{N}^2 \Psi_x &= b_t + J(b, \Psi) - \mathcal{N}^2 \Psi_x = 0
\end{aligned}
$$

- Jacobian determinant: $J(f, g) = f_x g_z - f_z g_x$

# Long's 1953 Theory



## Steady Flow

- buoyancy equation

$$J(b, \Psi) - \mathcal{N}^2 \Psi_x = J(\mathcal{N}^2 z + b, \Psi) = J(\theta, \Psi) = 0$$

  - $\theta$ is constant (isentropic) along steady streamlines
  - simplest linear correspondence: $\Psi = \dfrac{\mathcal{U}}{\mathcal{N}^2}\theta = \mathcal{U}\, z + \dfrac{\mathcal{U}}{\mathcal{N}^2} b = \mathcal{U} z + \psi$
  - uniform incident wind and streamfunction disturbance: $\psi = \dfrac{\mathcal{U}}{\mathcal{N}^2} b$

- vorticity equation ($b_x = \Psi_x$)

$$J(\eta, \Psi) + \frac{\mathcal{N}^2}{\mathcal{U}} \Psi_x = J\left(\nabla^2 \psi - \frac{\mathcal{N}^2}{\mathcal{U}} z, \mathcal{U} z + \psi\right) = 0$$

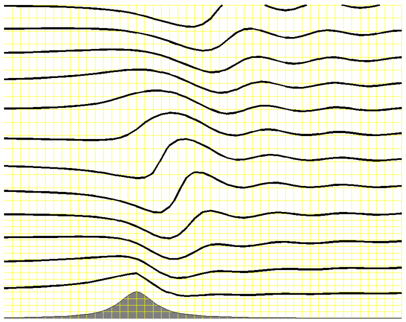
## Long's Theory

- linear Helmholtz equation for steady 2D streamfunction, $\psi(x, z)$

$$\nabla^2 \psi + \left(\frac{\mathcal{N}}{\mathcal{U}}\right)^2 \psi = 0$$

- exact nonlinear solution for constant stratification & uniform incident wind

## Scaling & Nondimensionalization



### Scaling

- ▷ simple topographic case: three length scales

  $L$, mountain width ; $H$, mountain height ; $\mathcal{U}/\mathcal{N}$, wave scale

- ▷ two dimensionless parameters

  $\sigma \equiv \frac{\mathcal{U}}{\mathcal{N}L}$, nonhydrostatic parameter ; $\mathcal{A} \equiv \frac{\mathcal{N}H}{\mathcal{U}}$, height parameter

- ▷ scalings: $x \sim L$ ; $z \sim \mathcal{U}/\mathcal{N}$ ; $\psi \sim \mathcal{U}^2/\mathcal{N}$ ; $b \sim \mathcal{U}\mathcal{N}$

### Nondimensionalization

- ▷ Helmholtz equation ($\sigma \longrightarrow 0$, hydrostatic case)

  $$\sigma^2\psi_{xx} + \psi_{zz} + \psi = 0$$

- ▷ fields: $u = 1 + \psi_z$ ; $w = -\sigma\psi_x$ ; $\Psi = \theta = z + \psi$

### Surface Boundary Condition at $z = \mathcal{A}h(x)$

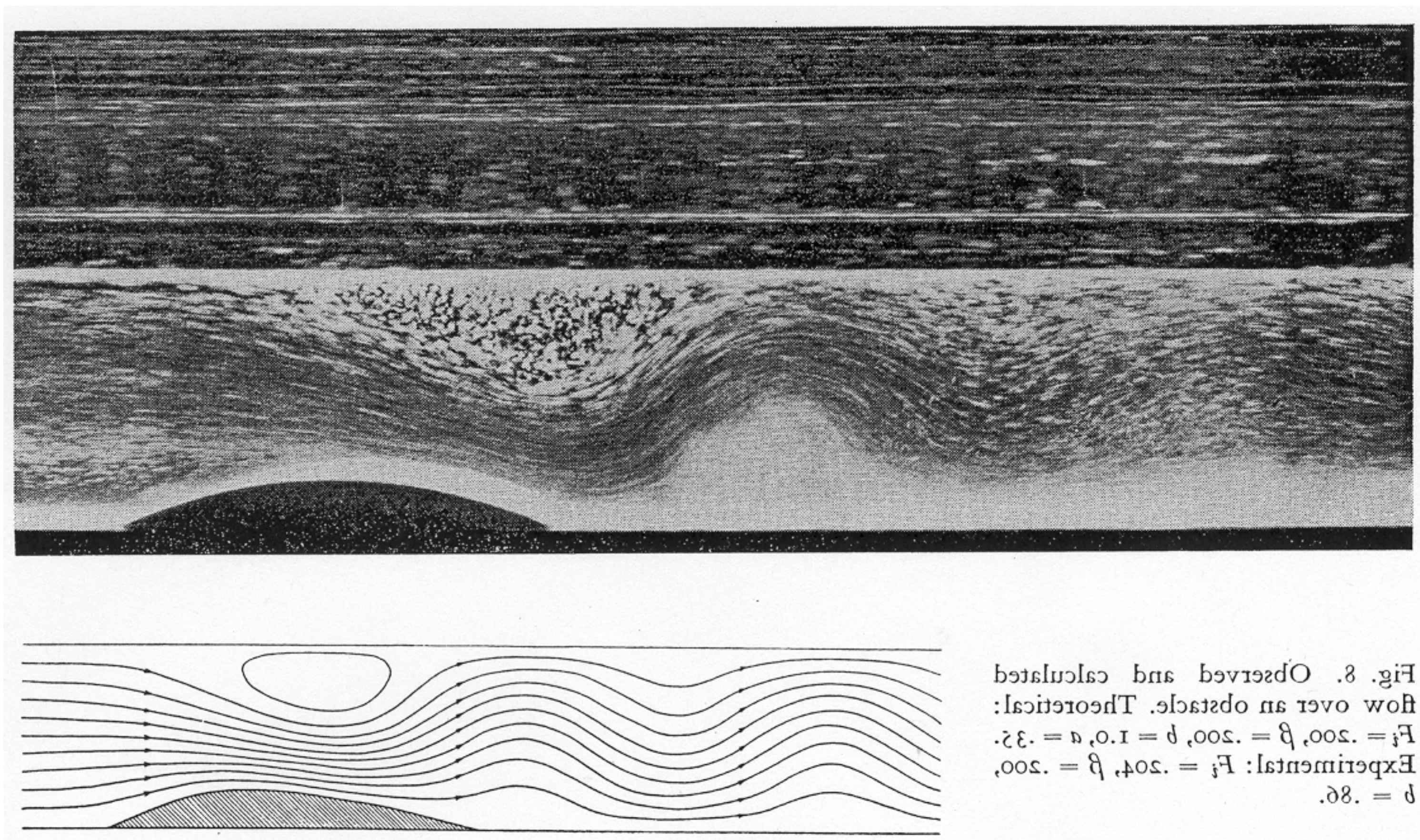
- ▷ zero total streamfunction: $\Psi(x, \mathcal{A}h(x)) = \mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = 0$

## Long 1955: Theory & Experiment

$$\sigma^2 \psi_{xx} + \psi_{zz} + \psi = 0$$

### Finite Amplitude Topography

- ▷ on streamline boundaries: $\psi = \mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = \text{constant}$




Long 1953

## A Fourier Approach

### Fourier Modes, $e^{i(kx+mz)}$

- steady dispersion relation: $m^2 = 1 - \sigma^2 k^2$
- sign choice $\longrightarrow$ far-field conditions: upward group velocity & decay (Lyra, 1940)

$$m(k) = \begin{cases} \mathrm{sign}(k)\,\sqrt{1-\sigma^2k^2} & \text{for } |\sigma k| \leq 1 \text{ (long scale radiation)} \\ i\,\sqrt{\sigma^2k^2-1} & \text{for } |\sigma k| \geq 1 \text{ (short scale decay)} \end{cases}$$

### General Helmholtz Solution

- Fourier integral representation satisfying far-field

$$\psi(x,z) \;=\; -\mathcal{A}\int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)z)}\; dk$$

- $z = \mathcal{A}h(x)$ surface condition: $\mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = 0$

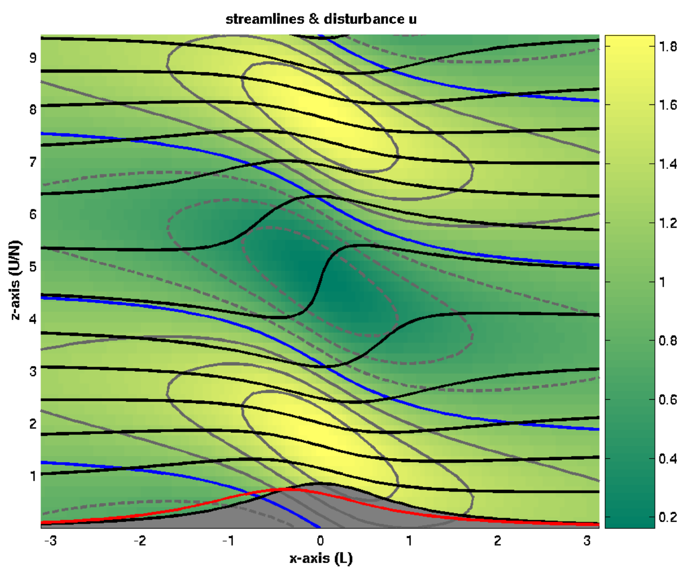
$$\boxed{h(x) \;-\; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)\mathcal{A}h(x))}\; dk \;=\; 0}$$

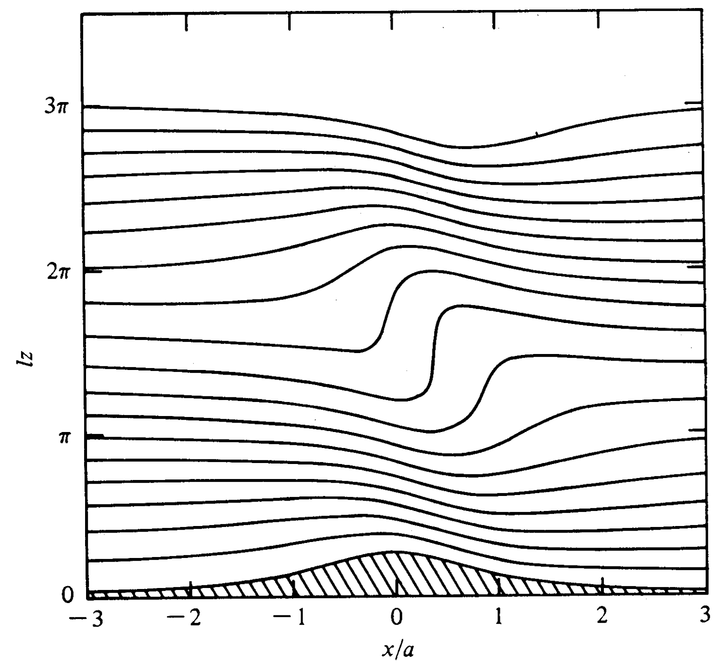
- linear integral operator on $\hat{c}(k)$ $\longrightarrow$ <u>Fredholm integral equation of first-kind</u>
- numerically equivalent to a matrix inversion

$$h(x) \;-\; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{ikx}\; dk \;=\; 0$$

$\mathcal{A}\!\rightarrow\!0$ "Linear" Limit is Fourier Inversion: $\hat{c}(k) = \hat{h}(k)$

- ▷ hydrostatic ($\sigma = 0$), critical overturning case ($\mathcal{A} = 0.85$); computed via FFTs




Lilly/Klemp 1979

- ▷ bottom (zero) streamline does not match topographic surface, $h(x) = 1/(1 + x^2)$
- ▷ FFT-based iterative solvers for exact surface condition (Raymond, 1972; Laprise et al, 1988)

# Direct Steady Solve

$$h(x) \; - \; \int_{-\infty}^{+\infty} \hat{c}(k) \; e^{i(kx + m(k) \mathcal{A} h(x))} \; dk \; = \; 0$$

## Numerical Discretization

- collocation points: $\{x_1 \; \ldots \; x_\alpha \; \ldots \; x_N\}$ & $N$ knowns: $h_\alpha = h(x_\alpha)$
- wavenumbers: $\{k_1 \; \ldots \; k_\beta \; \ldots \; k_N\}$ & $N$ unknowns: $\hat{c}_\beta \approx \hat{c}(k_\beta)$
- approximate integral at each $x_\alpha$ by quadrature (trapezoidal rule) over $\beta = 1 \ldots N$

$$h_\alpha \; - \; \sum_{\beta=1}^{N} \hat{c}_\beta \; \underbrace{e^{i(k_\beta x_\alpha + m(k_\beta) \mathcal{A} h(x_\alpha))} \; w_\beta \, \Delta k}_{\mathbf{K}_{\alpha,\beta}} \; = \; 0$$

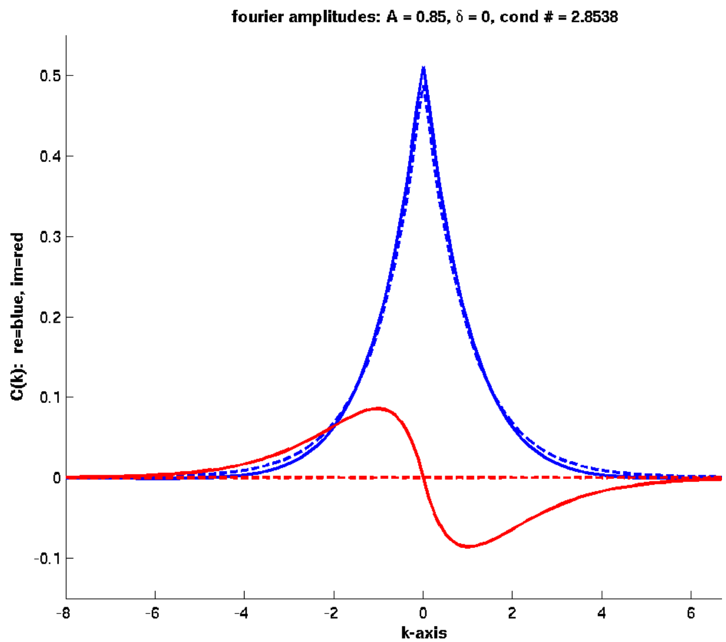
## Matrix Inversion

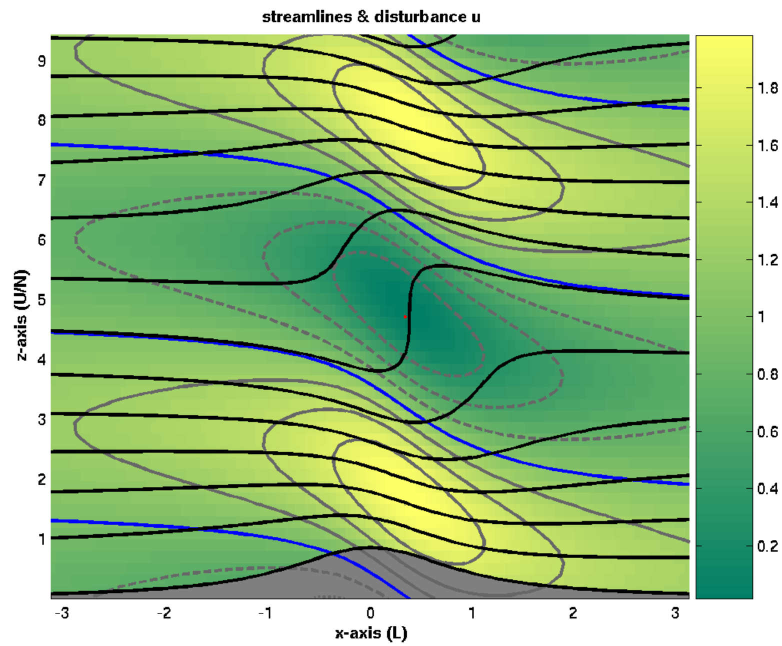
- $N$ linear equations in $N$ unknowns: $(\vec{h_\alpha}) = [\mathbf{K}_{\alpha,\beta}] \; (\vec{c_\beta})$
- $m(k)$ is discontinuous at $k = 0 \longrightarrow$ half-line integrals
- full matrix $\mathbf{K}$ can be ill-conditioned $\longrightarrow$ catastrophic loss of precision as $N$ increases

# Numerical Implementation

## Fourier Conditioning

- for $\mathcal{A} = 0$ linear theory, discrete Fourier transform is well-conditioned
- equi-spaced discretizations with $\Delta k \, \Delta x = 2\pi/N$ is essential



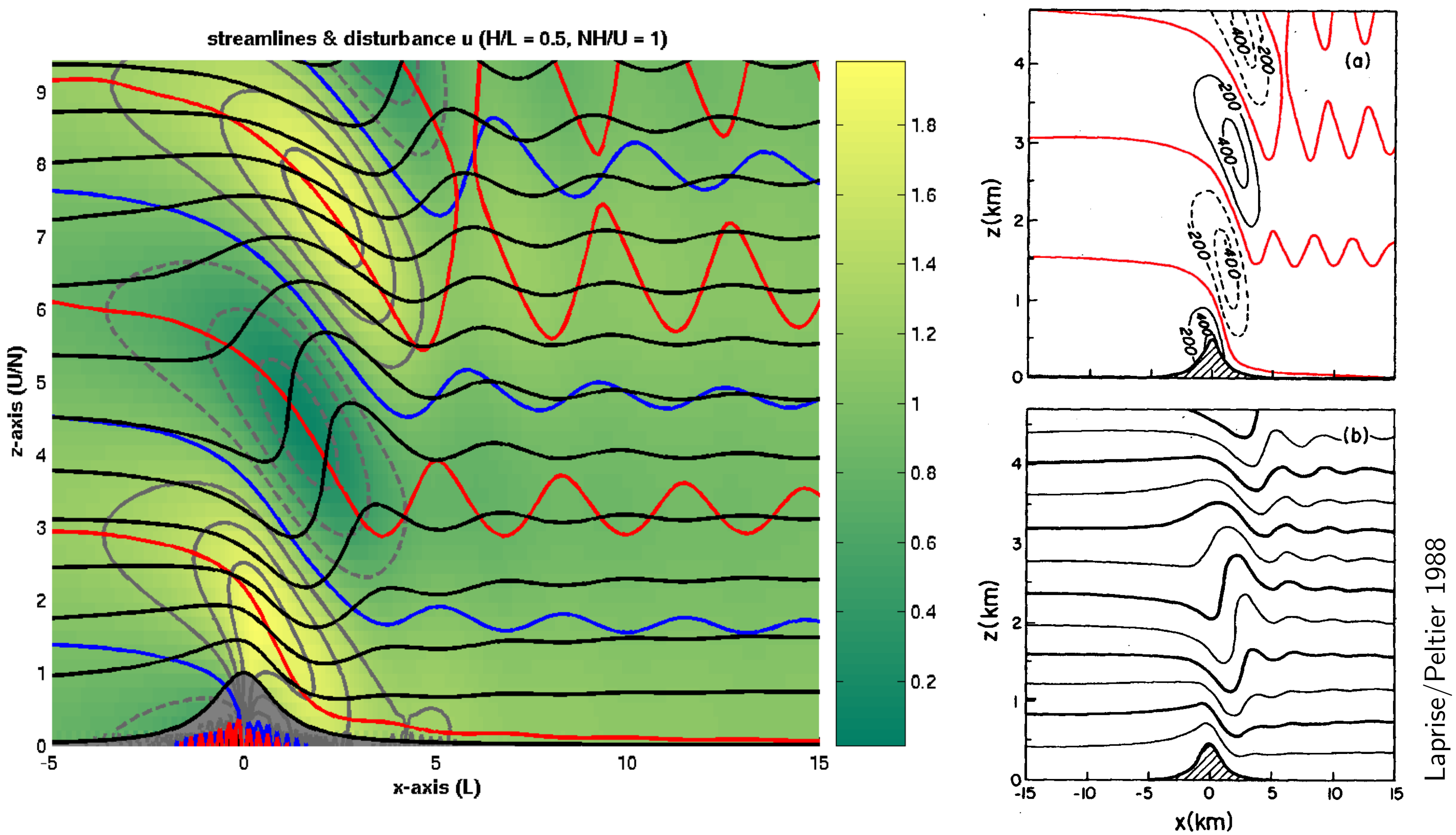




- hydrostatic critical overturning case (Lilly/Klemp 1979)
  - $N = 256, x_\infty = 8\pi$: 1.1s to solve & 2.0s to plot, log-condition number = 2.85
- Fourier representation allows periodic wrap-around $\longrightarrow$ large computational domains

# A Nonhydrostatic Example

## Laprise & Peltier, 1988

- predictor/corrector to obtain effective topography $c(x) \longrightarrow$ typically $50$ iterations

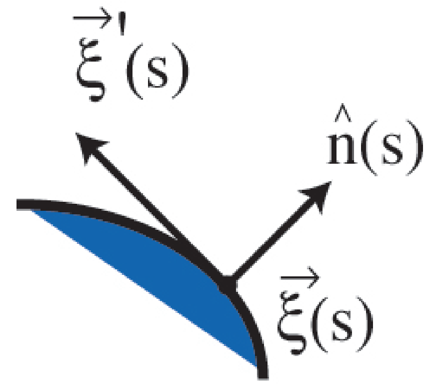



- large amplitude $\mathcal{A} = 1.0$ & moderately nonhydrostatic $\sigma = 0.5$
  - $N = 2048$, $x_\infty = 128$: 284s to solve, 89s to plot, log-condition number = 5.75

$$\mathcal{G}_{xx} + \mathcal{G}_{zz} + \mathcal{G} = \delta(\vec{x} - \vec{\xi})$$

## Helmholtz Free-Space Green's Function ($\sigma = 1$)

- radiating solution for a delta-function source at $\vec{\xi}$: $\mathcal{G}(\vec{x} - \vec{\xi})$
- classical, time-harmonic scattering problem in electromagnetics/acoustics
  - delta-function response in 2D involves Hankel functions: $J_0(r) \pm i\, Y_0(r)$
  - sign choice determined by far-field radiation condition (implied by time-harmonicity)




## General Surface Scattering Solution

- distribution of weighted, $\mu(s)$, Green's functions
- $\vec{\xi}(s)$, parametrization of surface boundary (clockwise)

$$\psi(\vec{x}) = -\mathcal{A} \int_{\mathcal{S}} \mu(s) \;\; 2\, \frac{\partial \mathcal{G}}{\partial n}(\vec{x} - \vec{\xi}(s))\; ds$$

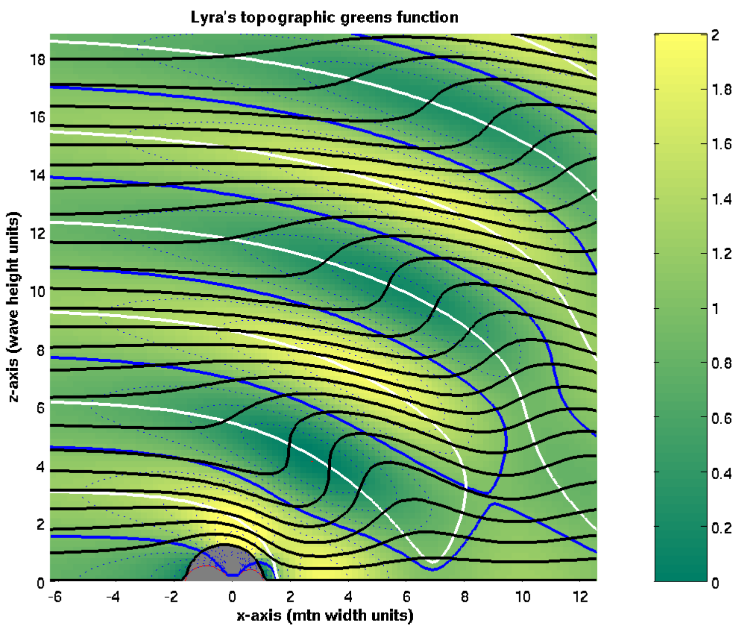
- need topographic Green's function $\mathcal{G}(\vec{x} - \vec{\xi})$ & weights $\mu(s)$

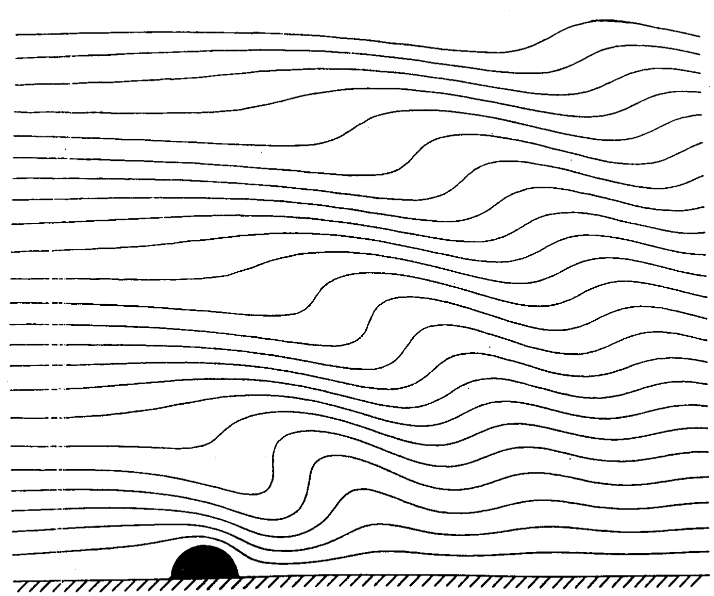
## Delta-Function Topography (linear theory)

- from Lyra 1940 & 1943 (via Alaka 1960) for $\sigma = 1$: Bessel series

$$\mathcal{G}_z(r,\theta) \;=\; \frac{1}{2}\,Y_1(r)\sin\theta \;+\; \frac{1}{\pi}\sum_{1}^{\infty}\,\frac{4n}{4n^2-1}\,J_{2n}(r)\sin 2n\theta$$

- critical overturning, $\Psi = z + (4.06)\mathcal{G}_z(r,\theta)$



Miles/Huppert 1968

- left/right asymmetric Greens function: waves must be downstream (Miles/Huppert 1968)

# Fredholm Integral Equation of Second-Kind

## Singular Integral

- singular integrand for points on surface: $\vec{x}_{\mathcal{S}}$

$$\psi(\vec{x}_{\mathcal{S}}) = -\mathcal{A}\ \mu(\vec{x}_{\mathcal{S}}) - \mathcal{A}\int_{\mathcal{S}} \mu(s)\ \ 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x}_{\mathcal{S}} - \vec{\xi}\,(s))\ ds$$

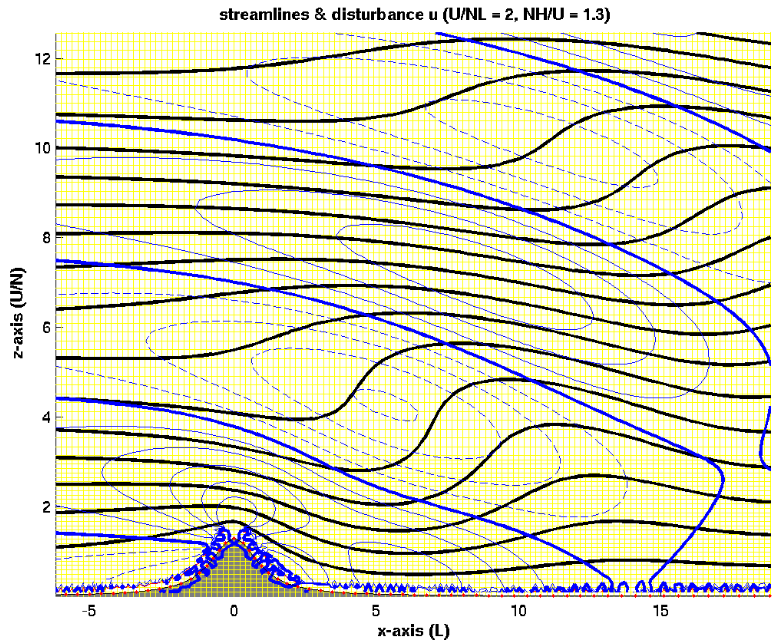
- surface boundary condition $\longrightarrow$ second-kind integral equation for $\mu(\vec{x}_{\mathcal{S}})$

$$\boxed{\mu(\vec{x}_{\mathcal{S}}) + \int_{\mathcal{S}} \mu(s)\ \ 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x}_{\mathcal{S}} - \vec{\xi}\,(s))\ ds \quad = \quad h(\vec{x}_{\mathcal{S}})}$$

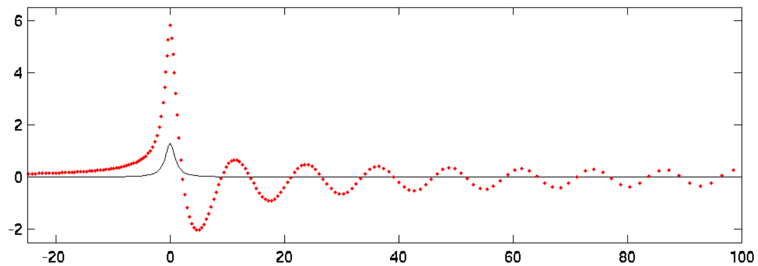
- kernel function is continuous at $\vec{x}_{\mathcal{S}} = \vec{\xi}\,(s)$, curvature contribution
- discretized quadrature gives diagonally-dominant matrix $\longrightarrow$ well-conditioned inversion
- amplitude parameter, $\mathcal{A}$, enters through surface parametrization: $\vec{\xi}\,(s) = \begin{pmatrix} x(s) \\ \mathcal{A}h(x(s)) \end{pmatrix}$
- small $\mathcal{A}$ limit: $\mu(\vec{x}_{\mathcal{S}}) \longrightarrow h(\vec{x}_{\mathcal{S}})$
- nonhydrostatic parameter, $\sigma$, handled by rescaling in $x$
- velocities obtained by differentiation: $u = 1 + \psi_z$ & $w = -\sigma\psi_x$

# A Strongly Nonhydrostatic Example

$\sigma = 2.0$ & Large Amplitude, $\mathcal{A} = 1.3$




- ▷ $N = 256$: 68s to solve, forever to plot, log-condition number = 2.24
- ▷ near-surface evaluations involve nearly-singular integrations



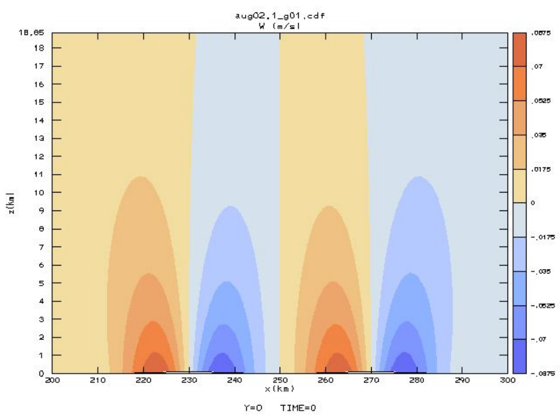
# Question of Stability

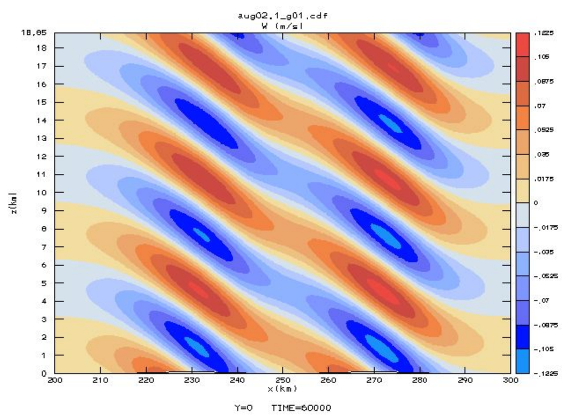
## Gravity Wave Instability

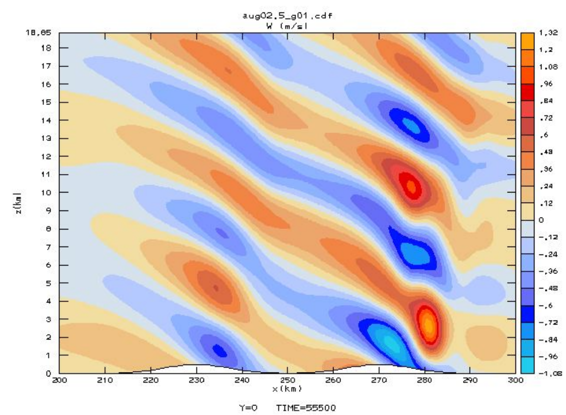
- Mied (1976), plane gravity waves are parametrically unstable
- Lilly/Klemp (1979), instability observed for sinusoidal topography
- Scinocca/Peltier (1994), unstable dynamics from critical overturning

## Time-Dependent Flow (Craig Epifanio, Texas A&M)

- twin peaks, hydrostatic ($\sigma = 0$), initialized from potential flow
- low-amplitude & high-amplitude comparison



# Linear Stability of Long's Steady Solutions

## Hydrostatic ($\sigma = 0$) Disturbance Equations (David Alexander & Youngsuk Lee, SFU)

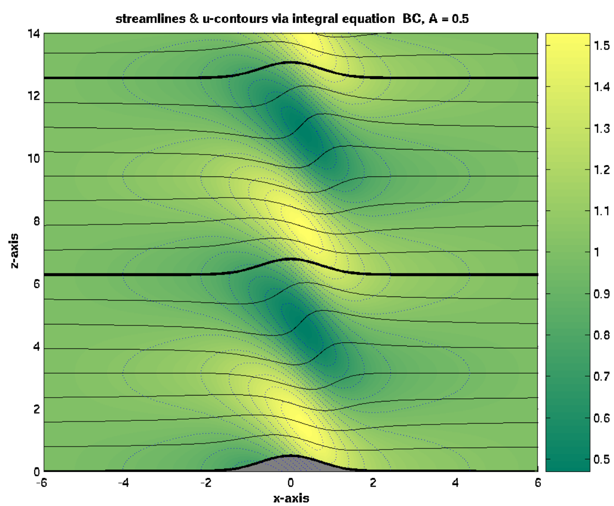
- non-constant coefficients from Long's streamfunction $\psi(x, z)$

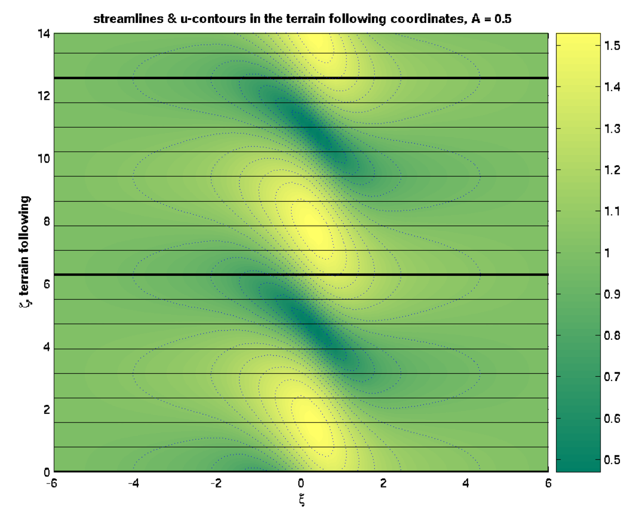
$$\begin{aligned} \tilde{\psi}_{zzt} - \tilde{\psi}_x + \tilde{\theta}_x &+ J(\tilde{\psi}_{zz} + \tilde{\psi}, z + \psi) = 0 \\ \tilde{\theta}_t &+ J(\tilde{\theta} - \tilde{\psi}, z + \psi) = 0 \end{aligned}$$

- 2D PDE eigenvalue problem for $\tilde{\psi} \rightarrow \tilde{\psi}(x, z)e^{\lambda t}$ & $\tilde{\theta} \rightarrow \tilde{\theta}(x, z)e^{\lambda t}$

## Numerical Linear Algebra

- isentropic coordinates for regular lower boundary
- self-adjoint formulation $\longrightarrow$ Arnoldi iterative search for eigenvalues

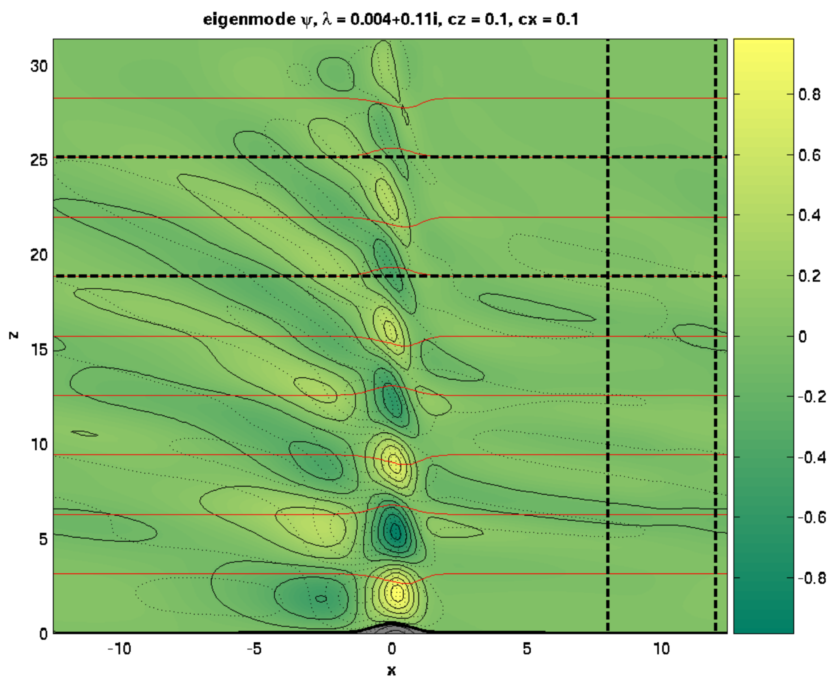


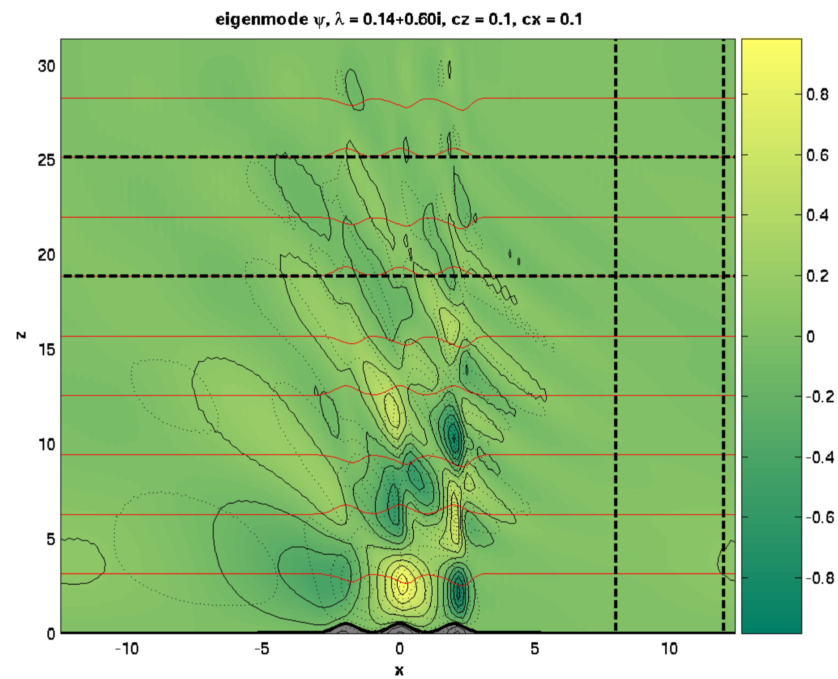



# Unstable Modes

## A Search for Eigenvalues

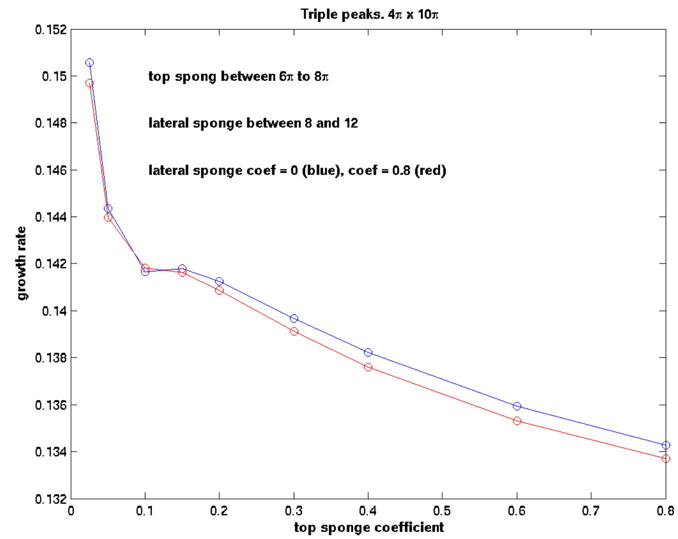
- ▷ one-, two- & three-peak configurations ($\mathcal{A} = 0.5$)
- ▷ most unstable eigenvalues:
$$\begin{aligned} \lambda_1 &\approx 0.004 + 0.11i \\ \lambda_2 &\approx 0.011 + 0.53i \\ \lambda_3 &\approx 0.142 + 0.61i \end{aligned}$$
- ▷ one- & 3-peak disturbance $\tilde{\psi}(x, z)$ eigenfunctions



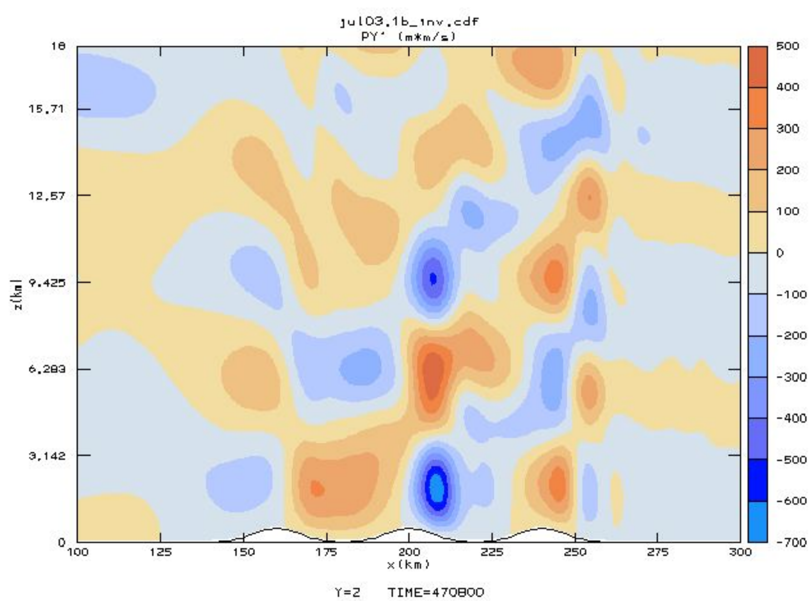



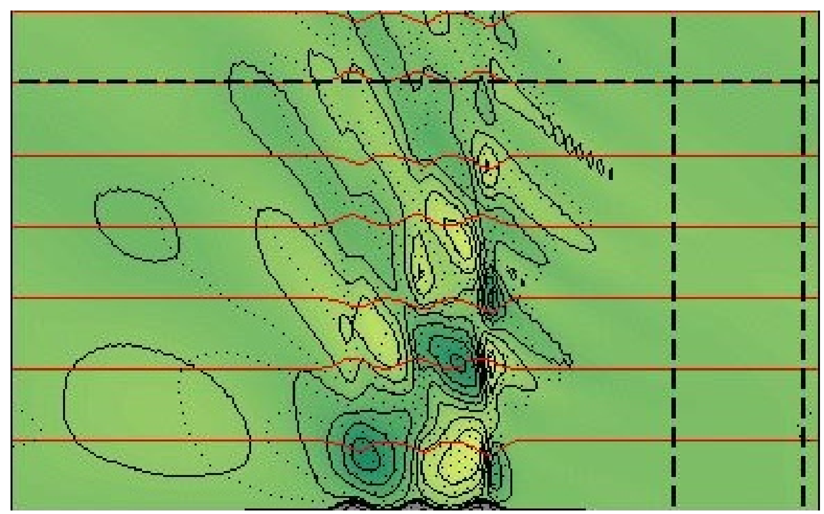
## The Black Art of Damping

- ▷ damping layers aloft & laterally




## Unstable Simulation

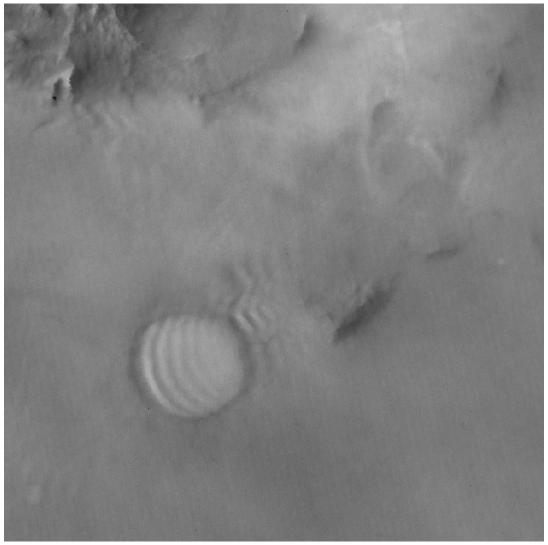


# In Closing

## Direct Steady Solve

- non-iterative formulations for exact topographic surface
  - Fourier-based 1st-kind solver: near-hydrostatic regime
  - Green's function-based 2nd-kind solver: hydrostatic regime (?)
- accurate solutions for linear stability analysis
- uniqueness issue $\longrightarrow$ zero eigenvalues?



## Linear Stability

- verify the existence of linear instabilities for multiply-peaked terrain
  - height & separation criterion for instability
  - mechanism for instability
- benchmark against time-dependent simulations

- implications for atmospheric wave drag estimates/parametrizations?