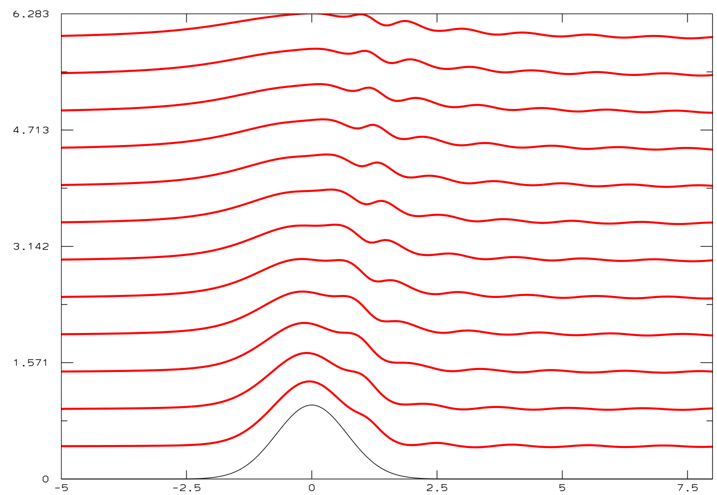
Nonlinear Topographic Wave Generation at Finite Rossby Number

- ▷ nonlinearity & rotational influences on wave generation
- ▷ a rotating version of Long's theory



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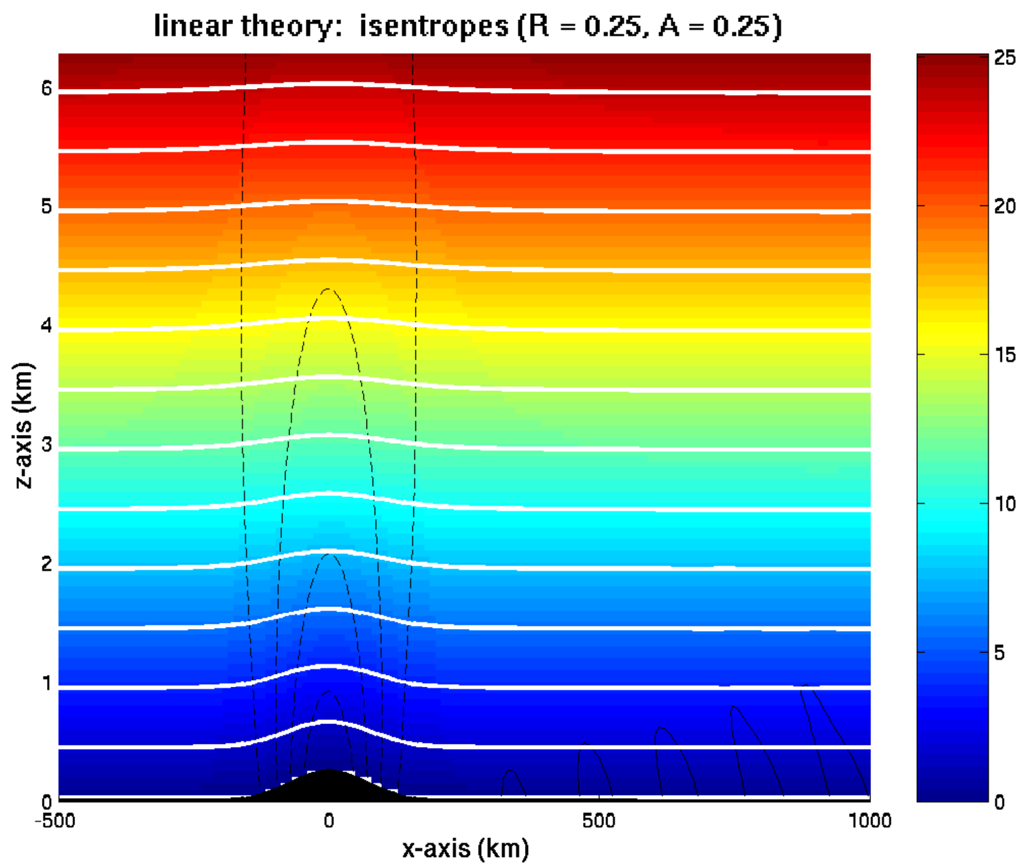
- ▷ Chris Snyder (NCAR Boulder)

Linear Theory: Tiny Rossby Number

Quasigeostrophic Flow Over A Ridge

- small height gaussian ridge ($\mathcal{A} = NH/U = 0.25$)
- predominantly balanced QG flow ($\mathcal{R} = U/fL = 0.25$)
- very weak wave anomalies near leeward surface (Pierrehumbert, 1984)



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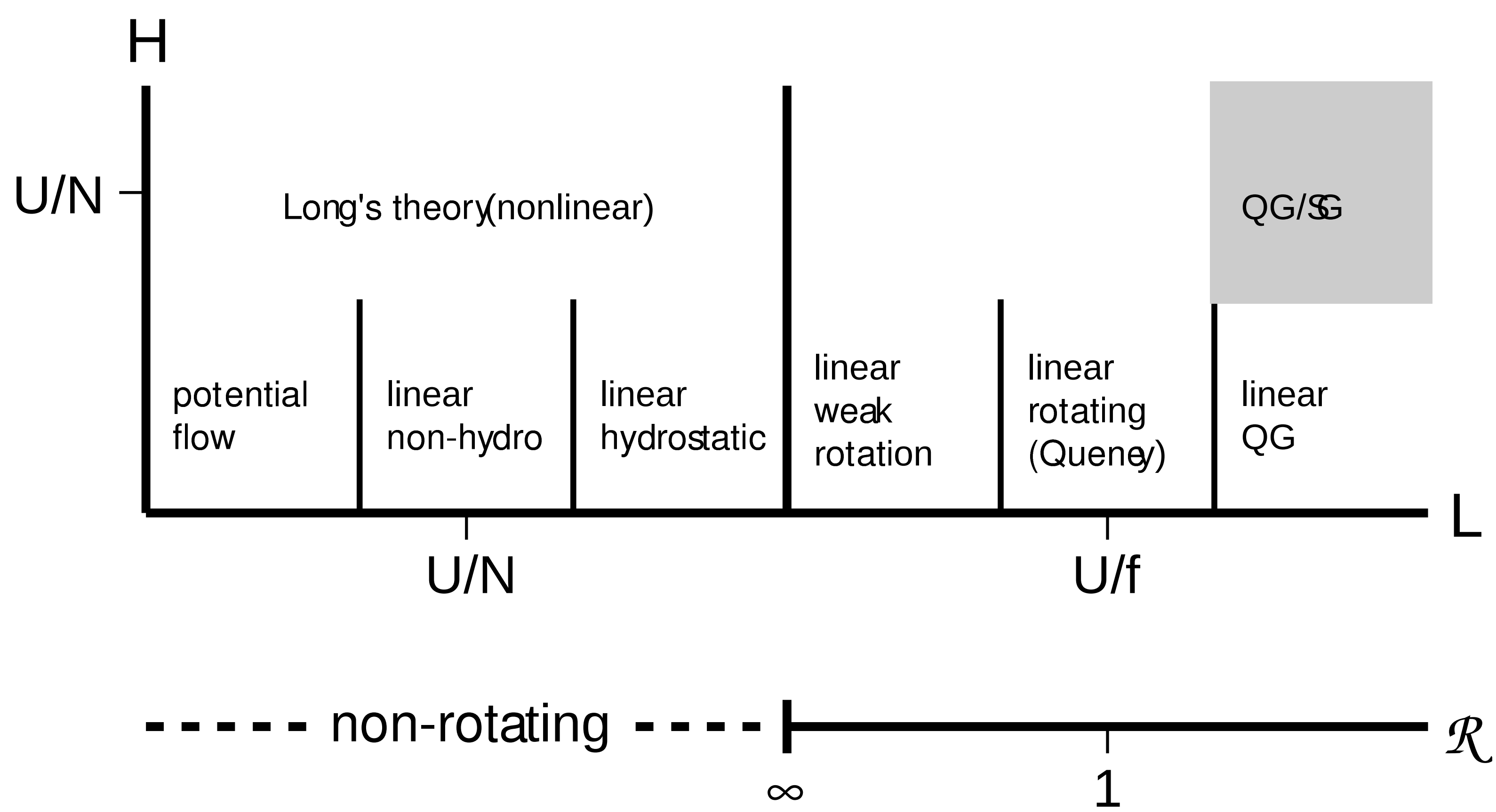
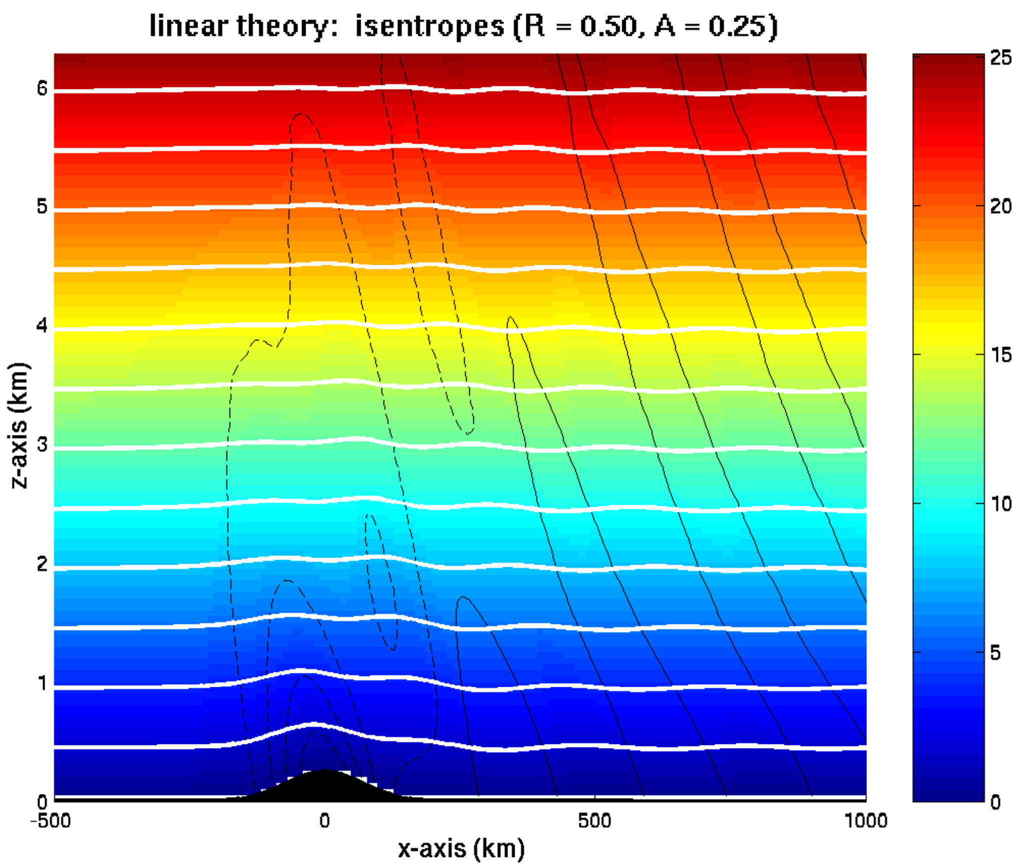
Regimes for 2D Steady Topographic Flows



Appearance of Waves

- ▷ steady uniform flow, constant stratification
- ▷ intermediate case: QG summit flow with short waves ($\mathcal{R} = 0.50$)
- ▷ development of downstream (dispersive) wavetrain



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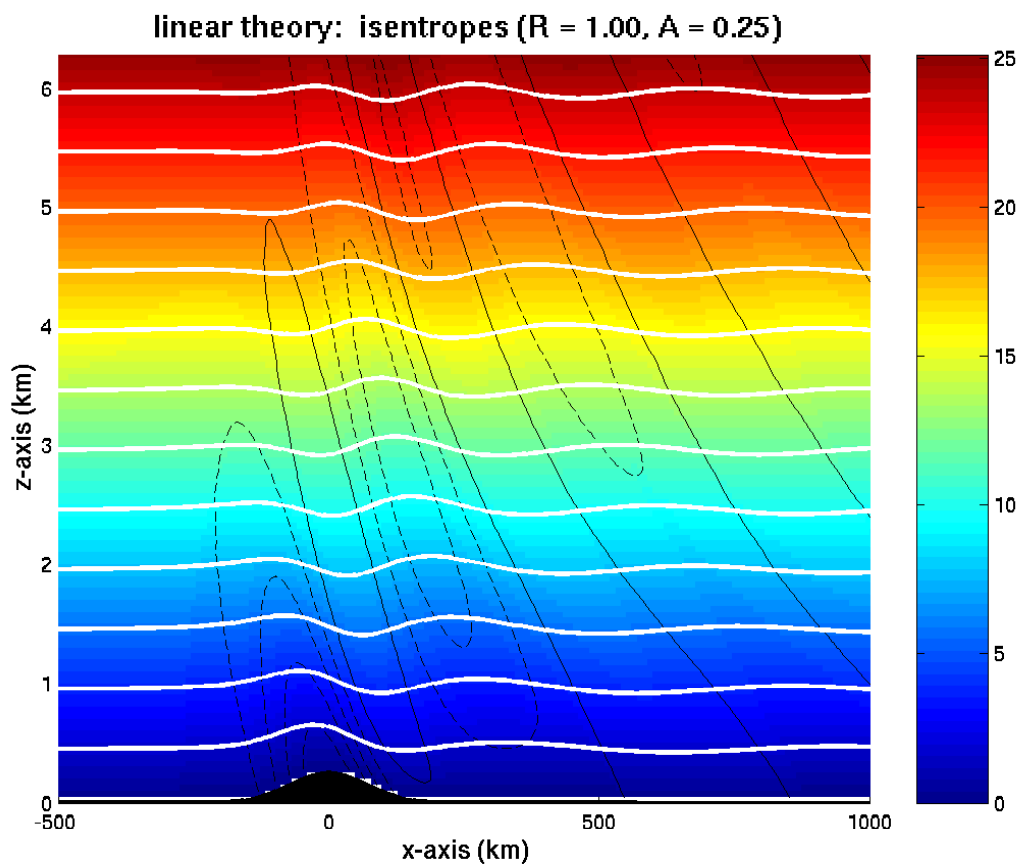
Linear Theory: Intermediate Rossby Number

Fully Developed Wave Field

- ▷ strong waves with similar scale to QG summit flow ($\mathcal{R} = 1.0$)
- ▷ significant wave radiation aloft



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- ▷ as $\mathcal{R} \nearrow$, waves grow in amplitude (exponentially) & wavelength (linearly)

Linear Theory: A Singular Numerical Problem

Fourier Integral Solution (Queney, 1947)

$$b(x,z) = -\frac{N^2}{\pi}\,\text{Real}\left\{\int_0^\infty \hat{h}(k)\; e^{ik\,x}\; e^{m(k)\,z}\; dk\right\}$$

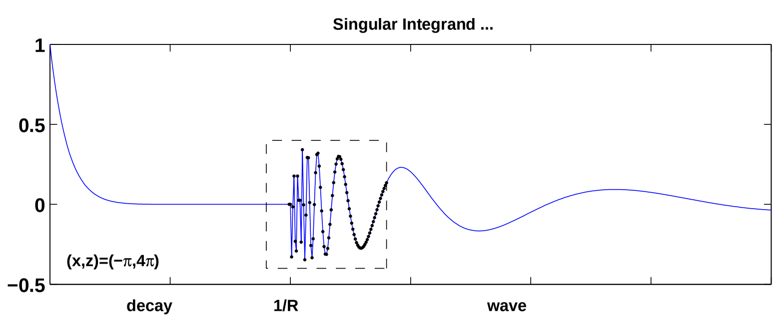
Buoyancy Anomaly

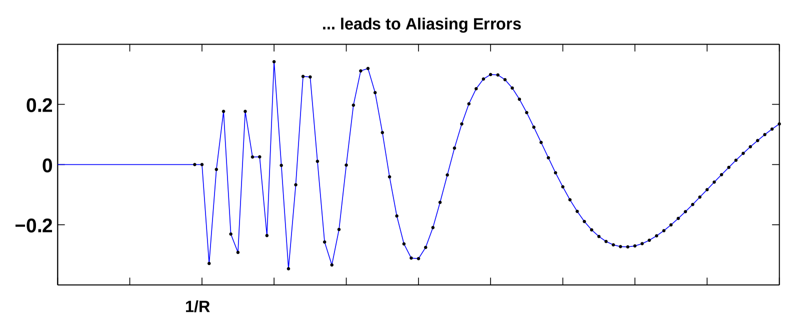
- linear waves with rotation, stratification & topography $h(x)$

$$\mathcal{A}^2\, b_{xx} + \mathcal{R}^{-2}\, b_{zz} + b_{xxzz} = 0 \quad ; \quad b(x,0) = -h(x)$$

- 2D linear dispersion relation gives a singular exponent at $k = \mathcal{R}^{-1}$

$$m(k) = \begin{cases} -\dfrac{\mathcal{A}\,k}{\sqrt{\mathcal{R}^{-2} - k^2}} & \text{for} \quad 0 \le k < \mathcal{R}^{-1} \quad \text{(vertical decay)} \\ i\,\dfrac{\mathcal{A}\,k}{\sqrt{\mathcal{R}^{-2} - k^2}} & \text{for} \quad \mathcal{R}^{-1} < k < \infty \quad \text{(outgoing waves)} \end{cases}$$





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- rotating wave case prone to severe numerical Fourier errors

Three Questions

a: Is There an Analog to Long's Theory that includes Coriolis Rotation?

- ▷ Long's theory (1953) for buoyancy anomaly
- ▷ steady, nonlinear & non-rotating flows are obtained exactly via linear Helmholtz solutions

  ⟶ no obvious extension to include rotation

b: What is the Nature of Pierrehumbert's Finite $\mathcal{R}$ Singularity?

- ▷ semi-geostrophic approximation: Pierrehumbert (1985)
- ▷ SG solutions have singular breakdown at finite Rossby number

  ⟶ a true finite amplitude flow transition, or merely a manifestation of SG approximation?

c: How can Waves be Generated at Small Rossby Number?

- ▷ Pierrehumbert/Wyman (1985) & Trüb/Davies (1995)
- ▷ wave generation by finite amplitude ridges at small $\mathcal{R}$
- ▷ relaxation of time-dependent flow computations

  ⟶ how does nonlinearity circumvent quasigeostrophic balance?

a: Long's Theory for Non-Rotating Topographic Waves

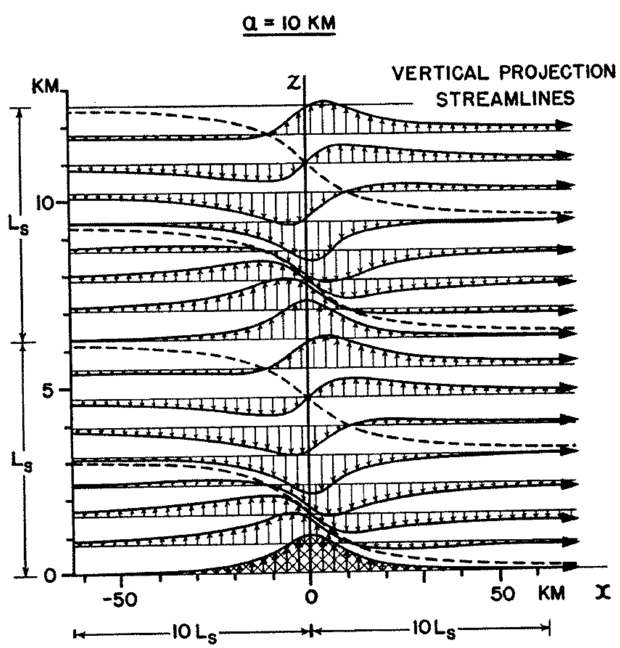
An Exact Nonlinear Theory for Buoyancy

- steady, non-rotating & hydrostatic/nonhydrostatic (Long, 1953)
- 2D helmholtz equation: stratified ($\mathcal{A} = NH/U$) & nonhydrostatic (δ^2)

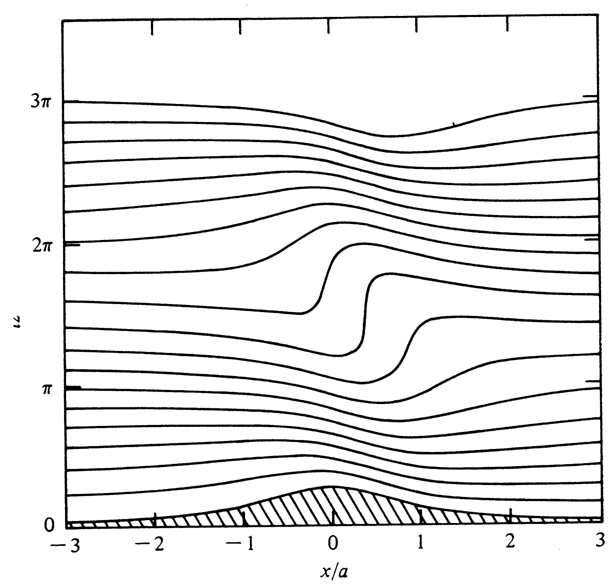
$$\mathcal{A}^2\, b + b_{zz} + \delta^2 \mathcal{A}^2\, b_{xx} = 0 \quad ; \quad b(x, h(x)) = 0$$

- downstream waves derive from radiation boundary conditions

$\longrightarrow$ except hydrostatic waves (δ^2=0) are nondispersive



Queney, 1947



Klemp & Lilly, 1979

- nonlinear fluid system reduces to a single equation for buoyancy

Isentropic Coordinates

2D Primitive Equations

- nondimensional: steady, rotating & nonhydrostatic
- potential temperature θ as vertical coordinate ($\theta_z = 1/z_\theta$)

$$
\begin{aligned}
\mathcal{D}u \quad - \mathcal{R}^{-1}\, v &= -\mathcal{A}^2 M_x \quad - \delta^2\, z_x\, \mathcal{D}w \\
\mathcal{D}v \quad + \mathcal{R}^{-1}\, u &= \mathcal{R}^{-1} \\
\delta^2\, z_\theta\, \mathcal{D}w \quad + \quad \mathcal{A}^2 z &= -\mathcal{A}^2 M_\theta \\
\mathcal{D}z \quad - \quad w &= 0
\end{aligned}
$$

- Montgomery potential: $M = \phi - z\theta$
- steady 2D advection: $\mathcal{D} = u\,\partial_x$; incident wind $u^\infty = 1$
- 2D divergence: $z_\theta\, u_x - z_x\, u_\theta + w_\theta = 0$

Steady Streamline Property

- divergence + thermodynamic $\longrightarrow \{u\, z_\theta\}_x = 0$

  $\longrightarrow$ squeezing isentropes (streamlines) accelerates flow
- velocity relations: $u = 1/z_\theta$; $w = z_x/z_\theta$
- across-ridge flow: $v_x = \mathcal{R}^{-1}\,(z_\theta - 1)$
- eliminating M through vorticity . . . then a miracle happens . . .

A Master Theory for Buoyancy

Vertical Displacement Equation

- includes both f-plane and non-hydrostatic effects

$$\mathcal{A}^2 z_{xx} + \mathcal{R}^{-2} z_{\theta\theta} - \eta_{xx} = 0 \quad ; \quad \eta = \frac{1}{2}\left(u^2 + \delta^2 w^2\right)_\theta - \delta^2 w_x$$

- surface condition: $z(x,0) = h(x)$ & radiation BCs
- nonlinearity in horizontal vorticity η
- equivalent to Long's equation without rotation ($\mathcal{R}^{-2} \to 0$)

Hydrostatic Buoyancy Equation ($\delta^2 = 0$)

- constant stratification: $z = \theta - b(x,\theta)$

$$\mathcal{A}^2 b_{xx} + \mathcal{R}^{-2} b_{\theta\theta} + \left\{u^3 b_{\theta\theta}\right\}_{xx} = 0 \quad ; \quad u = \frac{1}{1 - b_\theta}$$

- surface condition: $b(x,0) = -h(x)$ & radiation BCs
- linearizing ($u \sim 1$) recovers queney

Isentropic Coordinate Singularities

- ▷ breakdowns in coordinate inversion of $z = \theta - b(x, \theta)$

$$\theta_z = \frac{1}{z_\theta} = u = \frac{1}{1 - b_\theta} \longrightarrow \begin{cases} \infty & \text{isentrope collapsing, } u \longrightarrow \infty \\ 0 & \text{isentrope overturning, } u \longrightarrow 0 \end{cases}$$

Semigeostrophic Approximation

- ▷ small $\mathcal{R}$ extension of quasigeostrophy: Robinson (1960), Pierrehumbert (1985)
- ▷ SG truncation of *hydrostatic master theory*

$$\mathcal{A}^2\, b_{xx} \;+\; \mathcal{R}^{-2}\, b_{\theta\theta} = 0 \qquad ; \qquad b(x, 0) = -h(x)$$

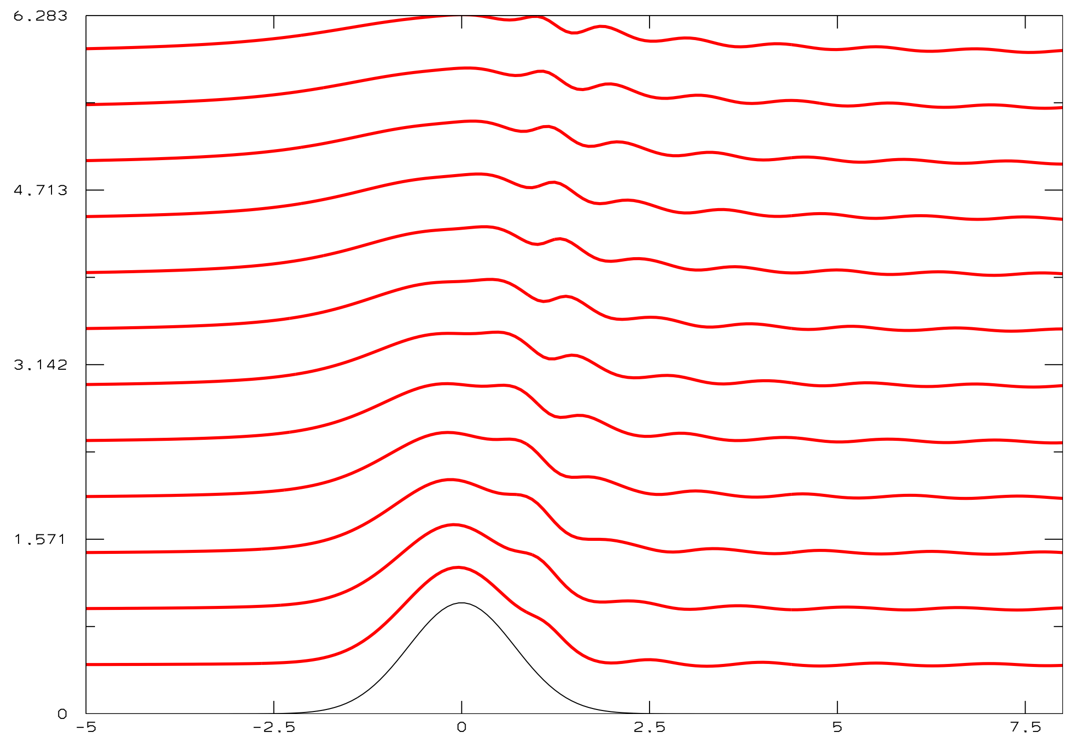
- ▷ isentrope collapse must occur above $h(x)$-dependent critical value of $\mathcal{R}\mathcal{A}$

Enhanced Wave Generation & Singularity Suppression?

- ▷ approach to collapse invalidates SG approximation, as nonlinearity u^3 must become large
- ▷ seems nonlinearity suppresses collapse singularity through enhanced wave generation

Nonlinear Wave Generation

- ▷ moderate height gaussian ridge ($\mathcal{A} = NH/U = 1.00$)
- ▷ tiny Rossby number flow ($\mathcal{R} = U/fL = 0.25$)
- ▷ time-transient computation to steady state



- ▷ how are these waves generated?

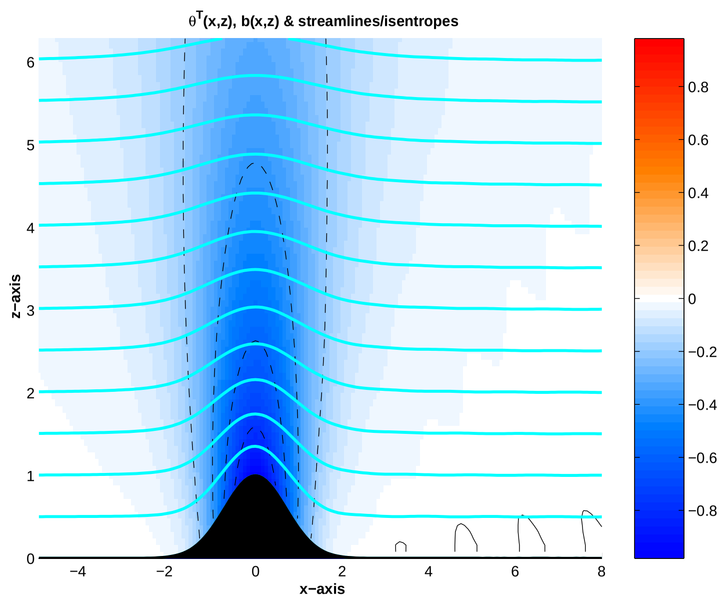
Direct Steady Solve

Solution at $\mathcal{R} = 0.25$; $\mathcal{A} = 1.00$

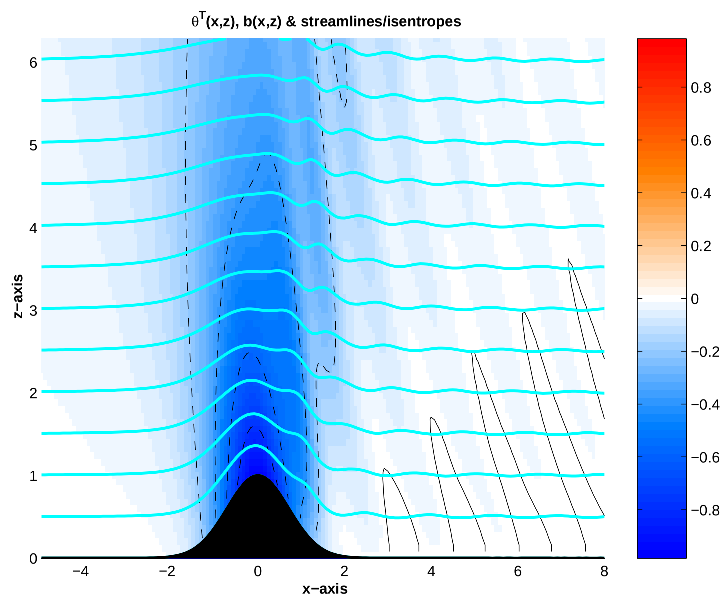
▷ iterate on nonlinearity in *hydrostatic master theory*: $u^{old} \longrightarrow b^{new} \longrightarrow u^{new}$

$$\mathcal{A}^2\, b^n_{xx} + \mathcal{R}^{-2}\, b^n_{\theta\theta} + \left\{ (u^o)^3\, b^n_{\theta\theta} \right\}_{xx} = 0 \qquad ; \qquad u^n = \frac{1}{1 - b^n_\theta}$$

linear solution: $b^0(x, \theta)$



waves after convergent iterations: $b(x, \theta)$



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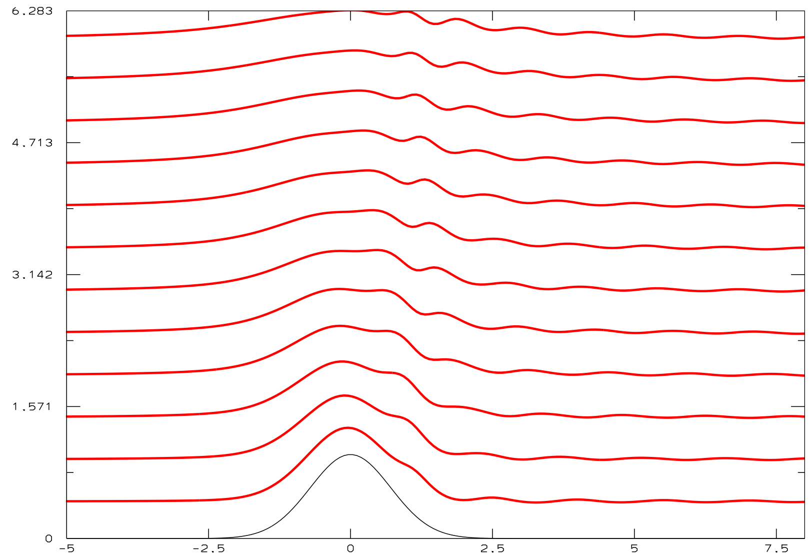
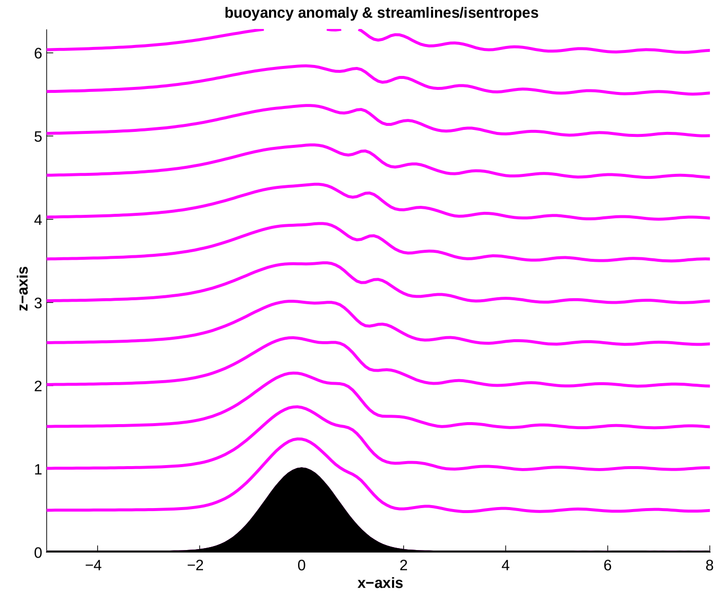
Streamline Comparison





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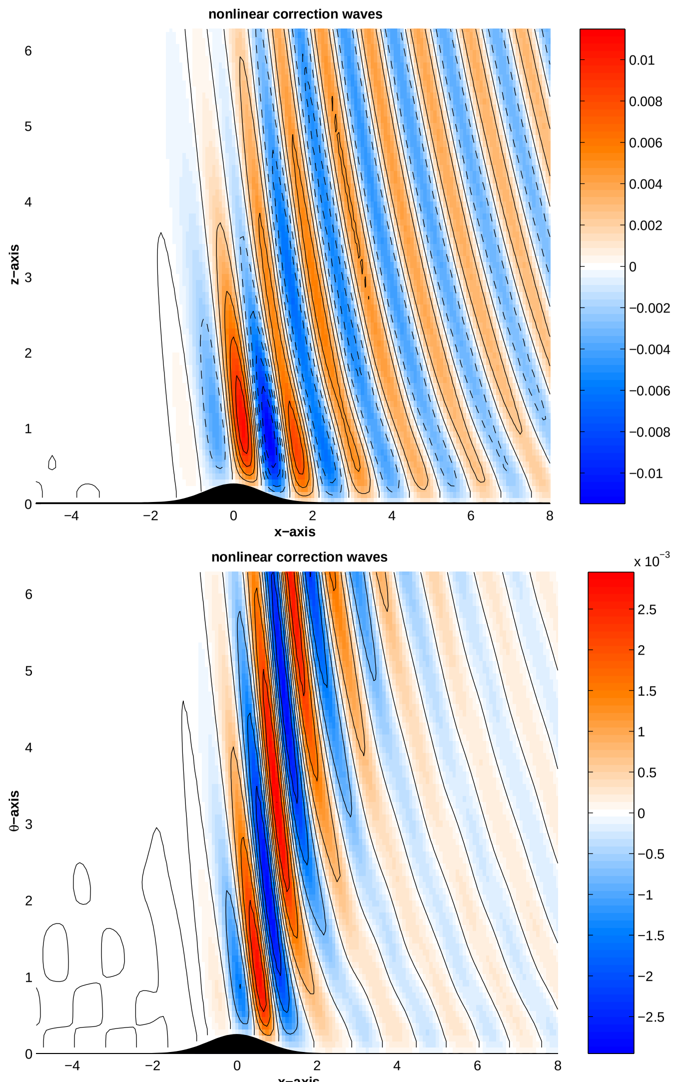
Possible Nonlinear Mechanisms

- ▷ nonlinear modification of local Rossby number
  - ⟶ enhanced topographic wave generation at ridge summit
  - ⟶ modification of wave propagation (rays) in interior
- ▷ nonlinear wave generation in interior?

$$\mathcal{A}^2\, b^n_{xx} + \mathcal{R}^{-2}\, b^n_{\theta\theta} + b^n_{xx\theta\theta} = -\left(\left((u^o)^3 - 1\right) b^o_{\theta\theta}\right)_{xx}$$

Generation/Enhancement/Refraction

Nonlinear Corrections



- ▷ 1^{st} correction:

  $\longrightarrow$ *new waves*

- ▷ remaining corrections:

  $\longrightarrow$ refraction by u^{QG}

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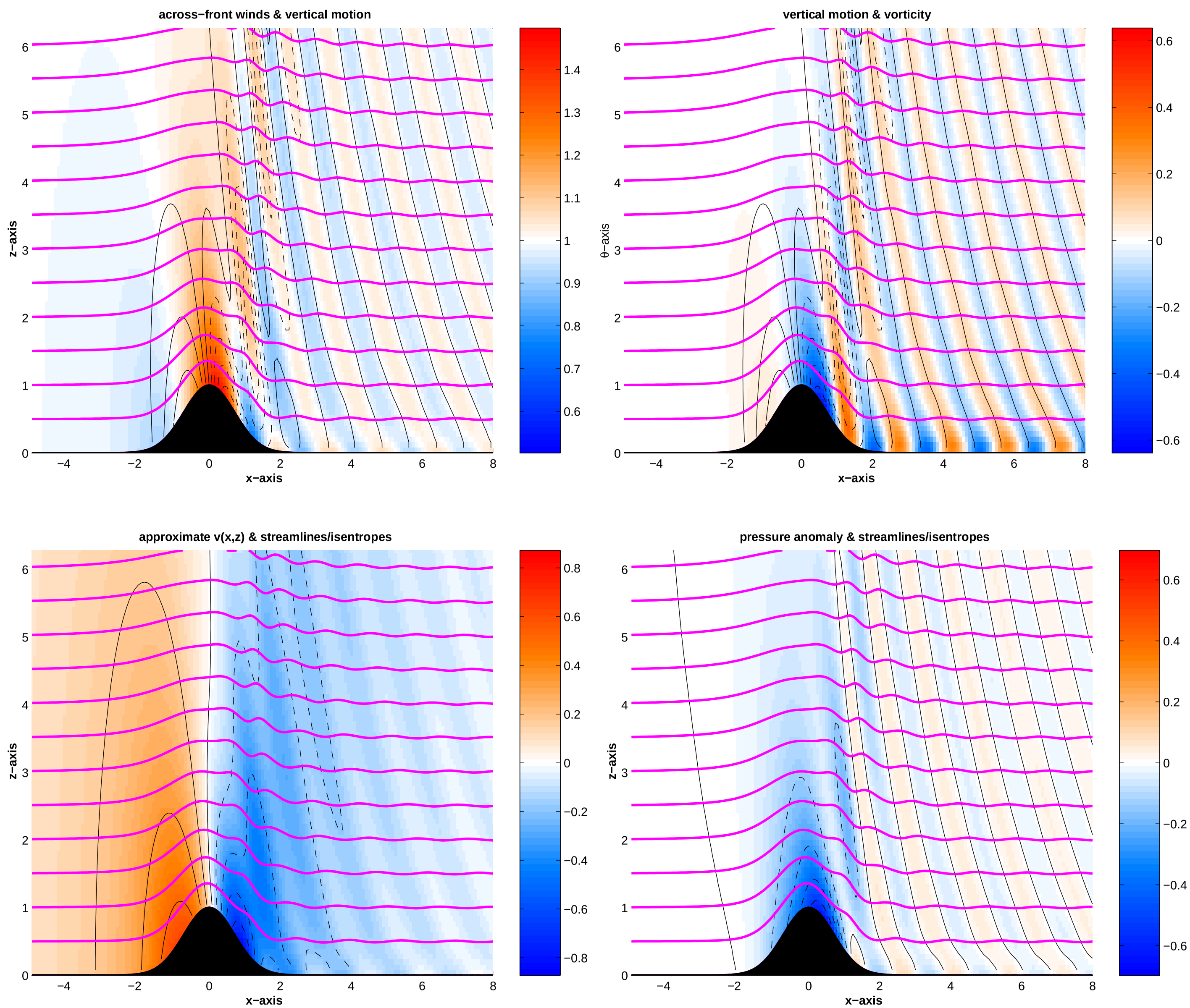
Other Fields



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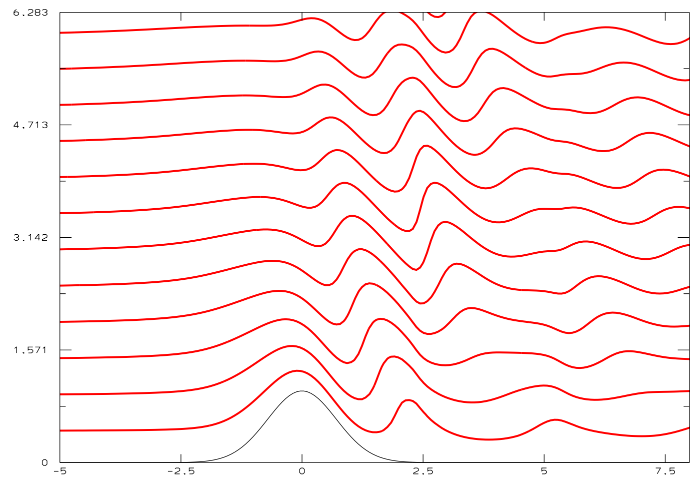
Nonlinear Waves at Small & Moderate Rossby Number

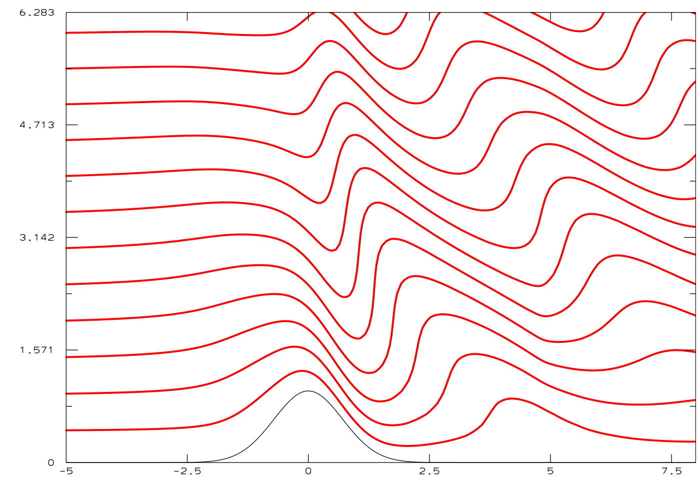
Nonlinear Wave Enhancement

- ▷ moderate height gaussian ridge ($\mathcal{A} = 1.00$)
- ▷ Rossby number flows ($\mathcal{R} = 0.50, 1.00$)
- ▷ time-transient computation to steady state

$\mathcal{R} = 0.50$



$\mathcal{R} = 1.00$



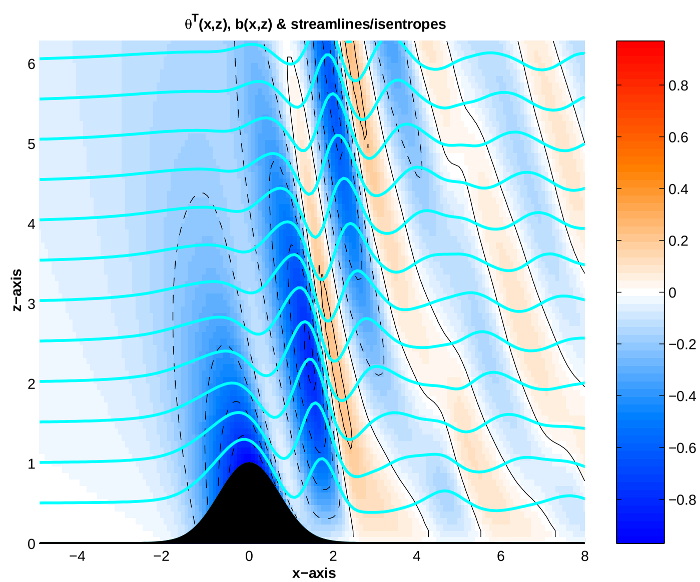
- ▷ wave amplitudes approach overturning as $\mathcal{R} \nearrow$

Summary

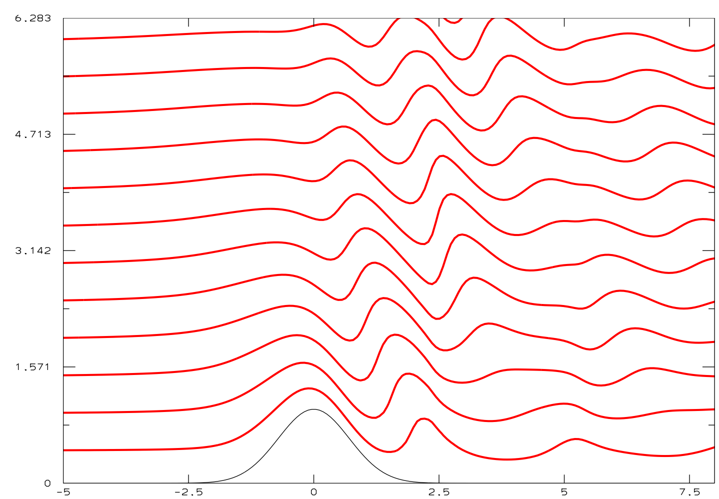
Master Theory for Buoyancy

- ▷ single equation for 2D topographic wave flow spanning non-hydrostatic to QG regimes
- ▷ quantitative tool for understanding nonlinear wave processes
- ▷ key issue: stability & accuracy of numerical solves

one iteration at $\mathcal{R} = 0.50$



time-transient relaxation



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