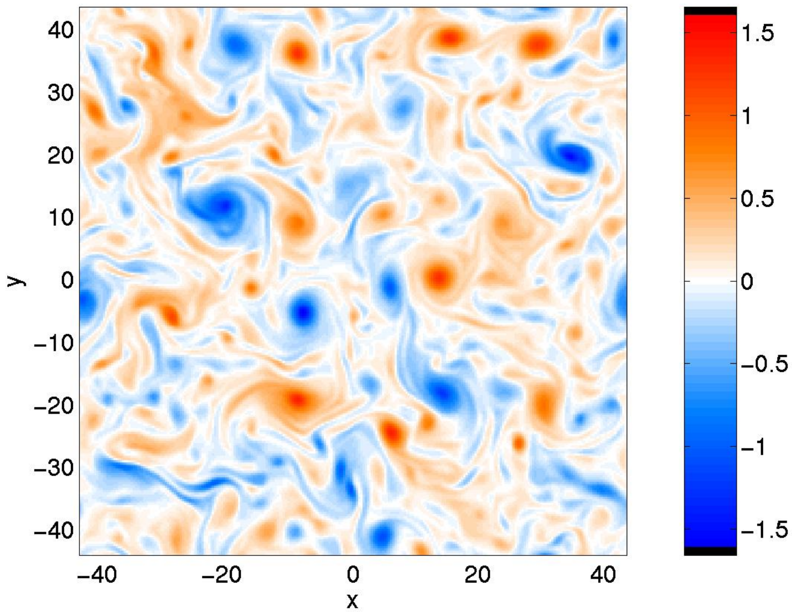
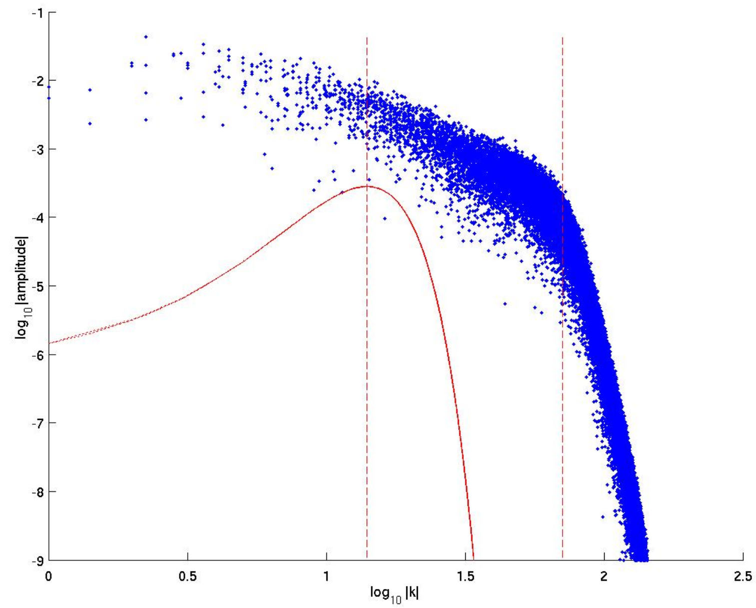
A Simple Illustration of a Spectral Cascade

- ▷ Fourier properties of the simplest nonlinear PDE: $u_t + u\, u_x = 0$
- ▷ nonlinear generation of small scales





▷ Dave Muraki, Simon Fraser University

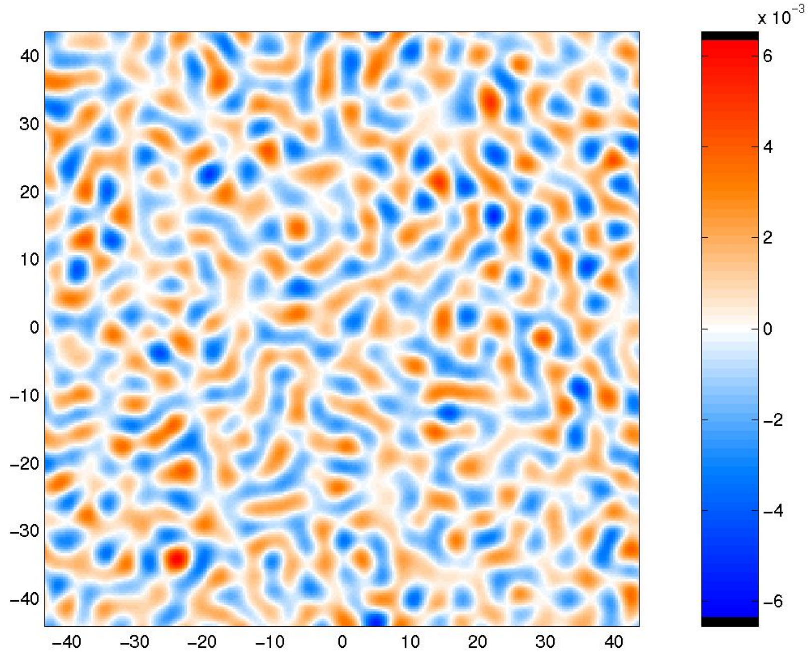
Cascade in a Turbulent Fluid

Atmospheric Tropopause Model

- ▷ 2D fluid advection with hyperdiffusion & random forcing for potential temperature $\theta(x, y, t)$

$$\theta_t + u\theta_x + v\theta_y + \nu\nabla^8\theta = F(\vec{x}, t)$$

- ▷ quadratic nonlinearity: u and v linearly determined from θ (via 3D Laplace solve)



Fourier Spectra: Forcing & Response

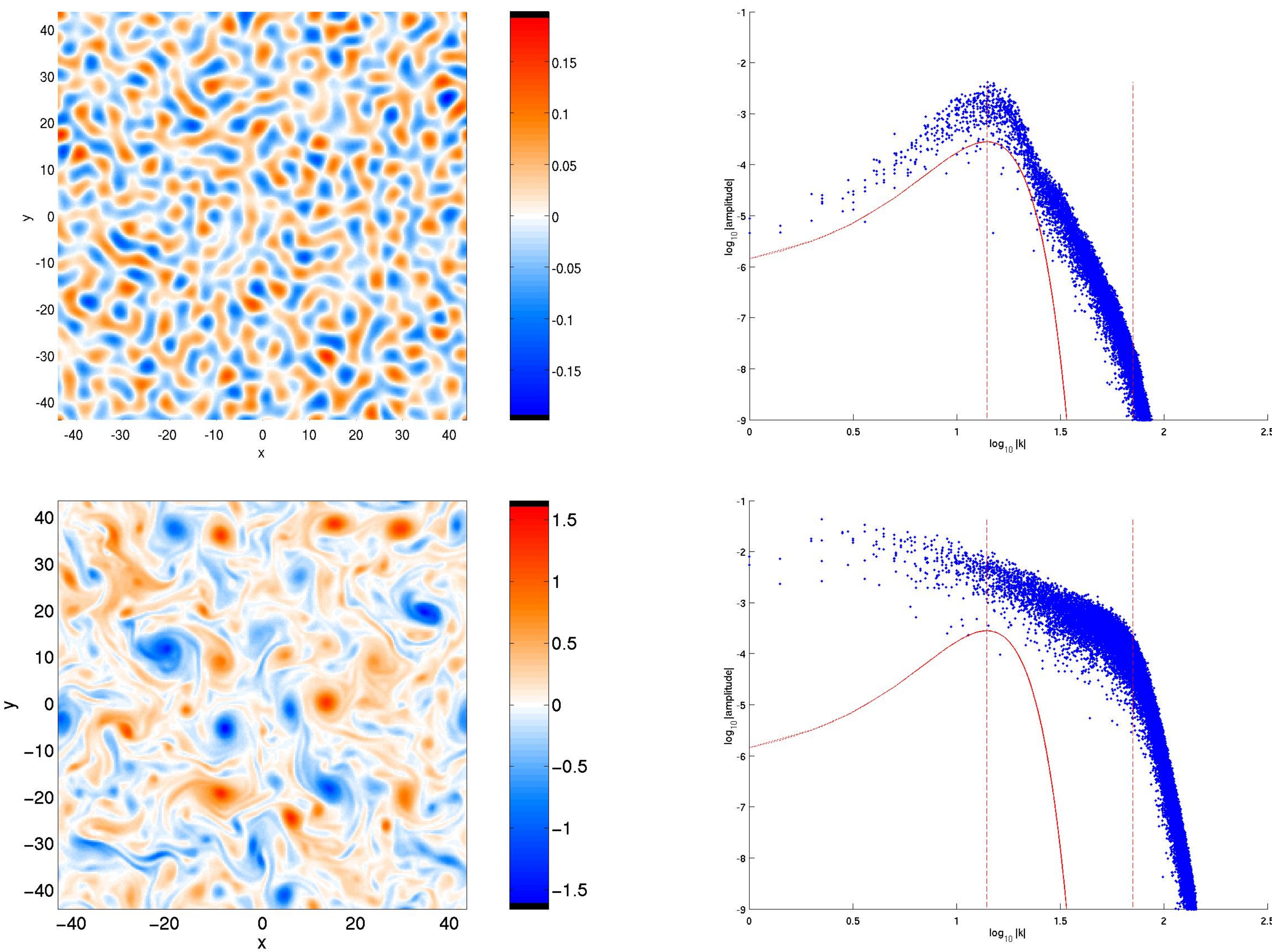
- ▷ $|\hat{F}(\vec{k}, t)|$ centred at intermediate scale k_0 with random phase

$$F(\vec{x}, t) = \iint \hat{F}(\vec{k}, t)\ e^{i\vec{k}\cdot\vec{x}}\ d\vec{k} \quad ; \quad \theta(\vec{x}, t) = \iint \hat{\theta}(\vec{k}, t)\ e^{i\vec{k}\cdot\vec{x}}\ d\vec{k}$$

Cascade in a Turbulent Fluid

Computational Results (Roy Wilds, 2003)

▷ early & late stages of vortex organization: $\hat{\theta}(\vec{k}, t)$ cascades to large & small scales



"Cascade" for a Nonlinear Eigenfunction

Linear Eigenvalue Problem

- linear boundary value problem

$$y'' + \lambda^2 y = 0 \quad ; \quad y(0) = y(\pi) = 0$$

- sinusoidal eigenfunctions, $y(x)$, for integers n

$$y(x) = A \sin nx \quad ; \quad \lambda^2 = n^2$$

Weakly Nonlinear Eigenvalue Problem

- nonlinear boundary value problem

$$y'' + \lambda^2 y = \epsilon y^3 \quad ; \quad y(0) = y(\pi) = 0$$

- exact solutions via elliptic functions
- perturbed eigenfunctions, $y(x)$, for $\epsilon A^2 \ll 1$

$$y(x) \sim A \sin nx + \frac{\epsilon A^2}{32n^2}\left(1 - \frac{3\epsilon A^2}{16n^2}\right) A \sin 3nx + \frac{3\epsilon^2 A^4}{512n^4} A \sin 5nx + \ldots$$

$$\lambda^2/n^2 \sim 1 + \frac{3\epsilon A^2}{4n^2} - \frac{9\epsilon^2 A^4}{64n^4} + \ldots$$

- nonlinear eigenfunction displays a cascade to full Fourier series for $\epsilon A^2 \neq 0$

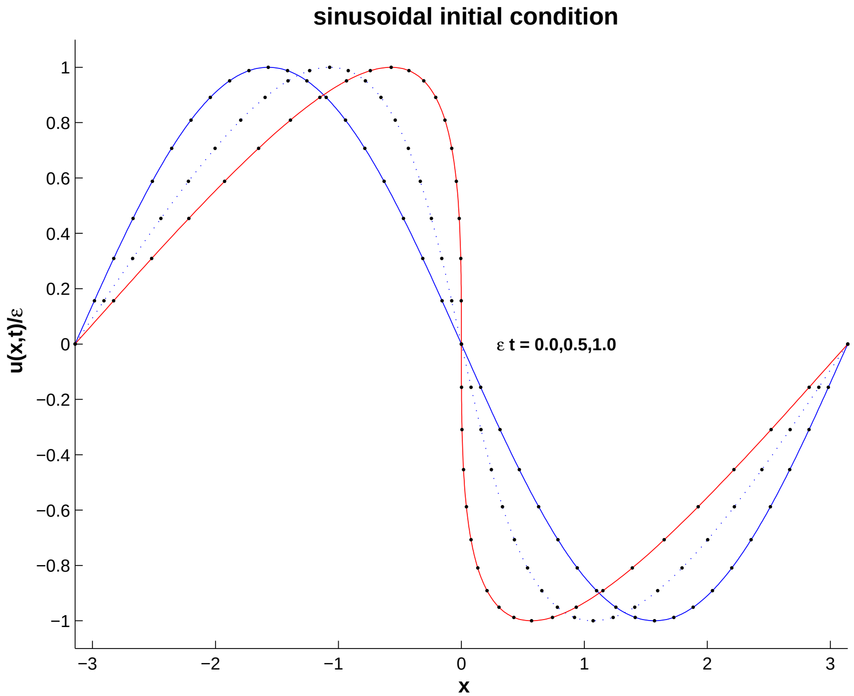
Simple Kinematic Wave Equation

Textbook Nonlinear PDE (Inviscid Burgers Equation)

- PDE of hyperbolic type, initial value problem for $u(x,t)$

$$u_t + u\,u_x = 0 \quad ; \quad u(x,0) = f(x)$$

- $exact$ solution by method of characteristics
- example of wave steepening & finite-time wavebreaking
- propagation of discontinuities determined by Rankine-Hugoniot conditions



- also embodies a simple one-dimensional cascade

An Exact Solution

Characteristic ODEs

- define characteristics as parametric curves $(x(s), t(s))$ originating from $(x_0, 0)$

$$\frac{dx}{ds} = u \quad ; \quad x(0) = x_0 \quad \longrightarrow \quad x = x_0 + s\, f(x_0)$$

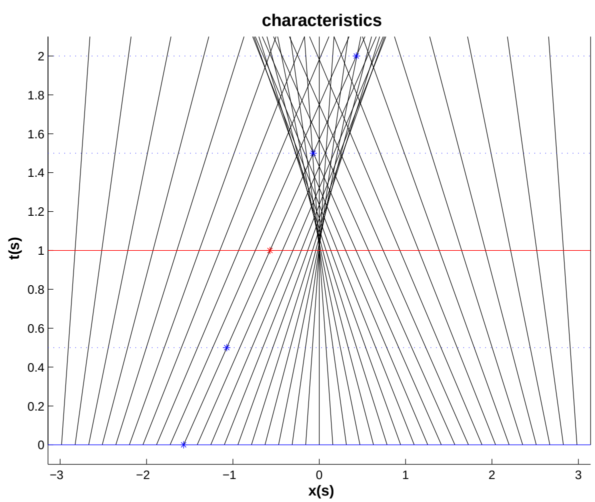
$$\frac{dt}{ds} = 1 \quad ; \quad t(0) = 0 \quad \longrightarrow \quad t = s$$

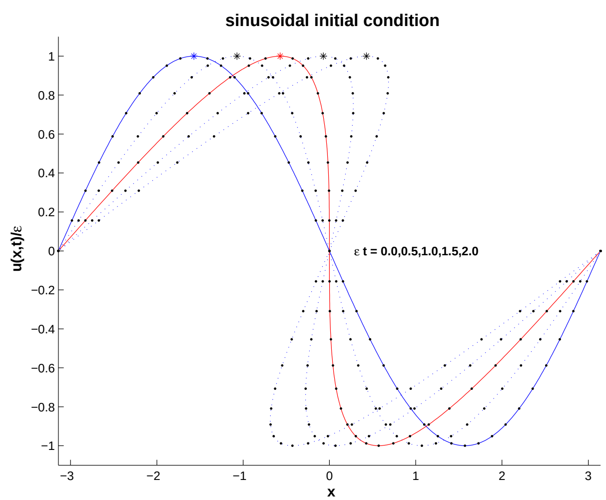
- PDE becomes ODE for $u(s)$ along each characteristic

$$\frac{du}{ds} = 0 \quad ; \quad u(0) = f(x_0) \quad \longrightarrow \quad u = f(x_0)$$

- parametric solution in terms of x_0 and t

$$u = f(x_0) \quad ; \quad x = x_0 + t\, f(x_0)$$





Series Representation

$$u(x,t) \quad = \quad a_0 + \sum_1^{\infty} \{a_n(t)\cos nx + b_n(t)\sin nx\}$$

Fourier Coefficients

- assume $u(x,t)$ continuous, prior to wavebreaking
- period integral of $u(x,t)$ is conserved $\longrightarrow$ time-independent mean

$$a_0 \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x_0)\ dx_0$$

- sine coefficient & integration by parts $(n \neq 0)$

$$b_n(t) \quad = \quad \frac{1}{\pi}\int_{-\pi}^{+\pi} u(x,t)\ \sin nx\ dx \quad = \quad \frac{1}{\pi n}\int_{-\pi}^{+\pi} u_x(x,t)\ \cos nx\ dx$$

- u_x via implicit differentiation of $x = x_0 + t\,u$

$$b_n(t) \quad = \quad \frac{1}{\pi n t}\int_{-\pi}^{+\pi} \left(1 - \frac{dx_0}{dx}\right) \cos nx\ dx \quad = \quad -\frac{1}{\pi n t}\int_{-\pi}^{+\pi} \cos nx\ dx_0$$

- introduce IVs via parametric solution $x = x_0 + t\,f(x_0)$

$$b_n(t) \quad = \quad -\frac{1}{\pi n t}\int_{-\pi}^{+\pi} \cos\left[n x_0 + nt\,f(x_0)\right]\ dx_0$$

$$a_n(t) \quad = \quad \frac{1}{\pi n t}\int_{-\pi}^{+\pi} \sin\left[n x_0 + nt\,f(x_0)\right]\ dx_0$$

Platzman's 1964 Solution

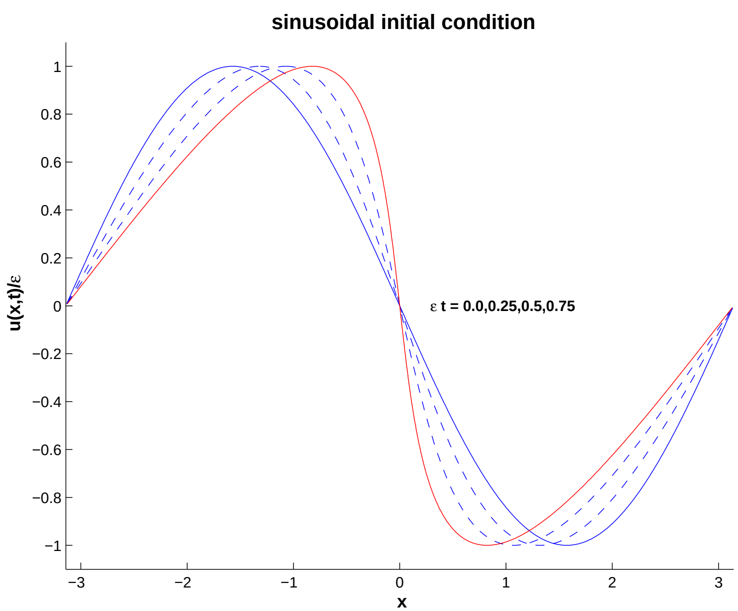
Sinusoidal Initial Condition: $f(x) = -\epsilon \sin x$

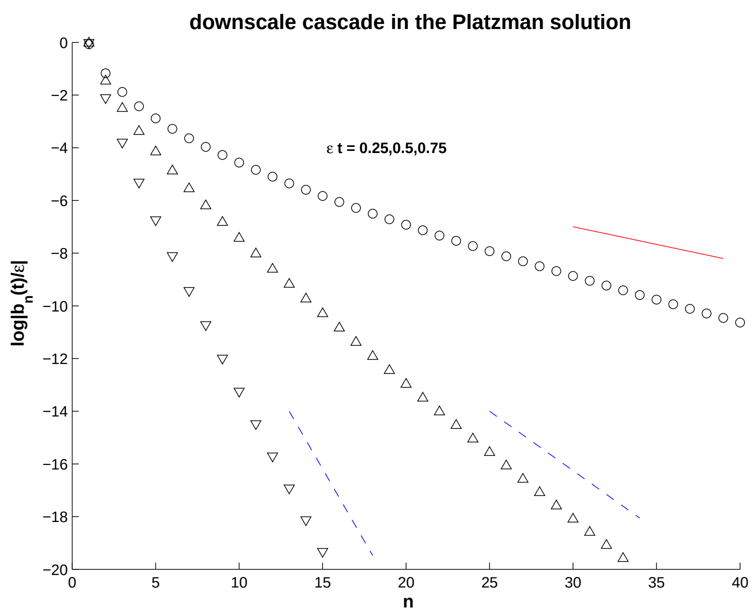
- solution remains a sine series for $t > 0$: $a_n(t) \equiv 0$
- sine coefficient is integral representation for $J_n(\cdot)$ Bessel function

$$b_n(t) = -\frac{1}{\pi n t}\int_{-\pi}^{+\pi} \cos\left[n x_0 - n\epsilon t \sin x_0\right]\, dx_0 = -2\,\frac{J_n(n\epsilon t)}{nt}$$

- exact series solution with time-dependent fourier coefficients $\longrightarrow$ cascade

$$u(x,t) = \sum_{1}^{\infty} -2\,\frac{J_n(n\epsilon t)}{nt}\, \sin nx$$





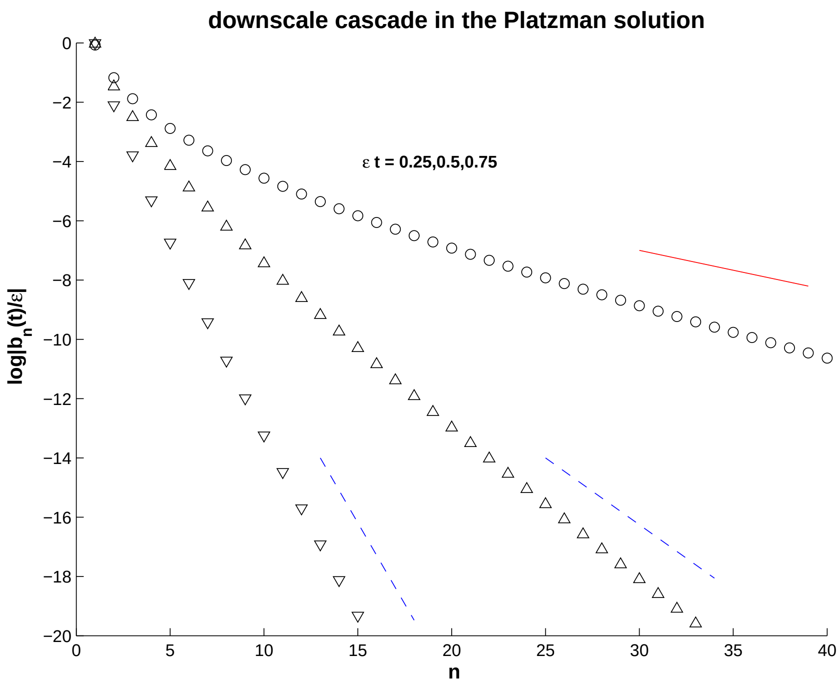
Large Wavenumber Asymptotics

- Bessel function asymptotics for large index & argument $(n \longrightarrow \infty, \epsilon t < 1)$

$$|b_n(t)| \sim \sqrt{\frac{2}{\pi \tanh \alpha}}\; t^{-1}\; n^{-3/2}\; e^{n(-\alpha + \tanh \alpha)} \quad ; \quad \cosh \alpha = \frac{1}{\epsilon t}$$

- spectral slope, $n \longrightarrow \infty$

$$\frac{\ln |b_n(t)|}{n} \sim \ln\left(\frac{\epsilon t}{2}\right) + \sqrt{1-\epsilon^2 t^2} - \ln\left(\frac{1+\sqrt{1-\epsilon^2 t^2}}{2}\right)$$



Spectral Dynamics

$$u(x,t) \;=\; \sum_1^\infty b_n(t)\sin nx$$

Direct Substitution into $u_t + uu_x = 0$

- identify all terms which produce $\sin nx$

$$\begin{aligned} \ldots\; b_n' \sin nx + \;\ldots\; + \sum_1^{n-1} &\quad k\, b_k\, b_{n-k}\; \cos kx\; \sin(n-k)x \quad + \;\ldots \\ \ldots\; + \sum_1^{\infty} &\quad k\, b_k\, b_{n+k}\; \cos kx\; \sin(n+k)x \quad + \;\ldots \\ \ldots\; + \sum_1^{\infty} &\quad (n+k)\, b_{n+k}\, b_k\; \cos(n+k)x\; \sin kx \quad + \;\ldots \;=\; 0 \end{aligned}$$

- spectral dynamics ODEs: triad resonances

$$b_n' \;=\; -\,\frac{n}{4}\sum_1^{n-1} b_k\, b_{n-k} \;+\; \frac{n}{2}\sum_1^{n-1} b_k\, b_{n+k} \;+\; \frac{n}{2}\sum_n^{\infty} b_k\, b_{n+k}$$

$\longrightarrow$ 1^{st}-sum: downscale transfer from smaller wavenumber, long waves

$\longrightarrow$ 2^{nd}-sum: mixing transfer from straddling wavenumbers

$\longrightarrow$ 3^{rd}-sum: upscale transfer from larger wavenumber, short waves

$$b_n' = -\frac{n}{4}\sum_{1}^{n-1} b_k\, b_{n-k} + \frac{n}{2}\sum_{1}^{\infty} b_k\, b_{n+k}$$

Solution Strategies

- Platzman solution is exact for initial conditions, $\{b_n(0)\} = \{-\epsilon, 0, 0, \ldots\}$

$$\text{slope} = \ln\left(\frac{\epsilon t}{2}\right) + \sqrt{1-\epsilon^2 t^2} - \ln\left(\frac{1+\sqrt{1-\epsilon^2 t^2}}{2}\right)$$

- downscale transfer only solution, asymptotically valid for $0 \leq \epsilon t \ll 1$

$$b_n(t) \sim -\epsilon\,\frac{n^{n-1}}{n!}\left(\frac{\epsilon t}{2}\right)^{n-1} \quad \longrightarrow \quad \text{slope} \approx \ln\left(\frac{\epsilon t}{2}\right) + 1$$

$\longrightarrow$ first Taylor term of Platzman solution for small ϵt & Stirling approximation

- small ϵt perturbation series

$$\left.\begin{aligned} b_1(t) &\sim -\epsilon + \epsilon\left(\frac{\epsilon t}{2}\right)^2 + \ldots \\ b_2(t) &\sim -\epsilon\left(\frac{\epsilon t}{2}\right) + \ldots \\ b_3(t) &\sim -\epsilon\,\frac{3}{2}\left(\frac{\epsilon t}{2}\right)^2 + \ldots \end{aligned}\right\} \longrightarrow \text{slope} \approx \ln\left(\frac{\epsilon t}{2}\right)$$

Cascade Solutions

▷ is there a general approach for constructing approximate solutions that embody spectral cascade?

Perturbation Expansion for $f(x) = O(\epsilon)$

▷ PDE for disturbance about initial condition: $u(x,t) = f(x) + \hat{u}(x,t)$

$$\hat{u}_t \;=\; -ff_x - (f\hat{u})_x - \hat{u}\hat{u}_x \quad ; \quad \hat{u}(x,0) = 0$$

▷ simple iterative construction $\hat{u} \;\sim\; \hat{u}_2(x,t) + \hat{u}_3(x,t) + \hat{u}_4(x,t) + \ldots$

$$\begin{aligned}
(\hat{u}_2)_t &= -\,ff_x \\
(\hat{u}_3)_t &= \qquad\quad -\,(f\hat{u}_2)_x \\
(\hat{u}_4)_t &= \qquad\quad -\,(f\hat{u}_3)_x \;-\; (\hat{u}_2)(\hat{u}_2)_x
\end{aligned}$$

⟶ generates polynomial-in-ϵt solutions; $\hat{u}_2$-error is $O(\epsilon^3)$

⟶ solution up to $\hat{u}_k$ contains wavenumbers up to k — partial cascade only

▷ 1^{st} newton iterate

$$(\hat{u}_2)_t + (f\hat{u}_2)_x \;=\; -\,ff_x$$

⟶ non-constant coefficient PDE; $\hat{u}_2$-error is $O(\epsilon^4)$

⟶ solution by characteristics

1^{st} Newton Iterate

Characteristic ODEs

- define characteristics as parametric curves $(x(s), t(s))$ originating from $(x_0, 0)$

$$\frac{dx}{ds} = f(x) \qquad ; \qquad x(0) = x_0$$

$$\frac{dt}{ds} = 1 \qquad ; \qquad t(0) = 0$$

- f times PDE becomes ODE for $f\hat{u}_2$ along each characteristic

$$\frac{d}{ds}(f\hat{u}_2) = -\frac{1}{2}f(f^2)_x = -\frac{1}{2}\frac{d}{ds}(f^2) \qquad ; \qquad u(0) = 0$$

- can integrate to obtain $\hat{u}_2$

$$\hat{u}_2 = \frac{1}{2}\,\frac{f^2(x_0) - f^2(x)}{f(x)}$$

$\longrightarrow$ but relation determining $x_0(x,t)$ requires solution to nonlinear ODE

Platzman to the Rescue (again)

- exact characteristics for $f(x) = -\,\epsilon \sin x$

$$\frac{dx}{dt} = -\epsilon \sin x \quad ; \quad x(0) = x_0 \qquad \longrightarrow \qquad \sin x_0 = \frac{\operatorname{sech} \epsilon t}{1 - \tanh \epsilon t\ \cos x} \sin x$$

Approximate Cascade Solution

Two Small ϵt Asymptotic Miracles

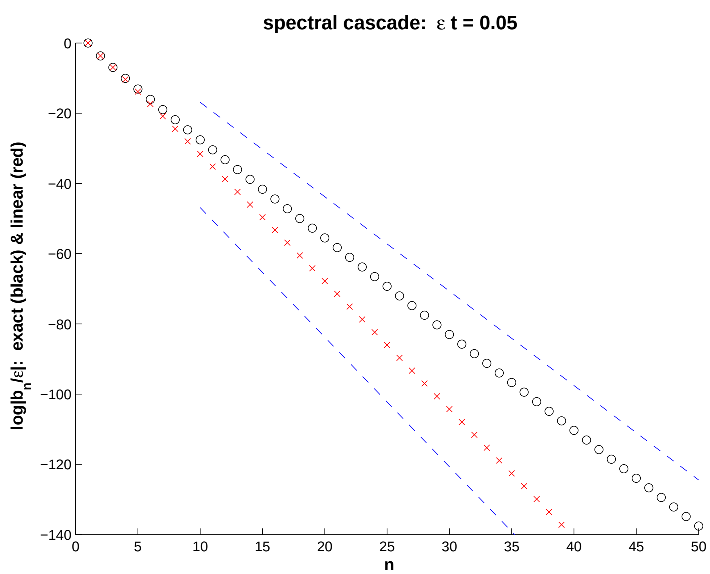
- formally $O(\epsilon^3)$-accurate solution from linearizing truncation

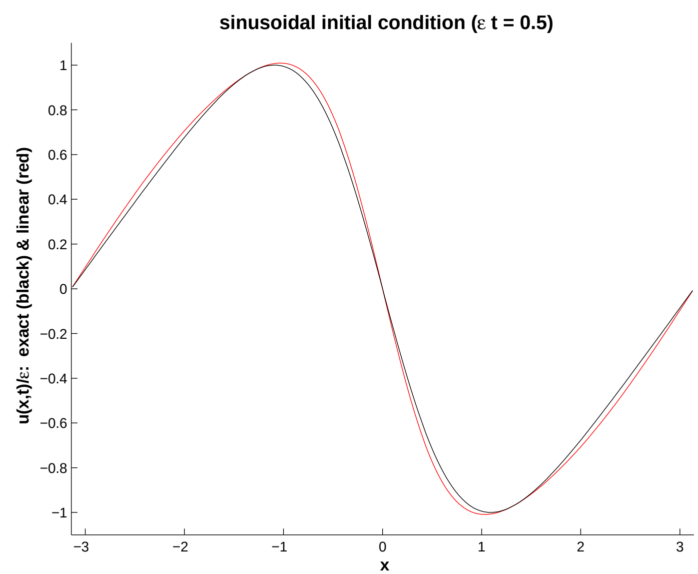
$$u(x,t) \sim -\epsilon \sin x + \frac{\epsilon}{2}\left(1 - \frac{\operatorname{sech}^2 \epsilon t}{(1 - \tanh \epsilon t\ \cos x)^2}\right) \sin x$$

- need Fourier series representation

$$u(x,t) \sim -\frac{\epsilon}{2}\left\{\sin x + \operatorname{sech}^2\left(\frac{\epsilon t}{2}\right) \sum_1^\infty n \tanh^{n-1}\left(\frac{\epsilon t}{2}\right) \sin nx\right\}$$

$\longrightarrow$ contains a cascade with spectral slope $= \ln\left(\tanh \frac{\epsilon t}{2}\right) \sim \ln\left(\frac{\epsilon t}{2}\right)$ for $0 < \epsilon t \ll 1$





Fourier Integral

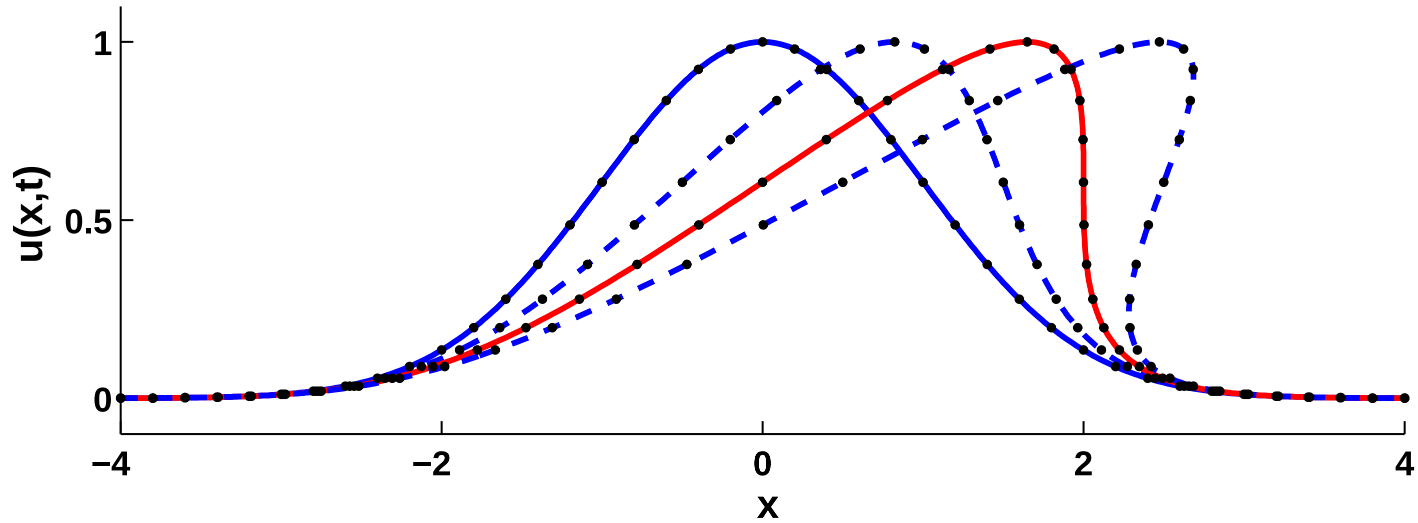
- integral representation

$$u(x,t) = \int_{-\infty}^{+\infty} c(k;t)\, e^{-ikx}\, dk$$

- modified formulation obtains convergent integral

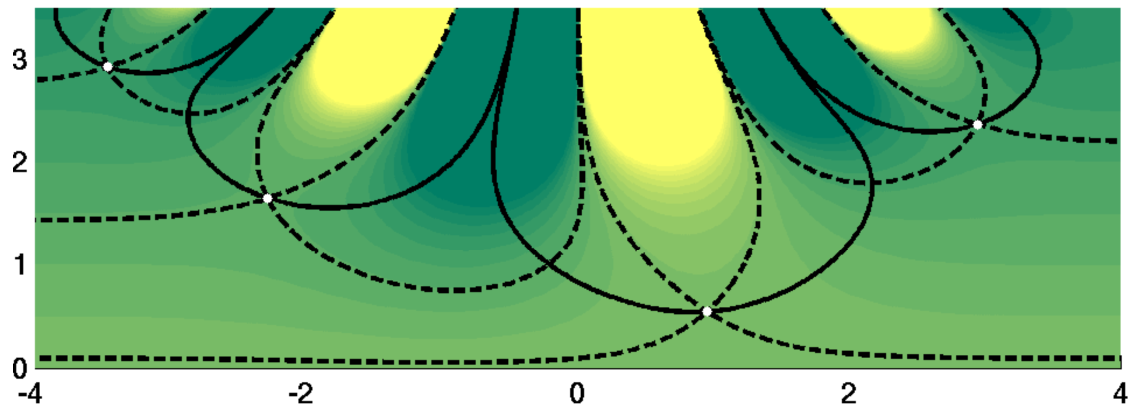
$$c(k;t) = \frac{i}{2\pi k} \int_{-\infty}^{+\infty} f'(x_0)\, \exp[ik(x_0 + t\, f(x_0))]\, dx_0$$

- gaussian initial condition: $f(x) = e^{-x^2/2}$

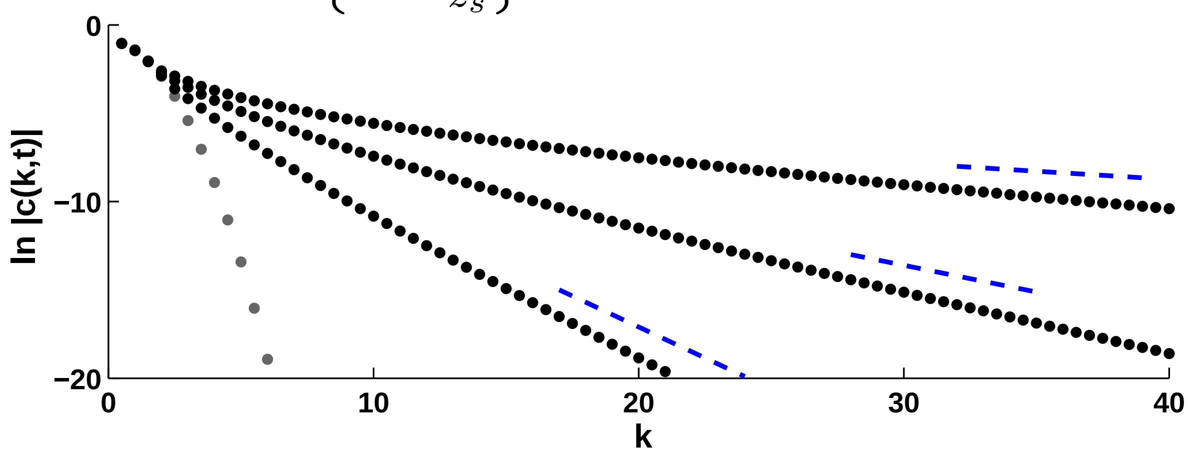


Gaussian Initial Condition: $f(x) = e^{-x^2/2}$

- complex contour integration & method of steepest descent
- largest contribution from saddlepoint nearest to real-axis: $(-z_s^2)e^{(-z_s^2)} = -1/t^2$



- spectral slope = -Im$\left\{ z_s + \frac{1}{z_s} \right\}$



Quantifying the Weak Cascade

Simple Quantitative Illustration of a Spectral Cascade

- ▷ perturbative construction of a solution containing approximate cascade
- ▷ use Platzman solution for kinematic wave equation as benchmark
- ▷ 1^{st}newton iterate uses non-constant coefficient in PDE to generate cascade

  $\longrightarrow$ robust methodology may be adapted to more difficult problems

  $\longrightarrow$ formal accuracy is not improved

  $\longrightarrow$ yet, full spectral content with leading behaviour of spectral slope
- ▷ likely new general results on an old problem: $u_t + uu_x = 0$

  $\longrightarrow$ fourier series solution for continuous evolutions

  $\longrightarrow$ exact characteristic solution of linearizing truncation about initial conditions

Emerging Applications to Atmospheric Fluid Dynamics

- ▷ generation of short-scale inertia-gravity waves by large-scale flows

Uplifted Streamlines & Downstream Wake

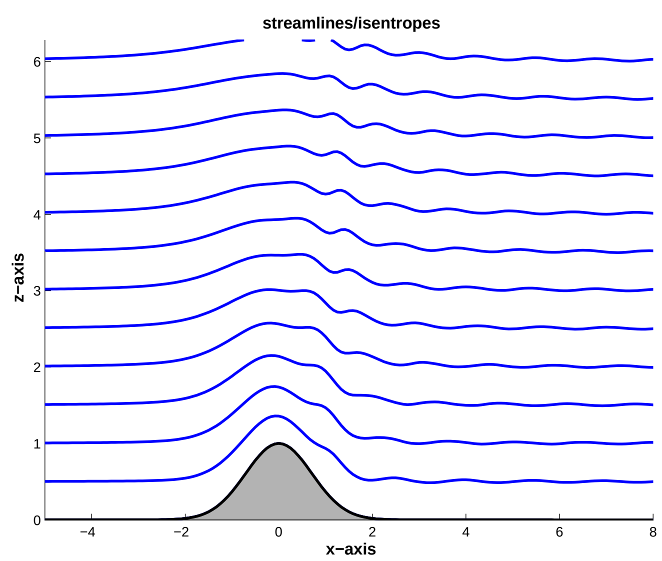
▷ primary issue: nonlinear coupling of large & wave-scales at small Rossby number

A New Equation for Topographic Waves

$$b_{xx} + b_{\theta\theta} + \mathcal{R}^2 \left\{ u^3 \, b_{\theta\theta} \right\}_{xx} = 0 \qquad ; \qquad u = \frac{1}{1 - \mathcal{R}\mathcal{A}\, b_\theta}$$

Rotating, Hydrostatic & Stratified Flow

- ▷ stratification allows isentropic "density" coordinates in vertical: θ (potential temperature)
- ▷ buoyancy anomaly from constant stratification: $b(x,\theta) = \theta/(\mathcal{R}\mathcal{A}) - z$
- ▷ surface condition: $b(x,0) = -h(x)$ with decay/radiation BCs
- ▷ topographic height scale ($\mathcal{A} = NH/U$) & Rossby number ($\mathcal{R} = U/fL$)



- ▷ a small Rossby ($\mathcal{R} = 0.25$) number mystery; nonlinear ($\mathcal{A} = 1.0$) waves

Waves at Small Scales

$$b_{xx} + b_{\theta\theta} + \mathcal{R}^2 \left\{ u^3\, b_{\theta\theta} \right\}_{xx} = 0 \qquad ; \qquad u = \frac{1}{1 - \mathcal{R}\mathcal{A}\, b_\theta}$$

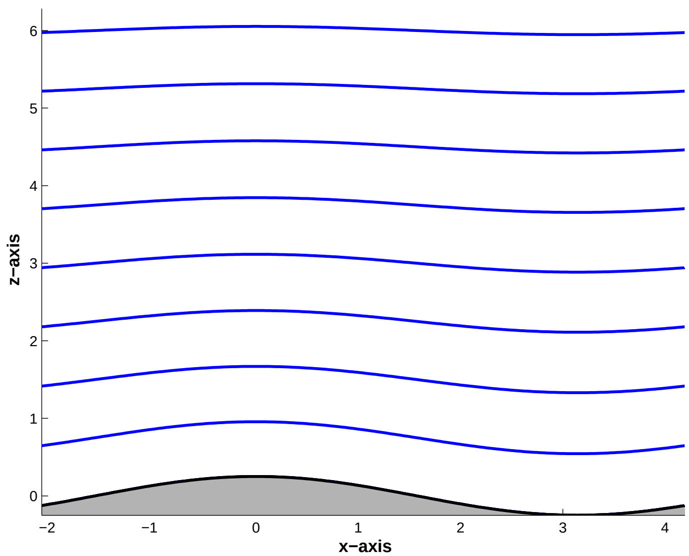
Linear Theory $(u \equiv 1)$, Queney 1947

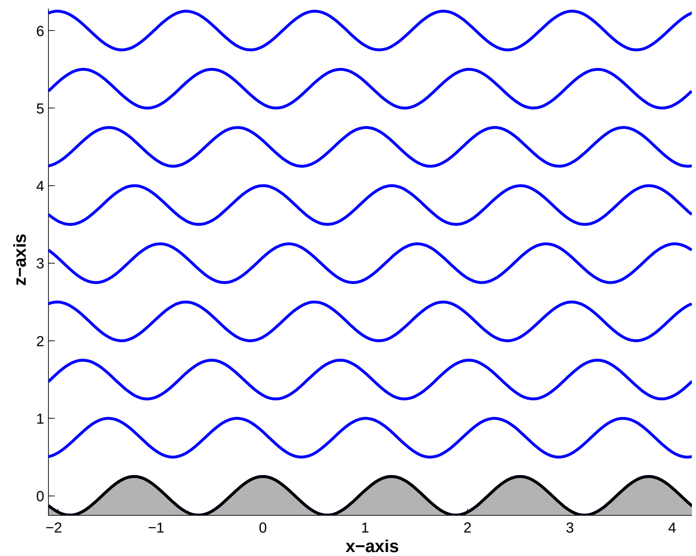
- ▷ small topographic amplitude: $\mathcal{A} \to 0$
- ▷ dispersion relation: $m^2 = -k^2/(1 - \mathcal{R}^2 k^2)$

  $\longrightarrow$ transition at inertial wavenumber: $k_i = 1/\mathcal{R}$

  $\longrightarrow$ long waves: upward decay & symmetric uplifting of streamlines

  $\longrightarrow$ short waves: waves & upstream phase tilt



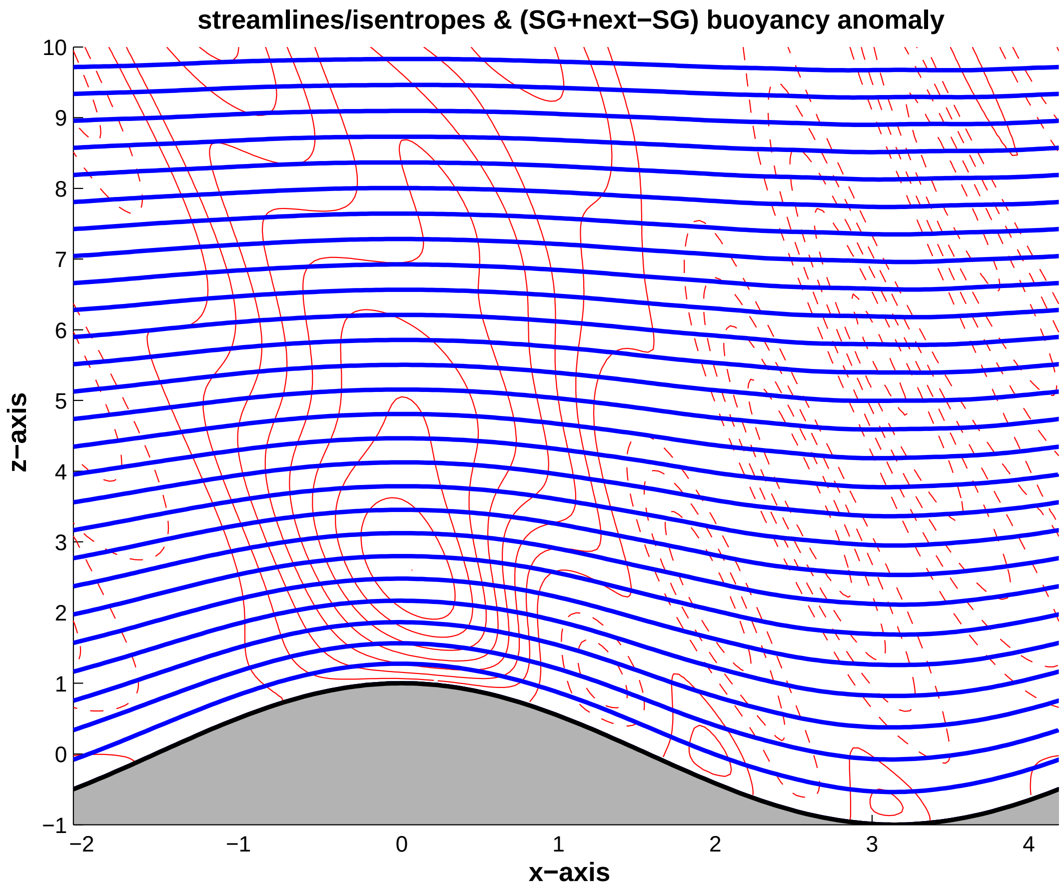


- ▷ quasigeostrophy: $\mathcal{R} \to 0$ limit banishes inertial wavenumber, $k_i = 1/\mathcal{R} \to \infty$

Nonlinear Wave Generation

First Iteration $(u = u_{linear})$

- ▷ sinusoidal topography with $\mathcal{R} = 0.25$; $\mathcal{A} = 1.00$
- ▷ first nonlinear correction well-approximates converged solution



- ▷ establish exponentially-small wave amplitude $(e^{-1/\mathcal{R}})$ via weak nonlinear cascade