Phys101 Lectures 28, 29

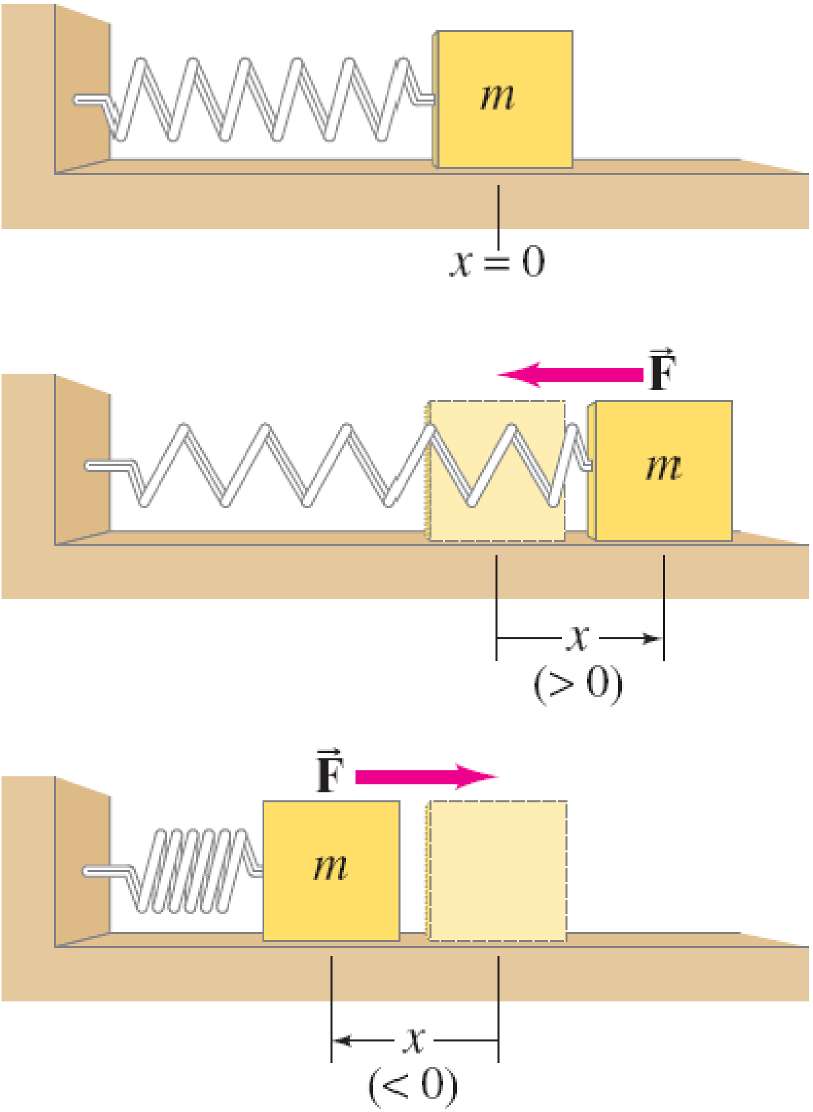
Oscillations

Key points:

- **Simple Harmonic Motion (SHM)**
- **SHM Related to Uniform Circular Motion**
- **The Simple Pendulum**

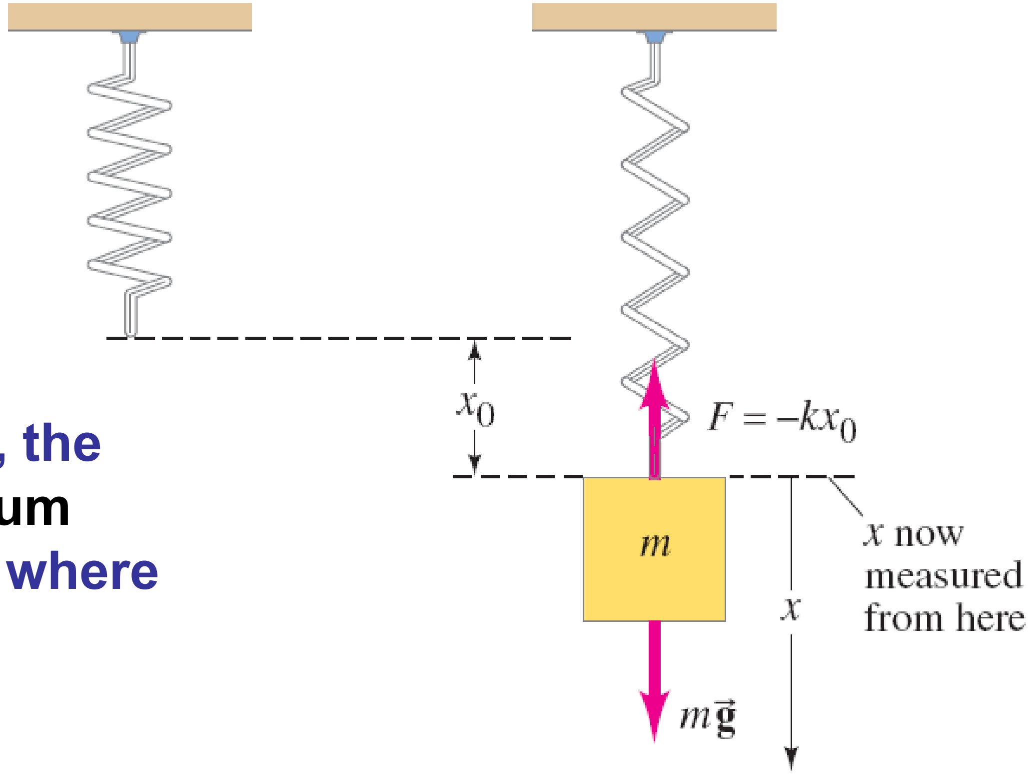
Ref: 11-1,2,3,4.

Oscillations of a Spring



If an object **oscillates** back and forth over the same path, each cycle taking the same amount of time, the motion is called **periodic**. The **mass** and **spring** system is a useful model for a periodic system.

Oscillations of a Spring



If the spring is hung **vertically**, the only change is in the **equilibrium position**, which is at the point where the spring force equals the gravitational force.

Demo

Oscillations of a Spring

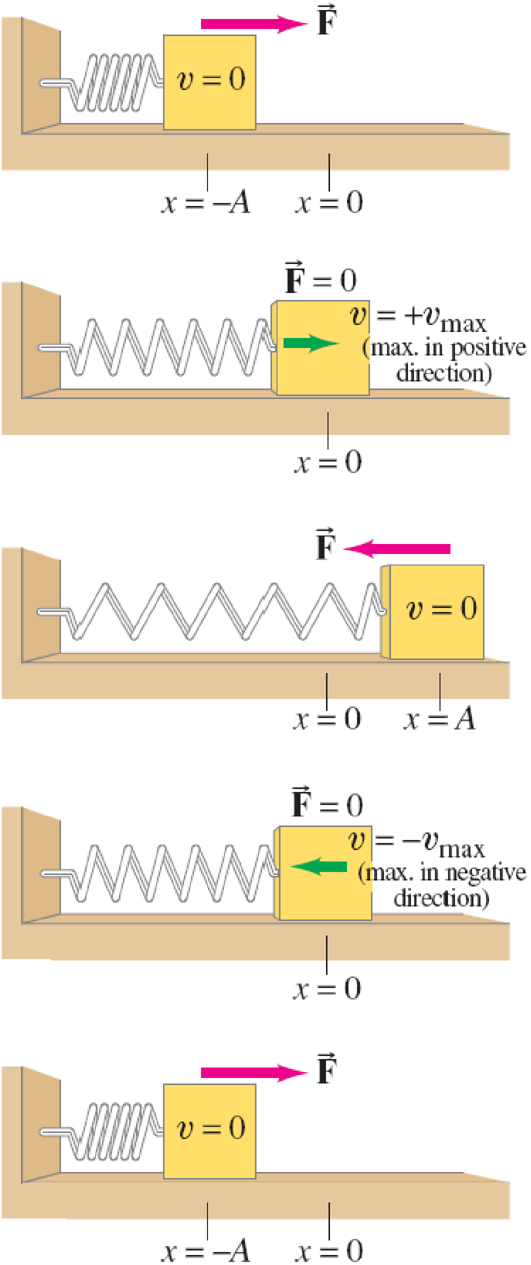
We assume that the system is frictionless. There is a point where the spring is neither stretched nor compressed; this is the equilibrium position. We measure displacement from that point (x = 0 on the previous figure).

The force exerted by the spring depends on the displacement:

$$F = -kx.$$

- The minus sign on the force indicates that it is a restoring force—it is directed to restore the mass to its equilibrium position.
- k is the spring constant.
- Since the force is not constant, the acceleration is not constant either.

Oscillations of a Spring



- **Displacement** is measured from the **equilibrium** point.
- **Amplitude** is the **maximum** displacement.
- A **cycle** is a full to-and-fro motion.
- **Period,** T, is the **time** required to complete one cycle.
- **Frequency, f,** is the **number** of cycles completed per second. The unit of frequency is Hz (cycles per second).

$$f = \frac{1}{T}$$

Simple Harmonic Motion

Any vibrating system where the restoring force is proportional to the negative of the displacement is in simple harmonic motion (SHM), and is often called a simple harmonic oscillator (SHO).

Substituting $F = -kx$ into Newton's second law gives the equation of motion:

$$-kx = ma = m\frac{d^2x}{dt^2}, \qquad i.e. \qquad a = -\frac{k}{m}x$$

$$\text{Or,} \qquad a = -\omega^2 x \qquad \text{where} \qquad \omega^2 = \frac{k}{m}$$

The solution has the form:

$$x = A\cos(\omega t + \phi).$$

How do you know?

We can guess

$$v = \frac{dx}{dt} = \frac{d}{dt}[A\cos(\omega t + \phi)] = -\omega A\sin(\omega t + \phi)$$

← Velocity

and then verify:

$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi) = -\omega^2 x$$

← Acceleration

Simple Harmonic Motion Related to Uniform Circular Motion

If we look at the **projection** onto the *x* axis of an object moving in a **circle** of radius *A* at a constant angular velocity ω, we find that the ***x* component** of the circular motion is in fact a SHM.

Demo

$$x = A\cos(\theta)$$

$$\theta = \omega t + \phi$$

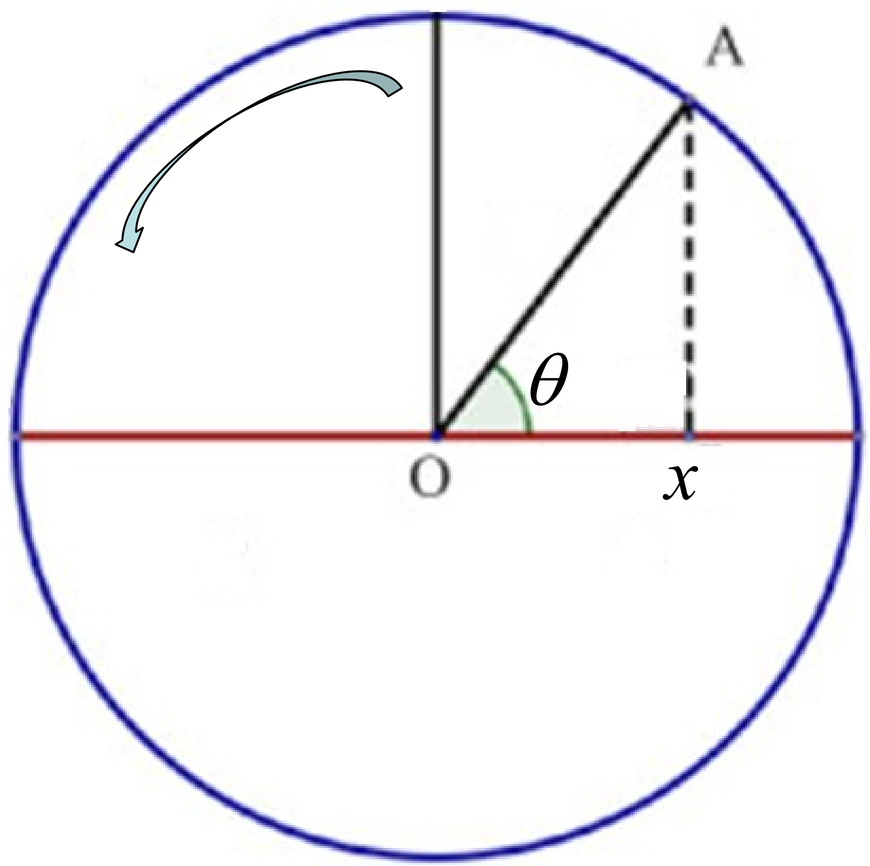
ϕ - initial angular position

$$x = A\cos(\omega t + \phi)$$

A - Amplitude

ω - Angular frequency

ϕ - initial phase

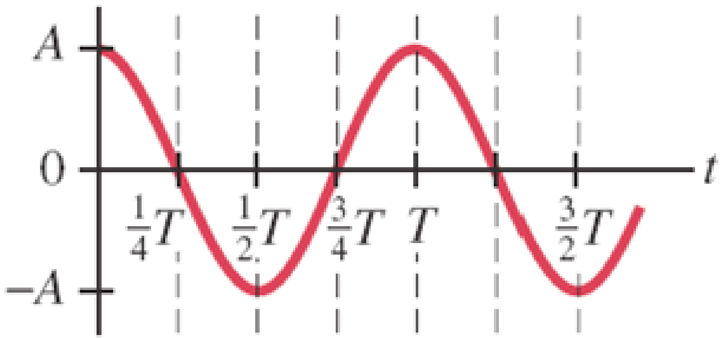


These figures illustrate the meaning of φ:

$$x = A\cos(\omega t + \phi)$$

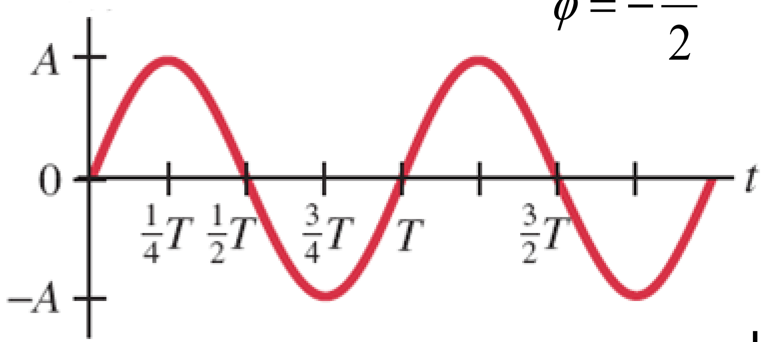
$$x = A\cos(\omega t)$$

$$\phi = 0$$

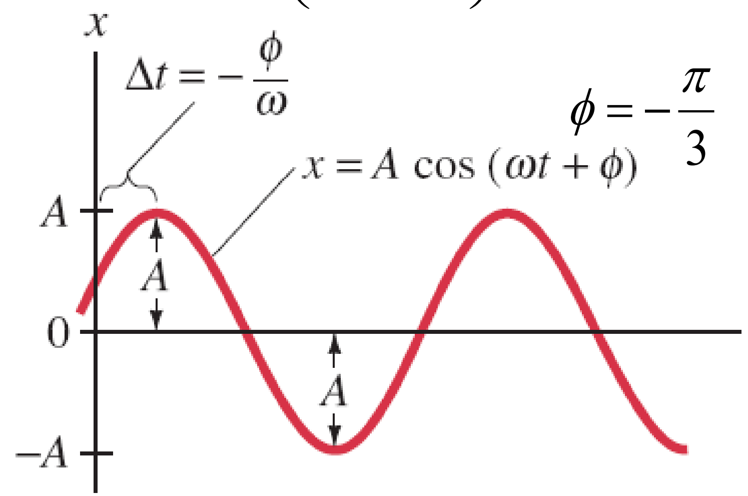


$$x = A\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\phi = -\frac{\pi}{2}$$



$$x = A\cos\left(\omega t - \frac{\pi}{3}\right)$$



Since $\cos 0 = 1$,

x_{max} occurs when $\omega t + \phi = 0$,

i.e., $t = -\dfrac{\phi}{\omega}$

φ < 0 if the nearest max is on the right.

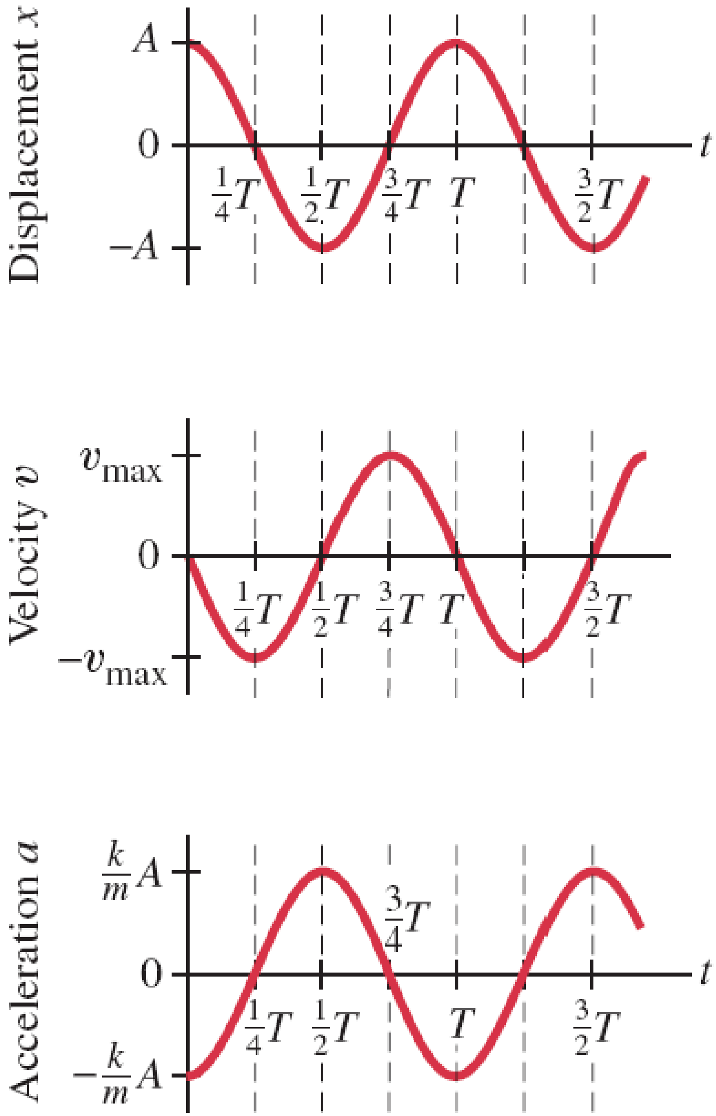
Simple Harmonic Motion

Because $\omega = 2\pi f = \sqrt{k/m}$, **then**

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}},$$

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Simple Harmonic Motion



The velocity and acceleration for simple harmonic motion can be found by differentiating the displacement:

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi).$$

Simple Harmonic Motion

Example: A vibrating floor.

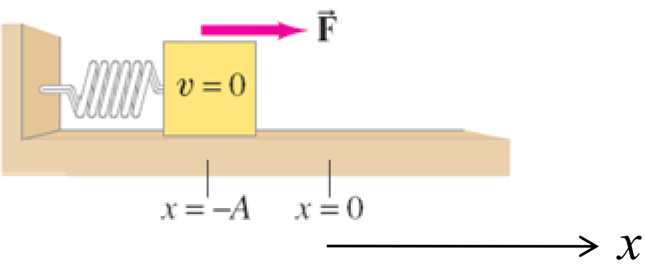
A large motor in a factory causes the floor to vibrate at a frequency of 10 Hz. The amplitude of the floor's motion near the motor is about 3.0 mm. Estimate the maximum acceleration of the floor near the motor.

[Solution] f=10Hz, $\omega=2\pi f=20\pi$; A=3.0 mm=3.0×10^{-3}m

$$a = -\omega^2 \mathrm{A}\cos(\omega t + \phi) \quad (\text{Here } a \text{ means } a_x)$$

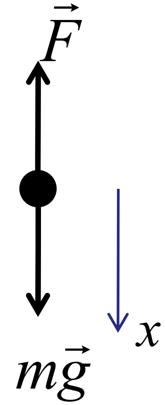
$$a_{max} = \omega^2 \mathrm{A}$$

$$a_{max} = (20\pi)^2 \times 3.0 \times 10^{-3} = 12 \mathrm{m/s}^2$$



Example 11-4: Spring calculations.

A spring stretches 0.150 m when a 0.300-kg mass is gently attached to it. The spring is then set up horizontally with the 0.300-kg mass resting on a frictionless table. The mass is pushed so that the spring is compressed 0.100 m from the equilibrium point, and released from rest. Determine: (a) the spring stiffness constant k and angular frequency ω; (b) the amplitude of the horizontal oscillation A; (c) the magnitude of the maximum velocity $v_{\max}$; (d) the magnitude of the maximum acceleration $a_{\max}$ of the mass; (e) the period T and frequency f; (f) the displacement x as a function of time; and (g) the velocity at t = 0.150 s.



(a)
$$\vec{F} + m\vec{g} = 0, \quad mg + F_x = 0, \quad mg - kx = 0$$

$$k = \frac{mg}{x} = \frac{0.3\times 9.8}{0.15} = 19.6\, N/m; \qquad \omega = \sqrt{\frac{k}{m}} = 8.08\, rad/s$$

(b) A=0.100m

(c) $v = -\omega A\sin(\omega t + \phi)$
$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A = 0.100\sqrt{\frac{19.6}{0.300}} = 0.808 m/s$$

(d) $a = -\omega^2 A\cos(\omega t + \phi)$ (Here a means a_x)

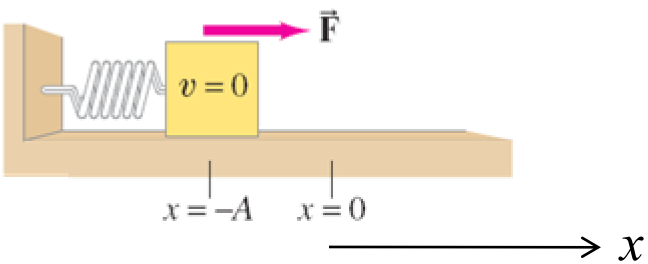
$$a_{max} = \omega^2 A = \frac{k}{m} A = \frac{19.6}{0.300} \times 0.100 = 6.53 m/s^2$$

(e)

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.6}{0.300}} = 8.08\ rad/s$$

$$f = \frac{\omega}{2\pi} = 1.29\ Hz \quad \text{(cycles/second)}$$

$$T = \frac{1}{f} = 0.777\ s$$



(f) In general, $x = A\cos(\omega t + \phi)$

From a specific given point, here for example, when t=0, x= -0.100;

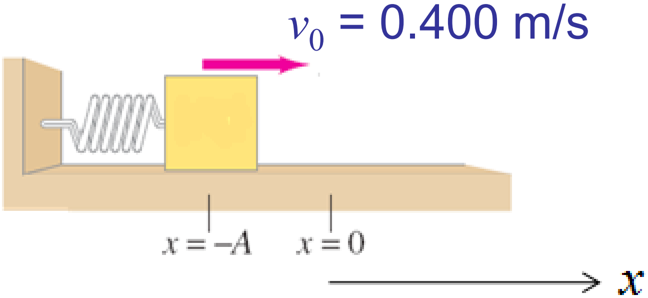
We can determine ϕ: $0.100 = -0.100\cos(\phi) \Rightarrow \phi = -\pi$ or π

$$\therefore \quad x = A\cos(\omega t + \phi) = 0.100\cos(8.08t - \pi) = -0.100\cos(8.08t)$$

(g) $v = \frac{dx}{dt} = (8.80)0.100\,sin(8.08t) = 0.808\,sin(8.08 \times 0.15) = 0.756 m/s$

Example: Spring is started with a push.

Suppose the spring of Example 11–4 (where ω = 8.08 s^{-1}) is compressed 0.100 m from equilibrium (x_0 = -0.100 m) but is given a shove to create a velocity in the +x direction of v_0 = 0.400 m/s. Determine (a) the phase angle ϕ, (b) the amplitude A, and (c) the displacement x as a function of time, $x(t)$.



[Solution] $x = A\cos(\omega t + \phi), \quad v = -\omega A \sin(\omega t + \phi)$

Use the initial conditions to determine A and ϕ:

$$\text{when } t = 0, x = -0.100\, m, \; v = 0.400\, m/s$$

$$i.e., \quad -0.1 = A\cos\phi \qquad (1)$$

$$0.4 = -\omega A \sin\phi \qquad (2)$$

$$\frac{(2)}{(1)}: \quad -4 = -\omega \tan\phi$$

$$\tan\phi = \frac{4}{\omega} = \frac{4}{8.08} = 0.495$$

$$\phi = 26.3°, \quad \text{or} \quad 206.3°$$

Since $\sin\phi < 0$, and $\cos\phi < 0$,

$$\phi = 206.3° = 3.60 \text{ rad}$$

$$\text{From (1)}, \quad A = \frac{-0.1}{\cos\phi} = \frac{-0.1}{\cos 206.3°} = 0.112m$$

$$x = 0.112 \cos(8.08t + 3.60)$$

Energy in the Simple Harmonic Oscillator

The **mechanical energy** of an object in simple harmonic motion is:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 .$$

i-clicker question 30-1:

Is the mechanical energy of a simple harmonic oscillator conserved?

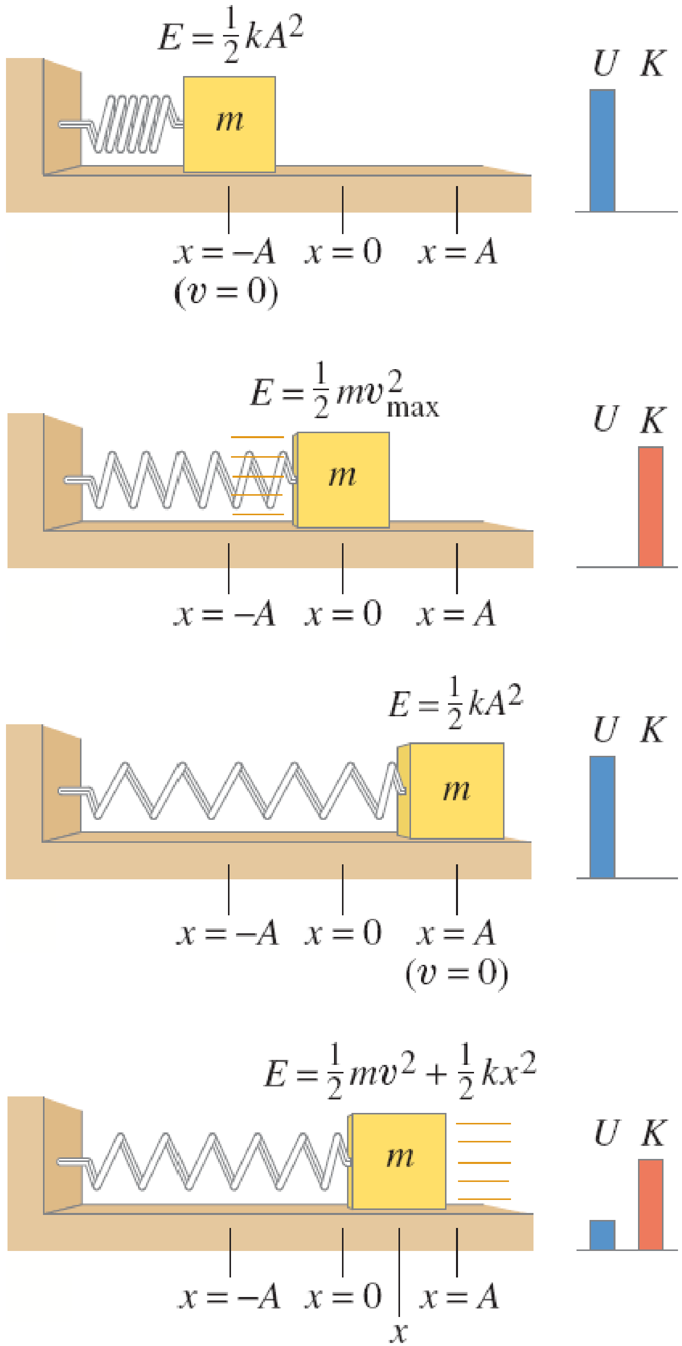
(A) Yes.

(B) No.

Why?

By definition, **simple** harmonic motion means no friction, more generally no energy loss.

Energy in the Simple Harmonic Oscillator



If the mass is at the **limits** of its motion, the energy is all **potential.**

If the mass is at the **equilibrium** point, the energy is all **kinetic.**

We know what the potential energy is at the turning points:

$$E = \frac{1}{2}kA^2.$$

Which is equal to the total mechanical energy.

Energy in the Simple Harmonic Oscillator

The **total energy** is, therefore, $\frac{1}{2}kA^2.$

And we can write:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2.$$

This can be solved for the **velocity** as a function of **position**:

$$v^2 = \frac{k}{m}A^2 - \frac{k}{m}x^2 = \frac{k}{m}A^2\left(1 - x^2\right)$$

$$v = \pm v_{\max}\sqrt{1 - \frac{x^2}{A^2}},$$

$$v = -\omega A \sin(\omega t + \phi)$$

where

$$v_{\max}^2 = (k/m)A^2.$$

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A$$

Example: Energy calculations.

For the simple harmonic oscillation of Example 11–4 (where k = 19.6 N/m, A = 0.100 m, x = -(0.100 m) cos(8.08t), and v = (0.808 m/s) sin 8.08t), determine (a) the total energy, (b) the kinetic and potential energies as a function of time, (c) the velocity when the mass is 0.050 m from equilibrium, (d) the kinetic and potential energies at half amplitude (x = ± A/2).

(a)

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(19.6)(0.1)^2 = 0.098J$$

(b)

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m[\omega A\, sin(\omega t + \phi)]^2 = \frac{1}{2}(0.3)[(0.808)sin(8.08t)]^2 = 0.098\, sin^2(8.08t)$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}k[A\, cos(\omega t + \phi)]^2 = \frac{1}{2}(19.6)[(0.808)cos(8.08t)]^2 = 0.098\, cos^2(8.08t)$$

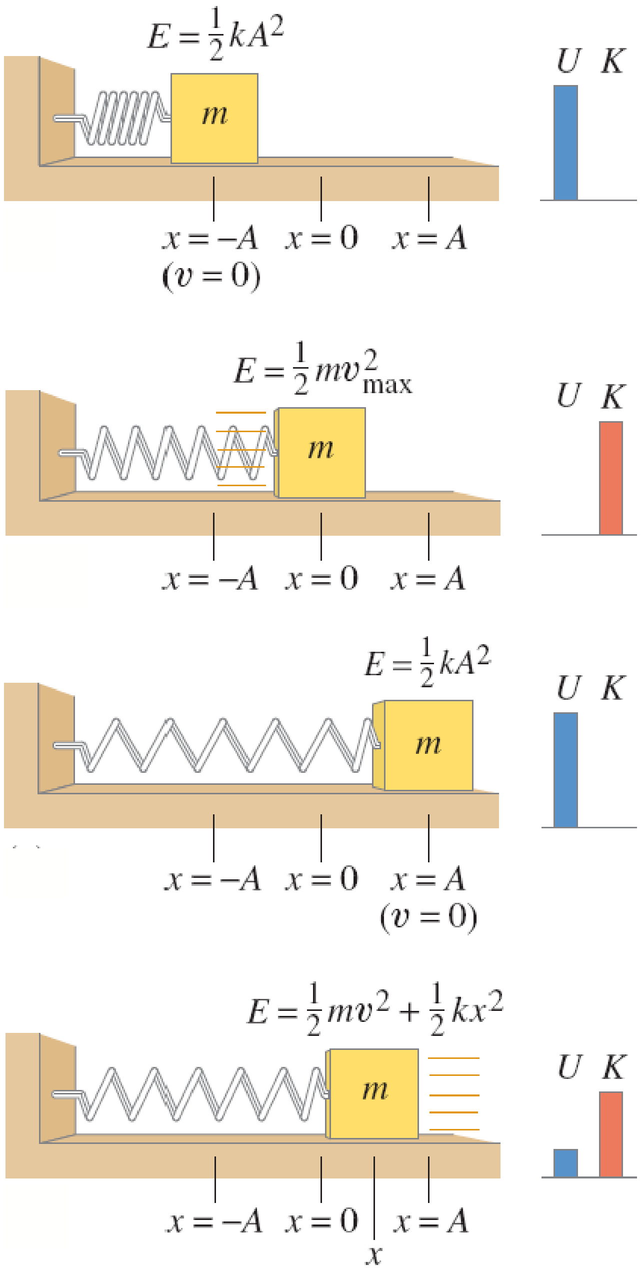
(c)

$$v = v_{max}\sqrt{1 - \frac{x^2}{A^2}} = 0.808\sqrt{1 - \left(\frac{1}{2}\right)^2} = 0.70\, m/s$$

(d)

$$U = \frac{1}{2}(19.6)(0.05)^2 = 0.0245\, J; \quad K = E - U = 0.098 - 0.0245 = 0.0735\, J.$$

Energy in the Simple Harmonic Oscillator



Conceptual Example: Doubling the amplitude.

Suppose this spring is stretched twice as far (to $x = 2A$).What happens to (a) the energy of the system, (b) the maximum velocity of the oscillating mass, (c) the maximum acceleration of the mass?

(a) The energy quadruples since

$$E \propto A^2$$

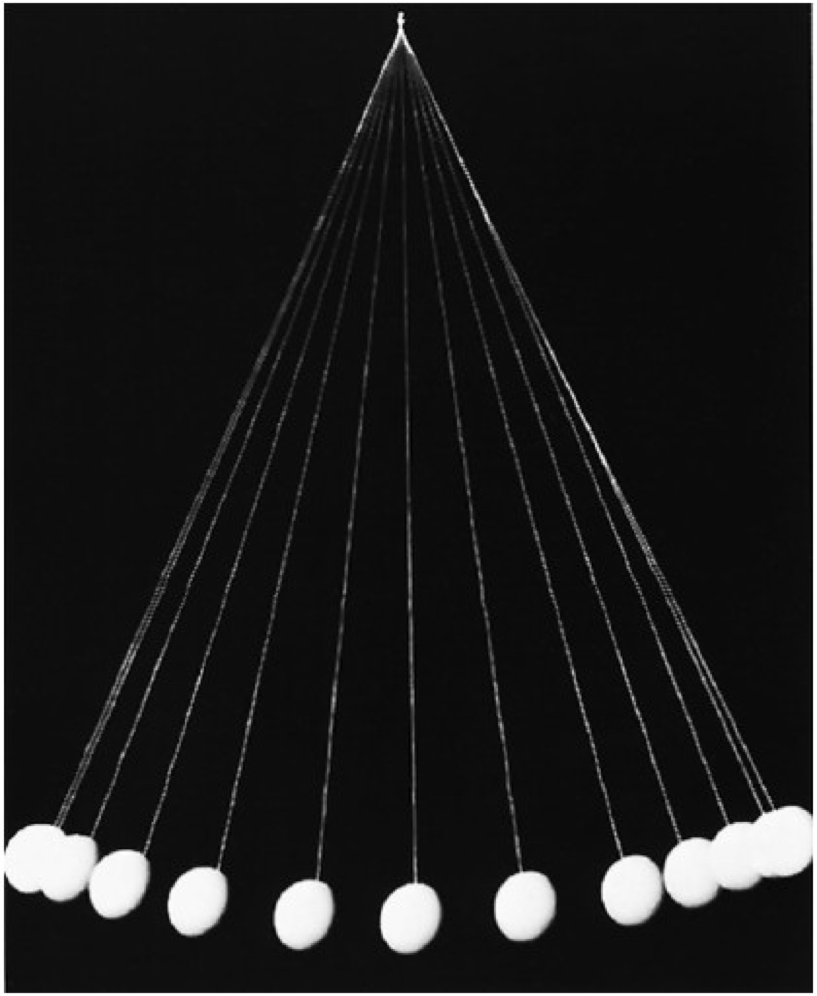
(b) The maximum velocity doubles

since $v_{max} \propto \sqrt{E}$ $\left(E = \frac{1}{2} m v_{max}^2 \right)$

(c) The maximum acceleration doubles since

$$a_{max} = \omega^2 A \qquad v_{max} = \omega A$$

The Simple Pendulum



A **simple pendulum** consists of a mass at the end of a lightweight cord. We assume that the cord does not **stretch**, and that its mass is **negligible**.

The Simple Pendulum

In order to be in SHM, the restoring force must be **proportional** to the negative of the **displacement**. Here we have:

$$F = -mg\sin\theta,$$

which is proportional to **sin *θ*** and not to *θ* itself.

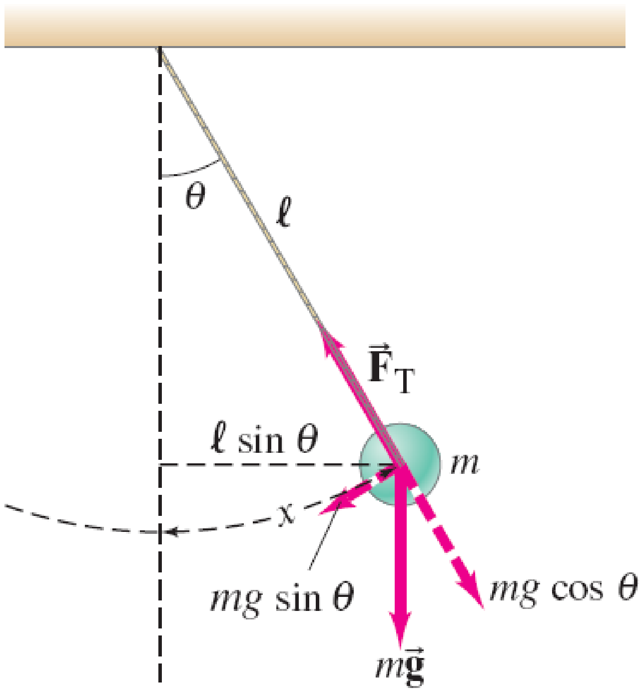
However, if the angle is **small**, **sin θ ≈ θ**.

Therefore, for small angles, we have

$$F \approx -\frac{mg}{l}x = m\frac{d^2x}{dt^2}$$

where $x = l\theta$

$$\text{Then, } \frac{d^2x}{dt^2} = -\frac{g}{l}x$$



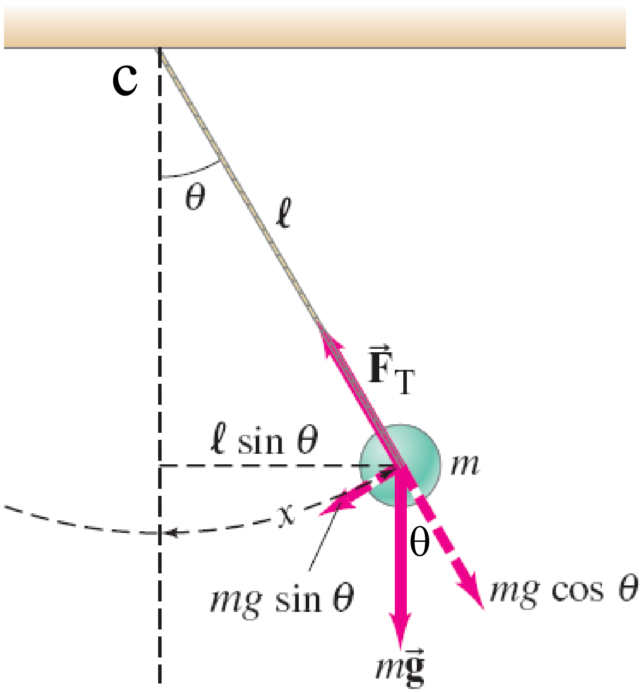
Therefore, it is a SHM with

$$\omega = \sqrt{\frac{g}{l}}$$

Another approach: from the angular point of view

The Simple Pendulum

From the angular point of view



$$\tau = I\alpha$$

$$-mgl\,sin\,\theta = ml^2\alpha$$

For small θ, $$sin\,\theta \approx \theta \quad (rad)$$

$$-g\theta = l\alpha$$

$$\alpha = -\frac{g}{l}\theta$$

"-" sign: the direction of τ is opposite to the direction of θ.

The angular acceleration is proportional to the negative angular displacement. Therefore, it's SHM.

Compared to, $$a = -\omega^2 x$$

We have $\omega = \sqrt{\frac{g}{l}}$

We can measure g!

The Simple Pendulum

Example: Measuring *g*.

A geologist uses a simple pendulum that has a length of 37.10 cm and a frequency of 0.8190 Hz at a particular location on the Earth. What is the acceleration of gravity at this location?

$$\omega = \sqrt{\frac{g}{l}}$$

$$g = \omega^2 l = (2\pi f)^2 l = (2\pi \times 0.8190)^2 (0.371) = 9.824\, m/s^2$$

Tutorials