

Phys101 Lectures 26, 27

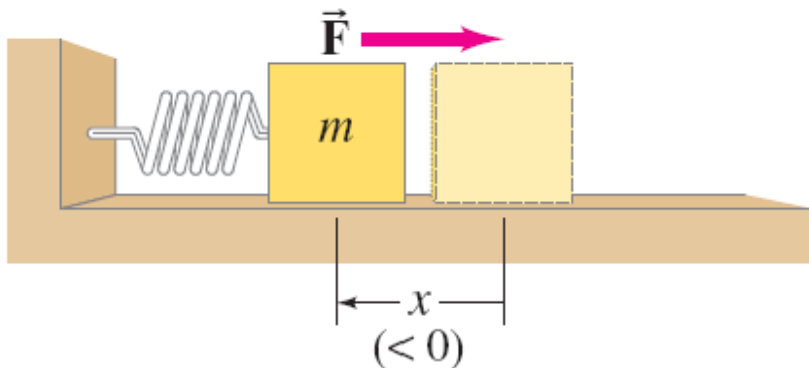
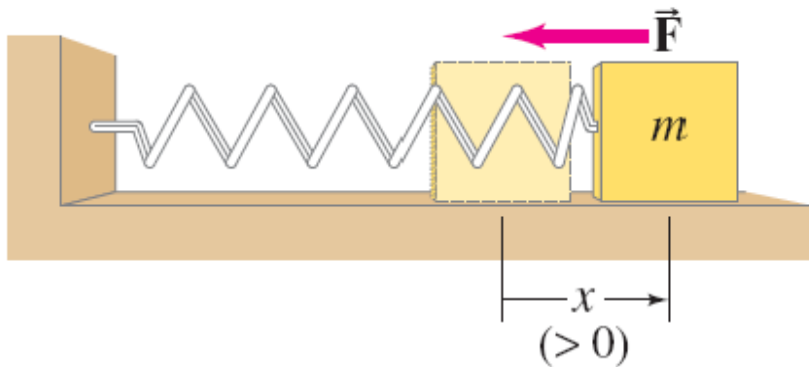
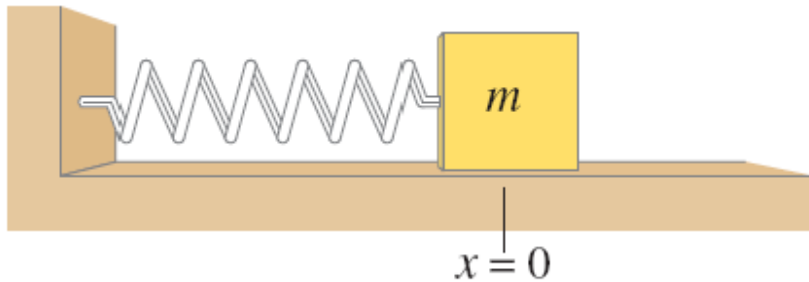
Oscillations

Key points:

- Simple Harmonic Motion (SHM)
- SHM Related to Uniform Circular Motion
- The Simple Pendulum

Ref: 11-1,2,3,4.

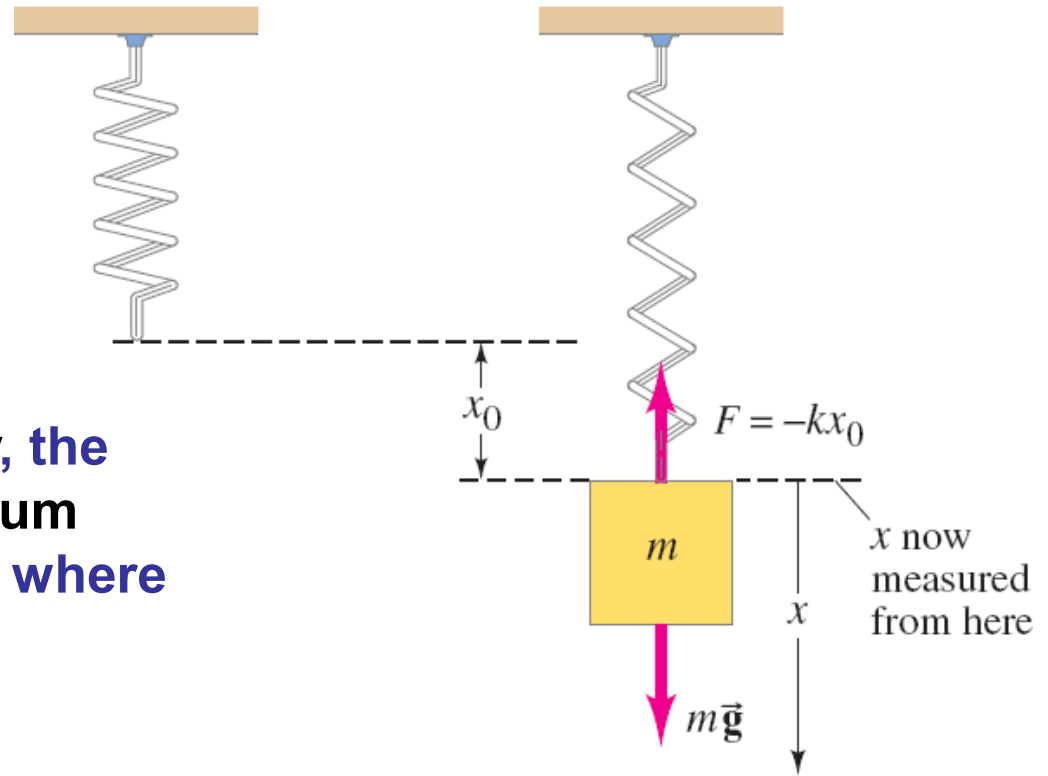
Oscillations of a Spring



If an object **oscillates** back and forth over the same path, each cycle taking the same amount of time, the motion is called **periodic**. The mass and spring system is a useful model for a periodic system.

Oscillations of a Spring

If the spring is hung vertically, the only change is in the equilibrium position, which is at the point where the spring force equals the gravitational force.



Demo

Oscillations of a Spring

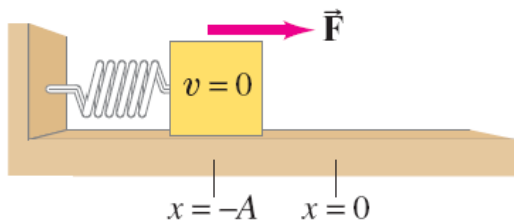
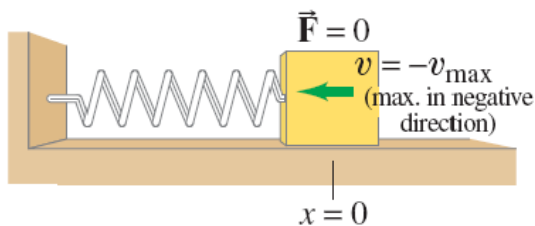
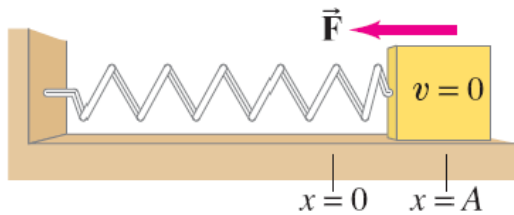
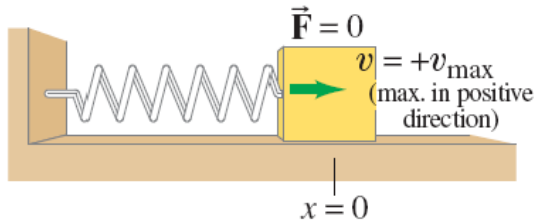
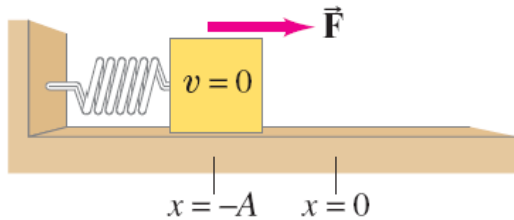
We assume that the system is **frictionless**. There is a point where the spring is neither stretched nor compressed; this is the **equilibrium position**. We measure **displacement** from that point ($x = 0$ on the previous figure).

The **force exerted by the spring depends on the displacement**:

$$F = -kx.$$

- The **minus sign on the force indicates that it is a restoring force—it is directed to restore the mass to its equilibrium position.**
- **k is the spring constant.**
- **Since the force is not constant, the acceleration is not constant either.**

Oscillations of a Spring



- Displacement is measured from the equilibrium point.
- Amplitude is the maximum displacement.
- A cycle is a full to-and-fro motion.
- Period, T , is the time required to complete one cycle.
- Frequency, f , is the number of cycles completed per second. The unit of frequency is Hz (cycles per second).

$$f = \frac{1}{T}$$

Simple Harmonic Motion

Any vibrating system where the restoring force is proportional to the negative of the displacement is in simple harmonic motion (SHM), and is often called a simple harmonic oscillator (SHO).

Substituting $F = -kx$ into Newton's second law gives the equation of motion:

$$-kx = ma = m \frac{d^2x}{dt^2}, \quad i.e. \quad a = -\frac{k}{m}x$$

$$\text{Or,} \quad a = -\omega^2 x \quad \text{where} \quad \omega^2 = \frac{k}{m}$$

The solution has the form:

$$x = A \cos(\omega t + \phi).$$

How do you know?

We can guess

$$v = \frac{dx}{dt} = \frac{d}{dt} [A \cos(\omega t + \phi)] = -\omega A \sin(\omega t + \phi) \quad \leftarrow \text{Velocity}$$

and then verify:

$$a = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi) = -\omega^2 x \quad \leftarrow \text{Acceleration}$$

Simple Harmonic Motion Related to Uniform Circular Motion

If we look at the **projection** onto the **x axis** of an object moving in a **circle of radius A** at a **constant angular velocity ω** , we find that the **x component of the circular motion** is in fact a **SHM**.

Demo

$$x = A \cos(\theta)$$

$$\theta = \omega t + \phi$$

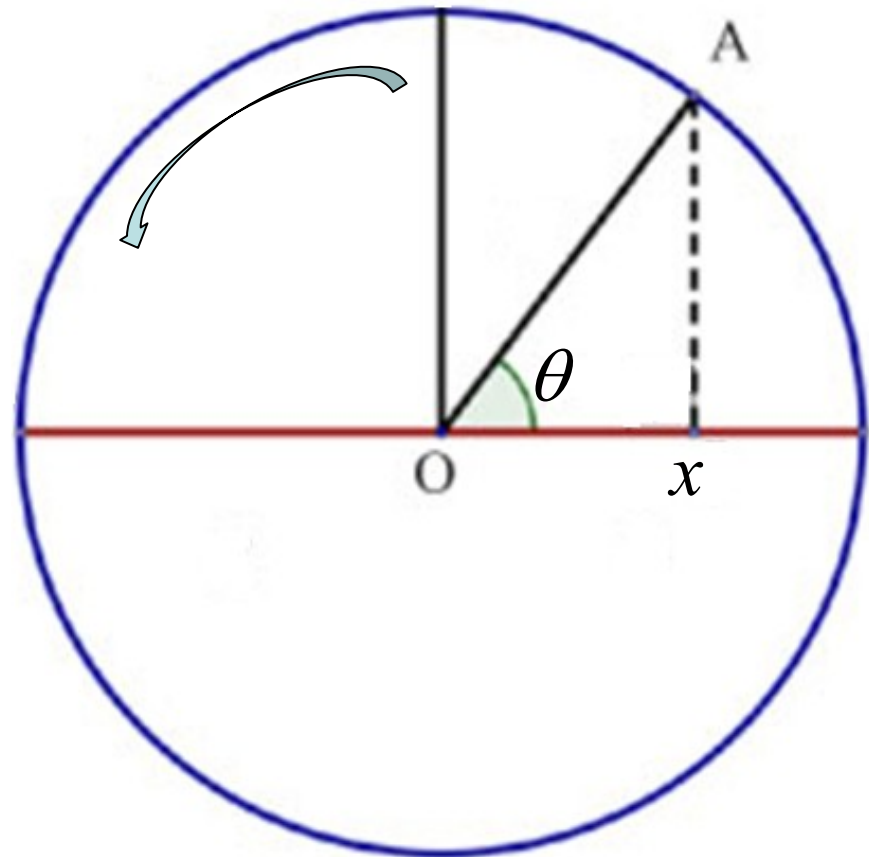
ϕ - initial angular position

$$x = A \cos(\omega t + \phi)$$

A - Amplitude

ω - Angular frequency

ϕ - initial phase

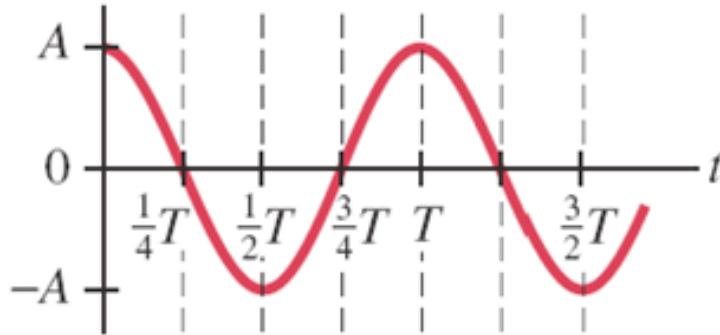


These figures illustrate the meaning of ϕ :

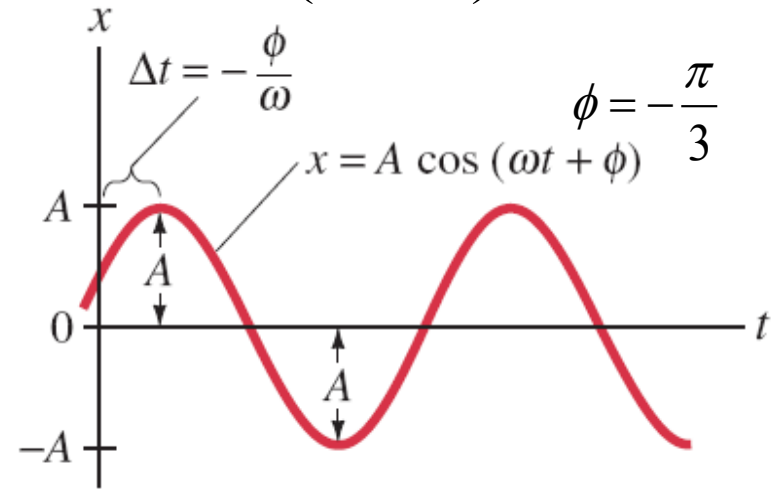
$$x = A \cos(\omega t + \phi)$$

$$x = A \cos(\omega t)$$

$$\phi = 0$$

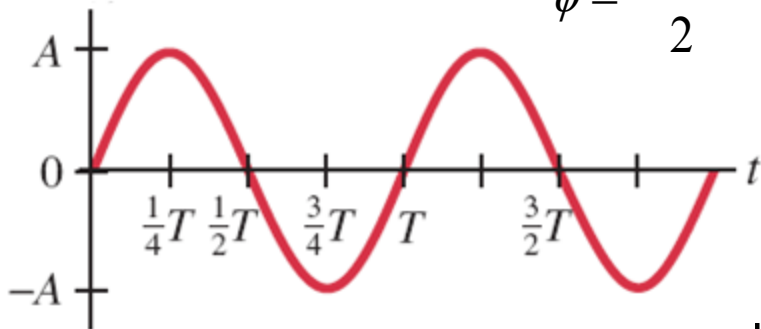


$$x = A \cos\left(\omega t - \frac{\pi}{3}\right)$$



$$x = A \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\phi = -\frac{\pi}{2}$$



Since $\cos 0 = 1$,

$x_{\max}$ occurs when $\omega t + \phi = 0$,

$$\text{i.e., } t = -\frac{\phi}{\omega}$$

$\phi < 0$ if the nearest max is on the right.

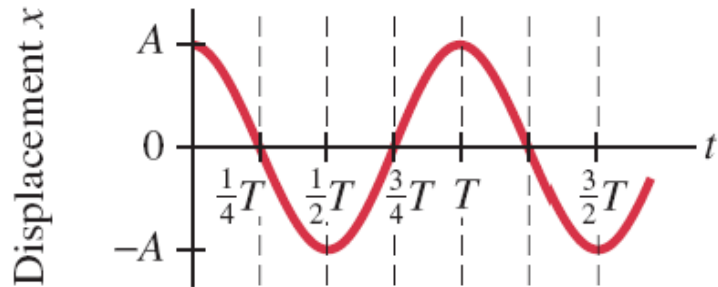
Simple Harmonic Motion

Because $\omega = 2\pi f = \sqrt{k/m}$, then

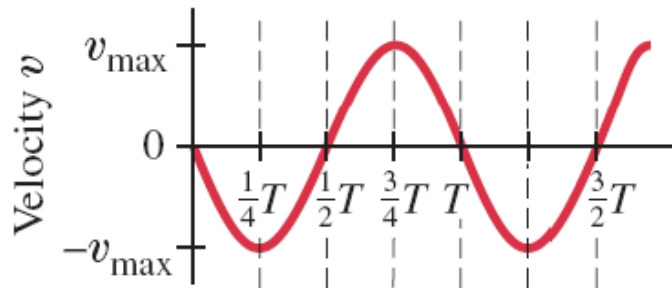
$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}},$$

$$T = 2\pi \sqrt{\frac{m}{k}}.$$

Simple Harmonic Motion

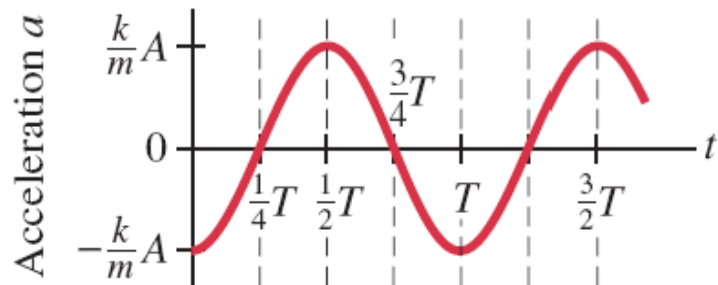


The velocity and acceleration for simple harmonic motion can be found by differentiating the displacement:



$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

$$v_{max} = \omega A$$



$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = -\omega^2 A \cos(\omega t + \phi).$$

$$a_{max} = \omega^2 A$$

Simple Harmonic Motion

Example: A vibrating floor.

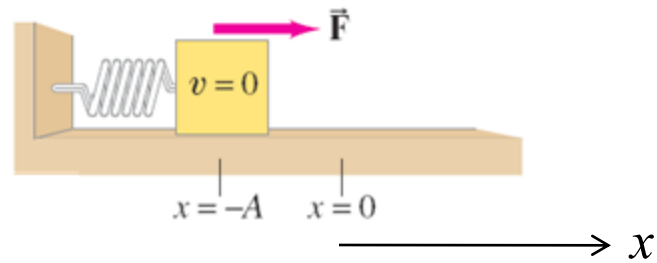
A large motor in a factory causes the floor to vibrate at a frequency of 10 Hz. The amplitude of the floor's motion near the motor is about 3.0 mm. Estimate the maximum acceleration of the floor near the motor.

[Solution] $f=10\text{Hz}$, $\omega=2\pi f=20\pi$; $A=3.0\text{ mm}=3.0\times 10^{-3}\text{m}$

$$a = -\omega^2 A \cos(\omega t + \phi) \quad (\text{Here } a \text{ means } a_x)$$

$$a_{\max} = \omega^2 A$$

$$a_{\max} = (20\pi)^2 \times 3.0 \times 10^{-3} = 12\text{m/s}^2$$



Example 11-4: Spring calculations.

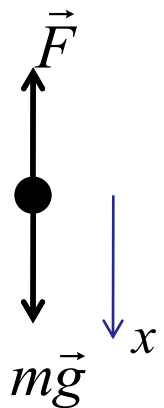
A spring stretches 0.150 m when a 0.300-kg mass is gently attached to it vertically. The spring is then set up horizontally with the 0.300-kg mass resting on a frictionless table. The mass is pushed so that the spring is compressed 0.100 m from the equilibrium point, and released from rest. Determine: (a) the spring stiffness constant k and angular frequency ω ; (b) the amplitude of the horizontal oscillation A ; (c) the magnitude of the maximum velocity $v_{\max}$; (d) the magnitude of the maximum acceleration $a_{\max}$ of the mass; (e) the period T and frequency f ; (f) the displacement x as a function of time; and (g) the velocity at $t = 0.150$ s.

(a) $\vec{F} + m\vec{g} = 0, \quad mg + F_x = 0, \quad mg - kx = 0$

$$k = \frac{mg}{x} = \frac{0.3 \times 9.8}{0.15} = 19.6 \text{ N/m}; \quad \omega = \sqrt{\frac{k}{m}} = 8.08 \text{ rad/s}$$

(b) $A = 0.100 \text{ m}$

(c) $v = -\omega A \sin(\omega t + \phi) \quad v_{\max} = \omega A = \sqrt{\frac{k}{m}} A = 0.100 \sqrt{\frac{19.6}{0.300}} = 0.808 \text{ m/s}$



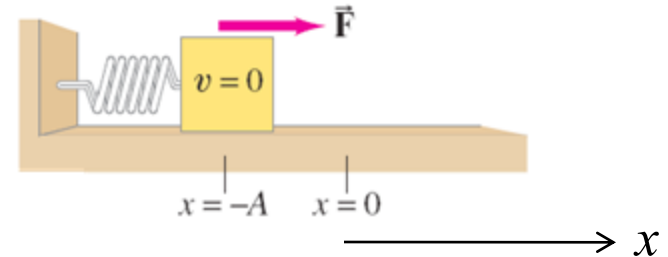
(d) $a = -\omega^2 A \cos(\omega t + \phi)$ (Here a means a_x)

$$a_{max} = \omega^2 A = \frac{k}{m} A = \frac{19.6}{0.300} \times 0.100 = 6.53 \text{ m/s}^2$$

(e) $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.6}{0.300}} = 8.08 \text{ rad/s}$

$$f = \frac{\omega}{2\pi} = 1.29 \text{ Hz (cycles/second)}$$

$$T = \frac{1}{f} = 0.777 \text{ s}$$



(f) In general, $x = A \cos(\omega t + \phi)$

From a specific given point, here for example, when $t=0$, $x = -0.100$;

We can determine ϕ : $-0.100 = 0.100 \cos(\phi) \Rightarrow \phi = -\pi$ or π

$$\therefore x = A \cos(\omega t + \phi) = 0.100 \cos(8.08t - \pi) = -0.100 \cos(8.08t)$$

(g) $v = \frac{dx}{dt} = (8.08)0.100 \sin(8.08t) = 0.808 \sin(8.08 \times 0.15) = 0.756 \text{ m/s}$

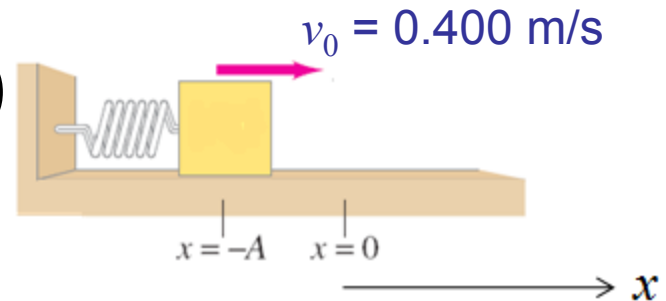
Example: Spring is started with a push.

Suppose the spring of Example 11–4 (where $\omega = 8.08 \text{ s}^{-1}$) is compressed 0.100 m from equilibrium ($x_0 = -0.100 \text{ m}$) but is given a shove to create a velocity in the $+x$ direction of $v_0 = 0.400 \text{ m/s}$.

Determine (a) the phase angle ϕ , (b) the amplitude A , and (c) the displacement x as a function of time, $x(t)$.

[Solution] $x = A \cos(\omega t + \phi)$, $v = -\omega A \sin(\omega t + \phi)$

Use the initial conditions to determine A and ϕ :



when $t = 0$, $x = -0.100 \text{ m}$, $v = 0.400 \text{ m/s}$

$$\text{i.e., } -0.1 = A \cos \phi \quad (1)$$

$$0.4 = -\omega A \sin \phi \quad (2)$$

$$\phi = 26.3^\circ, \text{ or } 206.3^\circ$$

Since $\sin \phi < 0$, and $\cos \phi < 0$,

$$\phi = 206.3^\circ = 3.60 \text{ rad}$$

$$\frac{(2)}{(1)}: -4 = -\omega \tan \phi$$

$$\tan \phi = \frac{4}{\omega} = \frac{4}{8.08} = 0.495$$

$$\text{From (1), } A = \frac{-0.1}{\cos \phi} = \frac{-0.1}{\cos 206.3^\circ} = 0.112 \text{ m}$$

$$x = 0.112 \cos(8.08t + 3.60)$$

Energy in the Simple Harmonic Oscillator

The mechanical energy of an object in simple harmonic motion is:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

i-clicker question 26-1:

Is the mechanical energy of a simple harmonic oscillator conserved?

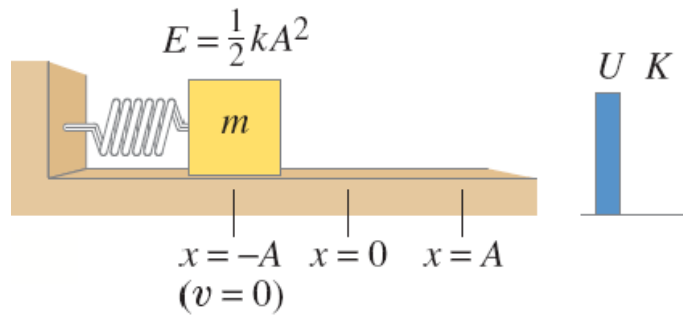
(A) Yes.

(B) No.

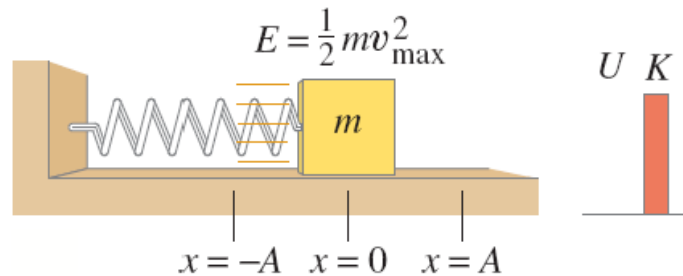
Why?

By definition, **simple** harmonic motion means no friction, more generally no energy loss.

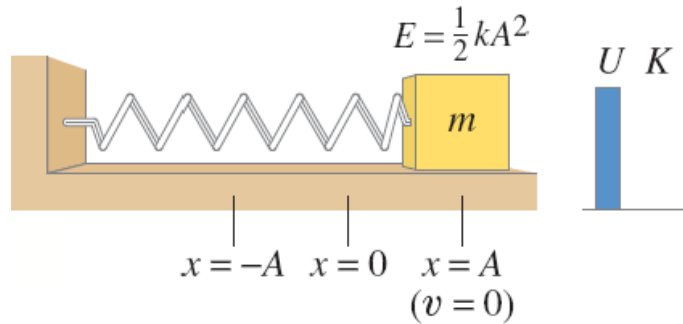
Energy in the Simple Harmonic Oscillator



If the mass is at the **limits** of its motion, the energy is **all** potential.

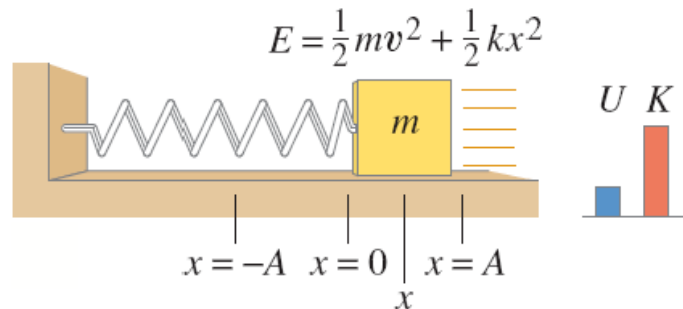


If the mass is at the **equilibrium** point, the energy is **all** kinetic.



We know what the potential energy is at the turning points:

$$E = \frac{1}{2} k A^2.$$



Which is equal to the **total** mechanical energy.

Energy in the Simple Harmonic Oscillator

The total energy is, therefore, $\frac{1}{2} k A^2$.

And we can write:

$$\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2.$$

This can be solved for the **velocity** as a function of **position**:

$$v^2 = \frac{k}{m} A^2 - \frac{k}{m} x^2 = \frac{k}{m} A^2 \left(1 - \frac{x^2}{A^2} \right)$$

$$v = \pm v_{\max} \sqrt{1 - \frac{x^2}{A^2}},$$

$$v = -\omega A \sin(\omega t + \phi)$$

where

$$v_{\max}^2 = (k/m) A^2.$$

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}} A$$

Example: Energy calculations.

For the simple harmonic oscillation of Example 11–4 (where $k = 19.6 \text{ N/m}$, $A = 0.100 \text{ m}$, $x = -(0.100 \text{ m}) \cos(8.08t)$, and $v = (0.808 \text{ m/s}) \sin 8.08t$), determine (a) the total energy, (b) the kinetic and potential energies as a function of time, (c) the velocity when the mass is 0.050 m from equilibrium, (d) the kinetic and potential energies at half amplitude ($x = \pm A/2$).

(a)

$$E = \frac{1}{2} k A^2 = \frac{1}{2} (19.6)(0.1)^2 = 0.098 \text{ J}$$

(b)

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m [\omega A \sin(\omega t + \phi)]^2 = \frac{1}{2} (0.3)[(0.808) \sin(8.08t)]^2 = 0.098 \sin^2(8.08t)$$

$$U = \frac{1}{2} k x^2 = \frac{1}{2} k [A \cos(\omega t + \phi)]^2 = \frac{1}{2} (19.6)[(0.808) \cos(8.08t)]^2 = 0.098 \cos^2(8.08t)$$

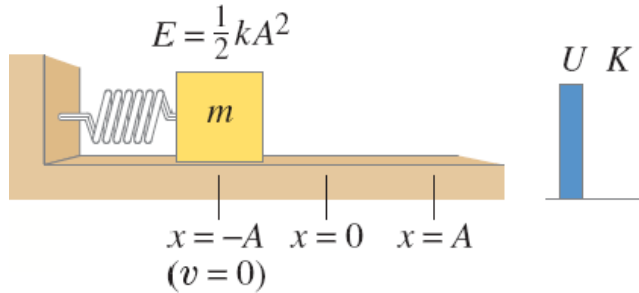
(c)

$$v = v_{\max} \sqrt{1 - \frac{x^2}{A^2}} = 0.808 \sqrt{1 - \left(\frac{1}{2}\right)^2} = 0.70 \text{ m/s}$$

(d)

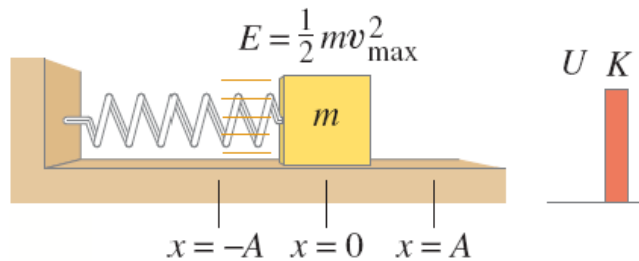
$$U = \frac{1}{2} (19.6)(0.05)^2 = 0.0245 \text{ J}; \quad K = E - U = 0.098 - 0.0245 = 0.0735 \text{ J}.$$

Energy in the Simple Harmonic Oscillator



Conceptual Example: Doubling the amplitude.

Suppose this spring is stretched twice as far (to $x = 2A$). What happens to (a) the energy of the system, (b) the maximum velocity of the oscillating mass, (c) the maximum acceleration of the mass?



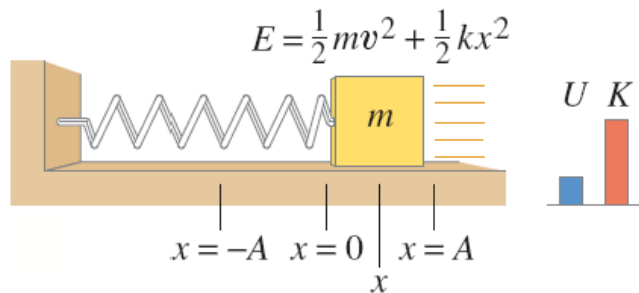
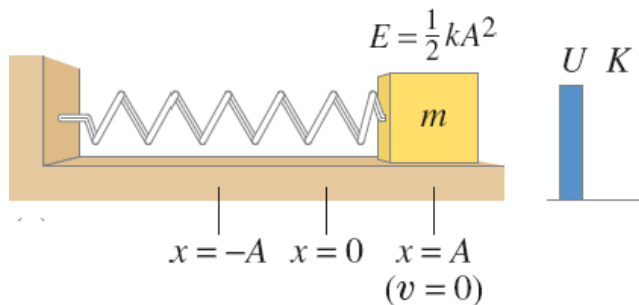
(a) The energy quadruples since

$$E \propto A^2$$

(b) The maximum velocity doubles

since $v_{\max} \propto \sqrt{E}$ $\left(E = \frac{1}{2}mv_{\max}^2 \right)$

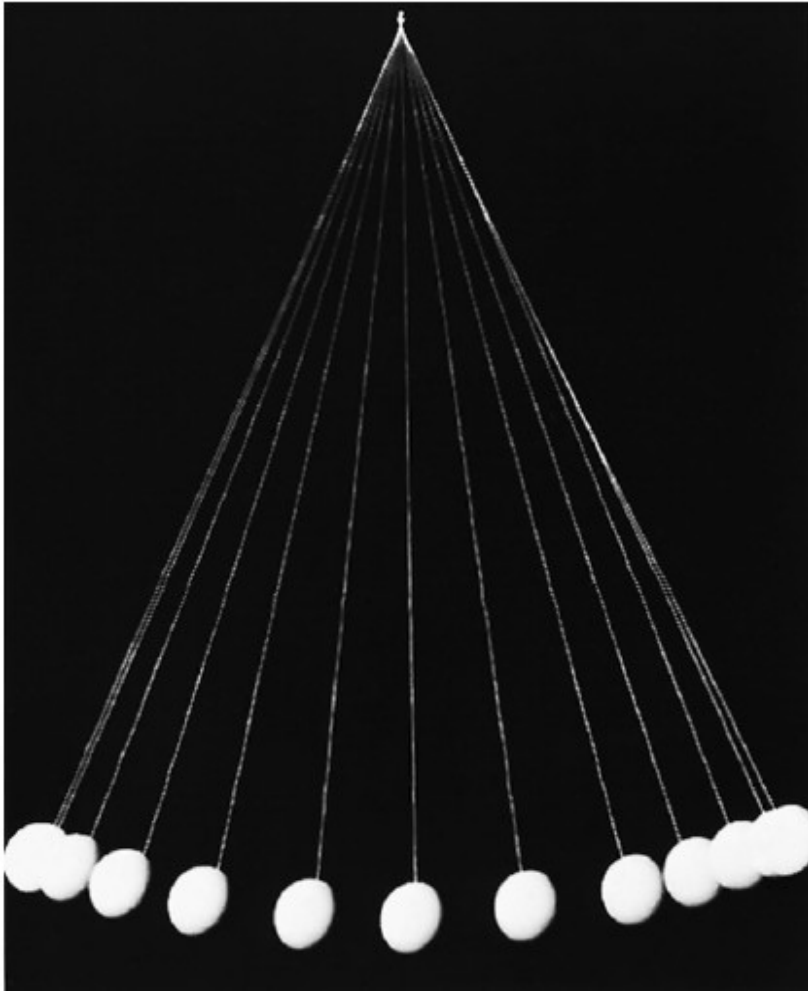
(c) The maximum acceleration doubles since



$$a_{\max} = \omega^2 A$$

$$v_{\max} = \omega A$$

The Simple Pendulum



A simple pendulum consists of a mass at the end of a lightweight cord. We assume that the cord does not stretch, and that its mass is negligible.

The Simple Pendulum

In order to be in SHM, the restoring force must be proportional to the negative of the displacement. Here we have:

$$F = -mg \sin \theta,$$

which is proportional to $\sin \theta$ and not to θ itself.

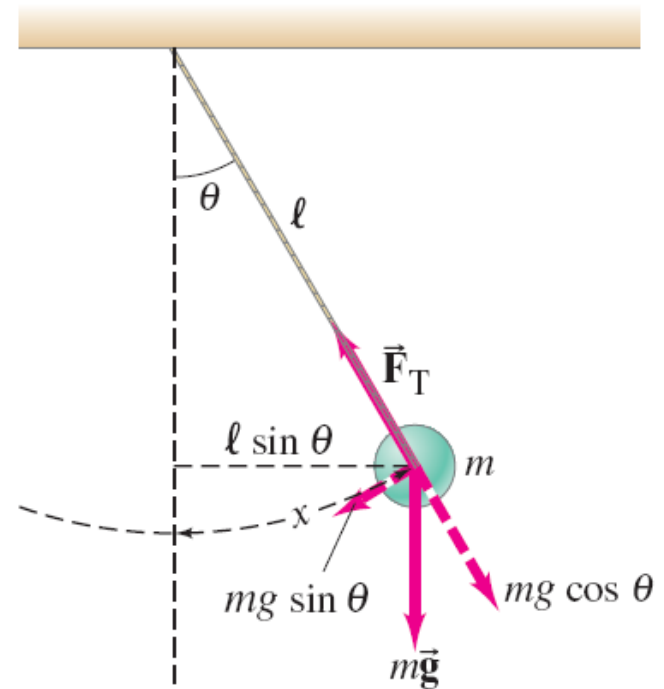
However, if the angle is small, $\sin \theta \approx \theta$.

Therefore, for small angles, we have

$$F \approx -\frac{mg}{l}x = m \frac{d^2x}{dt^2}$$

where $x = l\theta$

$$\text{Then, } \frac{d^2x}{dt^2} = -\frac{g}{l}x$$



Therefore, it is a SHM with

$$\omega = \sqrt{\frac{g}{l}}$$

Another approach: from the angular point of view

The Simple Pendulum

From the angular point of view

$$\tau = I\alpha$$

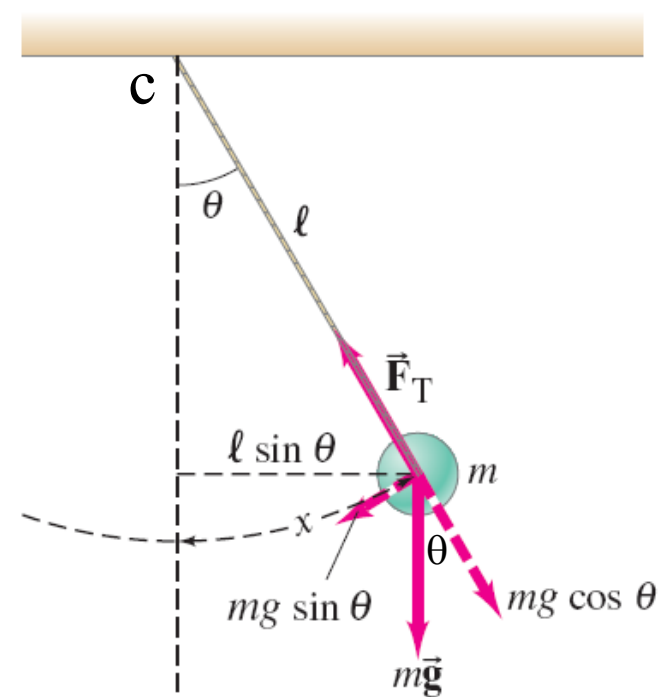
$$-mgl \sin \theta = ml^2 \alpha$$

For small θ , $\sin \theta \approx \theta$ (rad)

$$-g\theta = l\alpha$$

$$\alpha = -\frac{g}{l} \theta$$

“-” sign: the direction of τ is opposite to the direction of θ .



The angular acceleration is proportional to the negative angular displacement. Therefore, it's SHM.

Compared to,

$$a = -\omega^2 x$$

We have $\omega = \sqrt{\frac{g}{l}}$

We can measure g !

The Simple Pendulum

Example: Measuring g .

A geologist uses a simple pendulum that has a length of 37.10 cm and a frequency of 0.8190 Hz at a particular location on the Earth. What is the acceleration of gravity at this location?

$$\omega = \sqrt{\frac{g}{l}}$$

$$g = \omega^2 l = (2\pi f)^2 l = (2\pi \times 0.8190)^2 (0.371) = 9.824 \text{ m} / \text{s}^2$$