Phys 221. Lecture 2.

Vector Calculus. Ref: 3-1, 2; 4-1, 2, 3, 4, 5.

- Gradient of a scalar field $V(x,y,z)$.
  - e.g. electric potential.

$$\text{grad } V \triangleq \nabla V \triangleq \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}.$$

— The result is a vector.

— Represents the max. space rate of change of $V(x,y,z)$.
(both magnitude and direction).

e.g. gradient of temperature predicts the heat flow.

∇ — "del" — is a vector operator. $\nabla \triangleq \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$.

e.g. Electrostatics. V — electric potential.
$\vec{E}$ — electric field.

$$\vec{E} = -\nabla V, \quad \text{i.e., } E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}.$$

"$-$" sign — $\vec{E}$ direction is from higher V to lower V.

in 1-D: $E = -\frac{\Delta V}{\Delta x}$.

OR: $E \cdot d = -\Delta V = -(V_f - V_i)$

Times q: $W = -\Delta U$. (electric work = loss in potential energy)

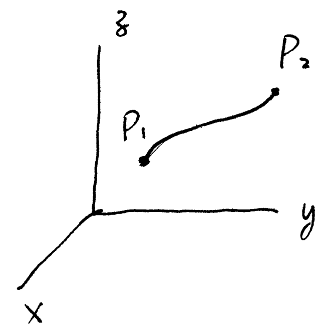
- Differential of $V(x, y, z)$:

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz .$$

$$= \nabla V \cdot \vec{dl}$$

Difference of V between locations P_1 and P_2

$\begin{pmatrix} \text{Imagine moving a charge } q \\ \text{from } P_1 \text{ to } P_2 \end{pmatrix}$.



$$V_2 - V_1 = \int_{P_1}^{P_2} dV = \int_{P_1}^{P_2} \nabla V \cdot \vec{dl}$$

$$\therefore \quad V_2 - V_1 = - \int_{P_1}^{P_2} \vec{E} \cdot \vec{dl} \qquad (\text{since } \vec{E} = -\nabla V)$$

(4-39).

Electric field $\vec{E}$ — $\vec{E} = \vec{F}/q$, electric force per unit charge.

Electric potential V — potential energy per unit charge.

(4-39) can be rewritten as.

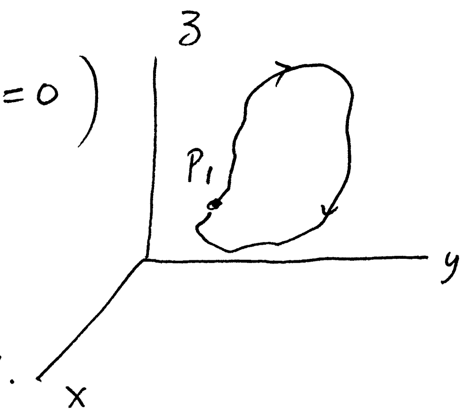
$$\underbrace{\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}}_{} = -(V_2 - V_1) = -\Delta V .$$

work done by $\vec{E}$-field = loss of potential energy.

(on an unit charge)

Along a closed path (starts from P_1 and ends at P_1)

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad (\because V_{P_1} - V_{P_1} = 0)$$



i.e. circulation of $\vec{E} = 0$.

meaning. $\vec{E}$ - field is conservative.

- work does not depend on path. only depends on initial and final positions.

  $\left(\text{e.g.: } \int_1^2 \vec{E} \cdot d\vec{l} = -(V_2 - V_1) \right)$

OR:

- work along any closed path $= 0$.

We just have shown: Any vector field (such as $\vec{E}$) that can be defined as a gradient of a scalar function is a conservative field.

$$\vec{E} = -\nabla V .$$

- Gradient operator in cylindrical and spherical coordinates.

cylindrical: $\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$.

spherical: $\nabla = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{R\sin\theta}\cdot\frac{\partial}{\partial \phi}$.

e.g: Given the electric potential of a point charge q located at the origin:

$$V = \frac{q}{4\pi\varepsilon_0}\frac{1}{R}$$

(We choose spherical coord. Then, V is independent of θ, ϕ, so is $\vec{E}$.)

Electric field:

$$\vec{E} = -\nabla V = -\hat{R}\frac{\partial}{\partial R}\left(\frac{q}{4\pi\varepsilon_0}\frac{1}{R}\right)$$

$$= \hat{R}\frac{q}{4\pi\varepsilon_0 R^2}.$$

(That's coulombe's Law)

• Divergence of a vector field.

Given a vector field $\vec{E}(x, y, z)$ its divergence is

$$\nabla \cdot \vec{E} \triangleq \operatorname{div} \vec{E} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right)$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}. \qquad (3\text{-}28)$$

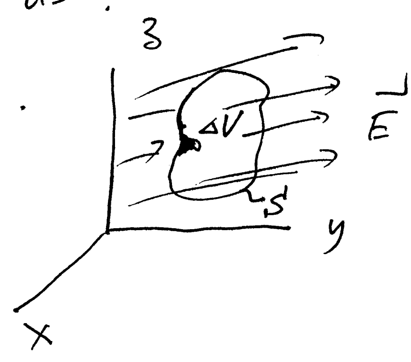
OR. (equivalent definition)

$$\nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{E} \cdot d\vec{s}}{\Delta V}.$$

ΔV — small volume. enclosed by S.

$\vec{E} \cdot d\vec{s}$ — flux of $\vec{E}$ through ds.

$\oint_S \vec{E} \cdot d\vec{s}$ — Total outward flux ~~from~~.



Meaning:

① . $\nabla \cdot \vec{E}$ is a scalar.

② It's the total outward flux of $\vec{E}$ per unit volume.

③ If $\nabla \cdot \vec{E} > 0$, it's a source.

If $\nabla \cdot \vec{E} < 0$, it's a sink.

④ . The magnitude of $\nabla \cdot \vec{E}$ reprensents the strength of the source (or sink).

e.g. For an electric field $\vec{E}$.

Gauss' Law: $\oint_S \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$.

then.

$$\nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{E} \cdot d\vec{s}}{\Delta V} = \lim_{\Delta V \to 0} \frac{\frac{Q}{\varepsilon_0}}{\Delta V} = \frac{\rho}{\varepsilon_0}.$$

ρ — charge density (charge per unit volume)

$+q$ — source of $\vec{E}$

$-q$ — sink of $\vec{E}$.

e.g: For a magnetic field. $\vec{B}$.

$\nabla \cdot \vec{B} = 0$. (4.1c Maxwell's equation).

why? Because there is no "magnetic charges". (monopoles)

- when $\nabla \cdot \vec{B} = 0$ everywhere,

We say that $\vec{B}$ is divergenceless (sourceless)

or <u>solenoidal</u>

└ the field lines go back to themselves. (never stop, never start at "charges").

- Gauss' theorem (The divergence theorem).

when the volume V is not small.

$$\int_V \nabla \cdot \vec{E}\, dV = \oint_S \vec{E} \cdot \vec{ds}$$

V — volume enclosed by surface S.

- calculation of divergence in different coordinate systems:

just plug in the formulas.

(need some practice).

e.g: 3-3. on page 55.

(a). $\vec{E} = \hat{x}\, 3x^2 + \hat{y}\, 2z + \hat{z}\, x^2 z$, find $\nabla \cdot \vec{E}$ at $(2, -2, 0)$
(x, y, z).

(b). $\vec{E} = \hat{R}\left(\frac{a^3 \cos\theta}{R^3}\right) - \hat{\theta}\left(\frac{a^3 \sin\theta}{R^2}\right)$, find $\nabla \cdot \vec{E}$ at $\left(\frac{a}{2}, 0, \pi\right)$
(R, θ, ϕ)

Next: 4-1, 2, 3, 4, 5