

Phys 221

Lecture 20.

Assignment #6.

Ref. 6-7*, 6-8, 6-9, 6-10.

6.1, 3, 5, 11, 18.

Time-dep. Maxwell's equations.

• Faraday's law: $\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$ $\nabla \times \vec{E} \neq 0$, because $\frac{\partial \vec{B}}{\partial t}$ contribute to $\nabla \times \vec{E}$.Question: Does $\frac{\partial \vec{E}}{\partial t}$ contribute to $\nabla \times \vec{B}$ as well?

Maxwell guess: it should!

Why?

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

(5.45) holds for magnetostatics

but can't be right when

fields and sources are

time-dependent

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J}$$

$$\frac{\partial \rho}{\partial t} \neq 0$$

 $\Downarrow$
0
(identity)

$$\Downarrow$$

 $-\frac{\partial \rho}{\partial t} \neq 0$

$$\Rightarrow \nabla \cdot \vec{J} \neq 0 \text{ since } \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

(charge conservation)

Then, (5.45) needs to be modified to allow $\frac{\partial \rho_v}{\partial t} \neq 0$.

How?

idea: replace $\nabla \cdot \vec{J}$ with $\nabla \cdot \vec{J} + \frac{\partial \rho_v}{\partial t} = \nabla \cdot \vec{J} + \nabla \cdot \frac{\partial \vec{E}}{\partial t} \cdot \epsilon_0$

$$(\because \nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0})$$

$$\therefore \nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{In media: } \nabla \times \vec{B} = \mu \vec{J} + \epsilon \mu \frac{\partial \vec{E}}{\partial t} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

$\frac{\partial \vec{D}}{\partial t}$ — displacement current density

(Based on a mechanical model — Don't worry about it)

$$\text{OR: } \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S}$$

(6.4)

• Potential functions (V, A) .

$$\text{Now } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \neq 0.$$

We can't define $\vec{E} = -\nabla V$ any more.

However, since $\nabla \cdot \vec{B} = 0$ is still valid.

we can still define $\vec{B} = \nabla \times \vec{A}$.

$$\text{Then, } \nabla \times \vec{E} = -\frac{\partial}{\partial t} (\nabla \times \vec{A})$$

$$\text{i.e. } \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

Now we can define V using.

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V$$

$$\text{i.e. } \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$\therefore \vec{E}$ can result from the spatial variations of V and/or the time variation of $\vec{A}$.

$\vec{E}$ and $\vec{B}$ are related via $\vec{A}$.

- What equations will V and $\vec{A}$ obey now?

(NOT Poisson's equation any more)

Since $\nabla \times \vec{B} = \mu \vec{J} + \epsilon \mu \frac{\partial \vec{E}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} + \epsilon \mu \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right)$$

$$\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \epsilon \mu \nabla \frac{\partial V}{\partial t} - \epsilon \mu \frac{\partial^2 \vec{A}}{\partial t^2}$$

i.e.

$$\nabla^2 \vec{A} - \epsilon \mu \frac{\partial^2 \vec{A}}{\partial t^2} = \mu \vec{J} + \nabla \left(\nabla \cdot \vec{A} + \epsilon \mu \frac{\partial V}{\partial t} \right)$$

In vacuum: $\nabla^2 \vec{A} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \nabla \left(\nabla \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial V}{\partial t} \right)$

Remember, $\vec{A}$ is not uniquely defined $\left(\begin{array}{l} \text{only require} \\ \nabla \cdot \vec{B} = 0 \end{array} \right)$

Now instead of choosing the ~~Coulomb~~ Coulomb

Gauge, we can choose the Lorentz Gauge:

$$\nabla \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial V}{\partial t} = 0$$

Then: $\nabla^2 \vec{A} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$

This is a wave equation (including 3 components), mathematically

well studied.

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Note: The Lorentz Gauge reduces to the Coulomb gauge.

When $\frac{\partial V}{\partial t} = 0$ (static case — for example).

• The wave equation for V .

$$\text{Since } \nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0}, \quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \cdot \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho_v}{\epsilon_0}$$

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$$

$$\text{Lorentz Gauge: } \nabla \cdot \vec{A} + \epsilon \mu \frac{\partial V}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{A} = -\epsilon \mu \frac{\partial V}{\partial t}$$

$$\therefore \nabla^2 V - \epsilon \mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon_0} \quad \left(\begin{array}{l} \text{when } \frac{\partial V}{\partial t} = 0 \\ \text{Lorentz Gauge becomes} \\ \text{Coulomb Gauge: } \nabla \cdot \vec{A} = 0 \end{array} \right)$$

• Both V and $\vec{A}$ can be described by wave equations.

$\vec{J}$ is the source of $\vec{A}$ wave,

ρ_v is the source of V wave.

Both waves travel at a speed of $c = \frac{1}{\sqrt{\epsilon_0 \mu}}$

$$\text{in vacuum: } c_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \times 10^8 \text{ m/s}$$

Note: It's the Lorentz gauge that separates $\vec{A}$ and V mathematically.

in

20-6.

• Source-free space, $\vec{J} = 0$, $\rho_v = 0$.

Wave Equations for $\vec{A}$ and V :

$$\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

$$\nabla^2 V - \epsilon\mu \frac{\partial^2 V}{\partial t^2} = 0.$$

General Wave equation: for any quantity $f(x, y, z, t)$

$$\nabla^2 f - \frac{1}{u^2} \cdot \frac{\partial^2 f}{\partial t^2} = 0 \quad \left(\begin{array}{l} \text{when there is} \\ \text{no source} \end{array} \right)$$

u — Speed of wave.

e.g: a plane wave: $f(x, t) = A \cos(\omega t - kx)$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} = -k^2 A \cos(\omega t - kx) \quad \text{OR} = A \cos[k(x - ut)] = A \cos(kx - \omega t)$$

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 A \cos(\omega t - kx)$$

$$\therefore -k^2 A \cos(\omega t - kx) - \frac{1}{u^2} (-\omega^2) A \cos(\omega t - kx)$$

$$= A (\cos \omega t - kx) \left(\frac{\omega^2}{u^2} - k^2 \right) = 0.$$

$$\left(\begin{array}{ll} \omega = 2\pi f = \frac{2\pi}{T}, & k = \frac{2\pi}{\lambda} \\ u = \frac{\lambda}{T} = \frac{\omega}{k} & \frac{\omega}{u} = k. \end{array} \right)$$