

Lecture 21

Ref: 7-1, 7-2, 7-3, 7-4.

 • $\vec{E}$ and $\vec{B}$ in source-free space ($\rho_v = 0$, $\vec{J} = 0$)

$$\left. \begin{aligned} \nabla \times \vec{E} &= - \frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \epsilon_r \mu_r \frac{\partial \vec{E}}{\partial t} \end{aligned} \right\} \vec{E} - \vec{B} \text{ coupling}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

Then,

$$\nabla \times (\nabla \times \vec{E}) = - \frac{\partial}{\partial t} (\nabla \times \vec{B}) = - \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

 $\Downarrow$

$$\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

 $\Downarrow$
0

$$\therefore \nabla^2 \vec{E} - \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\text{Similarly, } \nabla^2 \vec{B} - \epsilon \mu \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

 (1) $\vec{E}$ and $\vec{B}$ ~~are~~ obey "symmetric" equations.

 (2) $\vec{E}$ and $\vec{B}$ obey wave equations!

(They propagate as waves!)

• Plane wave propagation

e.g. $g(x, t) = A \cos(\omega t - \beta x + \phi_0)$

$$= A \cos\left(\frac{2\pi}{T} \cdot t - \frac{2\pi}{\lambda} x + \phi_0\right)$$

A - amplitude

T - period. $\omega = \frac{2\pi}{T}$ - angular frequency

$f = \frac{1}{T}$ - frequency ($\omega = 2\pi f$)

λ - wave length

$\beta = \frac{2\pi}{\lambda}$ - wave number
(phase constant)

ϕ_0 - reference phase

$(\omega t - \beta x + \phi_0) = \phi(x, t)$ - phase.

This wave propagates along x -direction.

(if it were $\omega t + \beta x$, then it travels in $-x$ direction)

"plane" wave: $g = \text{const}$ in y - z plane $\Rightarrow g(x, t)$
(No y - or z -dependence)

- Surface of constant phase:

$$(\omega t - \beta x + \phi_0) = \text{const.} \quad (y-z \text{ planes})$$

- phase velocity: The travelling speed of constant-phase surface.

$$\frac{d}{dt} (\omega t - \beta x + \phi_0) = 0$$

$$\omega - \beta \frac{dx}{dt} = 0$$

$$u_p = \frac{dx}{dt} = \frac{\omega}{\beta} = \frac{2\pi f \lambda}{2\pi} = f \lambda = \frac{\lambda}{T}$$

$$\left(\begin{array}{l} \text{It takes } T \text{ seconds for the wave} \\ \text{to travel } \lambda \text{ metres} \end{array} \right)$$

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

- In a lossy medium:

$$g(x, t) = A e^{-\alpha x} \cos(\omega t - \beta x + \phi_0)$$

α — attenuation constant.

$e^{-\alpha x}$ — attenuation factor.

• phasor

— a convenient math ~~form~~ descriptions
for time-harmonic functions

↳ Sinusoidal

If: $z(t) = \operatorname{Re} [\tilde{z} e^{j\omega t}]$

Then $\tilde{z}$ — phasor ($\tilde{z}$ — doesn't have t)
~~($\tilde{z}$ — doesn't have t)~~

e.g: $z(t) = A \cos \omega t$, $\tilde{z} = A$

$z(t) = A \cos(\omega t + \phi_0)$, $\tilde{z} = A e^{j\phi_0}$

$z(t) = A \cos(\omega t + \beta x + \phi_0)$, $\tilde{z} = A e^{j(\beta x + \phi_0)}$

$z(t) = e^{-\alpha x} \cos(\omega t + \beta x + \phi_0)$, $\tilde{z} = A e^{-\alpha x} e^{j(\beta x + \phi_0)}$
 $= A e^{-\alpha x + j(\beta x + \phi_0)}$

$\frac{d}{dt} [z(t)]$

$j\omega \tilde{z}$

(differentiation
→ multiplication by $j\omega$)

$\int z(t) dt$

$\frac{1}{j\omega} \tilde{z}$

(Integration
→ division by $j\omega$)

in time domain

in phasor domain.

- Time-harmonic function is very important in Solving ac circuits and waves.

— if the source is a harmonic function of Time with a frequency ω

Then the steady-state solutions will be a harmonic function with the same frequency ω .

— Any time-dependent quantity (e.g., source) can be expressed in terms of the sum of a set of time-harmonic functions (Fourier Series).

— The wave equation and Maxwell's equations are linear equations. So the linear combination of different solutions is still a solution.

(Therefore, we can deal with one frequency at a time, and ~~then~~ combine ~~the back~~ the solutions to get the final solution) .

• Time-harmonic fields

$$\vec{E}(x, y, z, t) = \text{Re} \left[\tilde{\vec{E}}(x, y, z) e^{j\omega t} \right]$$

$$\vec{H}(x, y, z, t) = \text{Re} \left[\tilde{\vec{H}}(x, y, z) e^{j\omega t} \right]$$

Maxwell's equations: (for the phasors)

$$\left\{ \begin{array}{l} \nabla \cdot \tilde{\vec{E}} = \tilde{\rho}_v / \epsilon \\ \nabla \times \tilde{\vec{E}} = -j\omega \mu \tilde{\vec{H}} \\ \nabla \cdot \tilde{\vec{H}} = 0 \\ \nabla \times \tilde{\vec{H}} = \tilde{\vec{J}} + j\omega \epsilon \tilde{\vec{E}} \end{array} \right. \quad \begin{array}{l} (7.57) \\ p. 209 \end{array}$$

• If $\tilde{\rho}_v = 0$, $\tilde{\vec{J}} = 0$ ~~($\tilde{\rho}_v = 0$)~~

Then:

$$\left. \begin{array}{l} \nabla \cdot \tilde{\vec{E}} = 0 \\ \nabla \times \tilde{\vec{E}} = -j\omega \mu \tilde{\vec{H}} \\ \nabla \cdot \tilde{\vec{H}} = 0 \\ \nabla \times \tilde{\vec{H}} = j\omega \epsilon \tilde{\vec{E}} \end{array} \right\} \begin{array}{l} \text{No charge, } \rho_v = 0 \\ \text{lossless, } \sigma = 0 \\ (\tilde{\vec{J}} = 0) \end{array}$$

- In a lossy medium, $\sigma \neq 0$, not perfect insulator
 $\sigma \neq \infty$, not perfect conductor.

Then $\vec{J} \neq 0$ if $\vec{E} \neq 0$. Since $\vec{J} = \sigma \vec{E}$.

(The medium will consume energy)

$$\begin{aligned} \text{Then: } \nabla \times \vec{H} &= (\sigma + j\omega\epsilon) \vec{E} \\ &= j\omega \left(\frac{\sigma}{j\omega} + \epsilon \right) \vec{E} \\ &= j\omega \left(\epsilon - j\frac{\sigma}{\omega} \right) \vec{E} \end{aligned}$$

can be written as:

$$\nabla \times \vec{H} = j\omega\epsilon_c \vec{E} \quad (7.63)$$

$$\begin{aligned} \epsilon_c &\equiv \epsilon - j\frac{\sigma}{\omega} \quad \text{--- complex permittivity.} \\ &\equiv \epsilon' - j\epsilon'' \end{aligned}$$

$$\epsilon' = \epsilon$$

$$\epsilon'' = \frac{\sigma}{\omega}.$$

For lossless medium: $\sigma = 0$, $\epsilon_c = \epsilon$