

Lecture 22, 23, 24

- Time-harmonic fields

$$\vec{E}(x, y, z, t) = \text{Re} \left[\tilde{\vec{E}}(x, y, z) e^{j\omega t} \right]$$

$$\vec{H}(x, y, z, t) = \text{Re} \left[\tilde{\vec{H}}(x, y, z) e^{j\omega t} \right]$$

varies
(need to work on)

always the same.

- Charge-free, lossless Maxwell's equations:

$$(\rho_v = 0, \quad \vec{J} = 0)$$

$$\sigma = 0$$

$$\nabla \cdot \tilde{\vec{E}} = 0$$

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

$$\nabla \cdot \tilde{\vec{H}} = 0$$

$$\nabla \times \tilde{\vec{H}} = j\omega\epsilon \tilde{\vec{E}}$$

- Charge-free, lossy medium $\left\{ \begin{array}{l} \sigma \neq 0, \quad \rho_v = 0 \\ \text{but } \underline{\vec{J}} = \sigma \tilde{\vec{E}} \neq 0 \end{array} \right.$

$$\nabla \cdot \tilde{\vec{E}} = 0$$

$$\nabla \times \tilde{\vec{E}} = -j\omega\mu \tilde{\vec{H}}$$

$$\nabla \cdot \tilde{\vec{H}} = 0$$

$$\nabla \times \tilde{\vec{H}} = j\omega\epsilon_c \tilde{\vec{E}}$$

$$\epsilon_c = \epsilon' - j\epsilon''$$

$$\epsilon' = \epsilon, \quad \epsilon'' = \frac{\sigma}{\omega}$$

- Wave equations for charge-free medium.

$$\nabla \times \nabla \times \tilde{\mathbf{E}} = \nabla \times (-j\omega\mu \tilde{\mathbf{H}})$$

$$\nabla(\nabla \cdot \tilde{\mathbf{E}}) - \nabla^2 \tilde{\mathbf{E}} = -j\omega\mu \nabla \times \tilde{\mathbf{H}}$$

$$\downarrow$$

$$0 - \nabla^2 \tilde{\mathbf{E}} = \omega^2 \mu \epsilon_c \tilde{\mathbf{E}}$$

$$\therefore \nabla^2 \tilde{\mathbf{E}} + \omega^2 \mu \epsilon_c \tilde{\mathbf{E}} = 0 \quad (7.68)$$

(p.210)

define : propagation constant γ :

$$\gamma^2 \triangleq -\omega^2 \mu \epsilon_c$$

Then, wave equation's for $\tilde{\mathbf{E}}$:

$$\boxed{\nabla^2 \tilde{\mathbf{E}} - \gamma^2 \tilde{\mathbf{E}} = 0} \quad (7.70)$$

Similarly, wave equation's for $\tilde{\mathbf{H}}$:

$$\nabla^2 \tilde{\mathbf{H}} - \gamma^2 \tilde{\mathbf{H}} = 0 \quad (7.71)$$

Note :

How to solve (7.70) ?

① depends on boundary. (shape, properties.)

② by guessing: e.g. $\begin{cases} (e^{-j\gamma x})' = -j\gamma e^{-j\gamma x} \\ (e^{-j\gamma x})'' = -\gamma^2 e^{-j\gamma x} \end{cases}$

- Lossless media : $\sigma = 0$, $\epsilon'' = 0$,
 $\epsilon_c = \epsilon' = \epsilon$.
 $\gamma^2 = -\omega^2 \mu \epsilon$.

define : wavenumber k : $k = \omega \sqrt{\mu \epsilon}$. (7.73)

$$(k^2 = -\gamma^2)$$

Then, wave equations :

$$\nabla^2 \vec{\tilde{E}} + k^2 \vec{\tilde{E}} = 0 \quad (7.74)$$

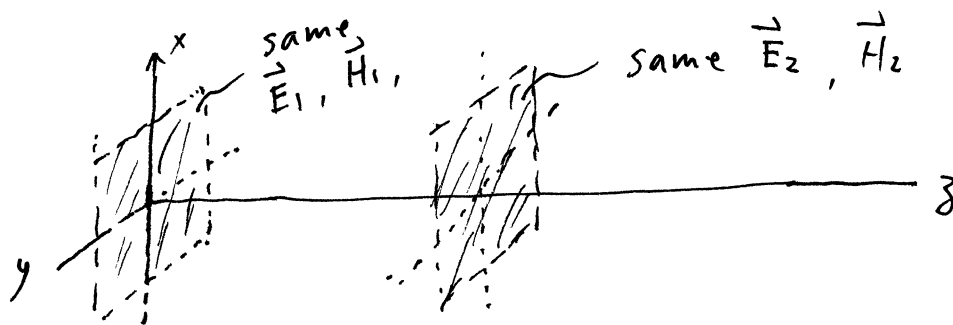
$$\nabla^2 \vec{\tilde{H}} + k^2 \vec{\tilde{H}} = 0$$

Called Helmholtz's equations .
 (well studied)

- Plane waves in lossless media :

$\vec{E}$ — same at every point on a plane . ($\vec{H}$ too) .

e.g. $\vec{E}$ and $\vec{H}$ — (uniform on x-y plane) ind. of (x,y) .



Then, $\tilde{E}(x, y, z) = \tilde{E}(z)$

$\tilde{H}(x, y, z) = \tilde{H}(z)$

Wave equations: $\frac{d^2 \tilde{E}}{dz^2} + k^2 \tilde{E} = 0$

$\left(\frac{d^2 \tilde{H}}{dz^2} + k^2 \tilde{H} = 0 \right)$

Solutions: $\tilde{E}_0^- e^{jkz}$

OR $\tilde{E}_0^+ e^{-jkz}$

$\tilde{E}^-(z)$

$\tilde{E}^+(z)$ ($\tilde{E}_0^+$ constant - (No $-z$)

↑
propagates in $-z$ direction

↑ propagates in $+z$ direction

$\tilde{E}_0^+ = \tilde{E}_0^+ e^{j\phi_0}$

$\tilde{E}^+(z) = \tilde{E}_0^+ e^{-jkz}$

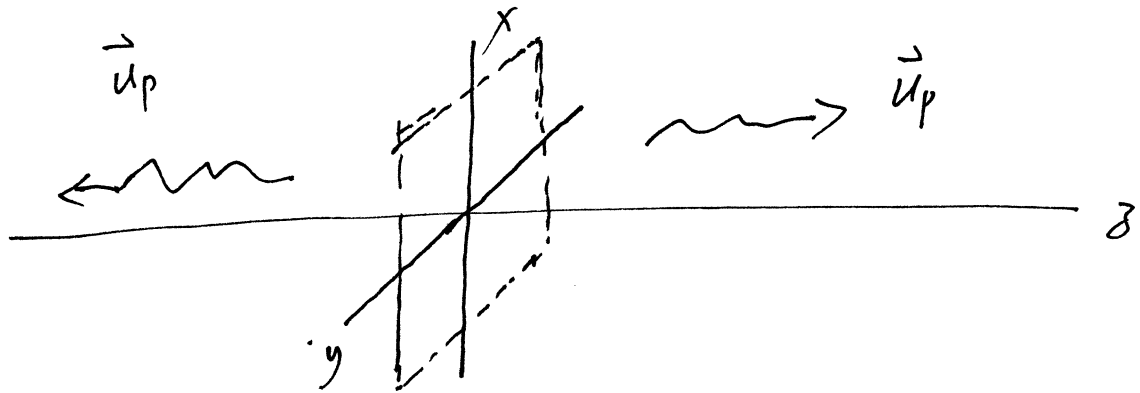
$\vec{E}^+(z, t) = \text{Re} \left[\tilde{E}_0^+ e^{-jkz} \cdot e^{j\omega t} \right] = \vec{E}_0^+ \cos(\omega t - kz + \phi_0^+)$

$\vec{E}^-(z, t) = \vec{E}_0^- \cos(\omega t + kz + \phi_0^-)$

~~Simplified notation: $\vec{E}(z, t) = \vec{E}_0 \cos(\omega t - kz + \phi_0)$~~

General solution: $\vec{E}(z, t) = \vec{E}_0^+ \cos(\omega t - kz + \phi_0^+) + \vec{E}_0^- \cos(\omega t + kz + \phi_0^-)$

ie, it's a wave propagating in $+z$ or $-z$ directions ²⁴⁻⁵.



• Direction of $\vec{E}$? direction of $\vec{H}$? (in x-y plane).
 (It turns out $(\vec{E} \times \vec{H}) \parallel \vec{u}_p$)

How do we know?

$$\nabla \times \vec{H} = j\omega \epsilon \vec{E}$$

$$(\nabla \times \vec{H})_z = j\omega \epsilon E_z$$

$$\frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} = j\omega \epsilon \tilde{E}_z$$

$$\downarrow \quad \downarrow$$

$$0 \quad 0$$

(No x-y dependence)

$$\therefore \tilde{E}_z = 0 \Rightarrow \vec{E} \perp \hat{z}, \text{ i.e. } \vec{E} \perp \vec{u}_p$$

$$\text{Similarly, } \tilde{H}_z = 0 \Rightarrow \vec{H} \perp \hat{z}, \text{ i.e. } \vec{H} \perp \vec{u}_p$$

$$(\text{since } \nabla \times \vec{E} = -j\omega \mu \vec{H})$$

e.g. When $\vec{E}$ is along x-axis, we say that

the E & M wave is polarized along x-direction.

if it travels along +z direction :

$$\vec{\tilde{E}}(z) = \hat{x} E_0 e^{-j k z}$$

Then, what about $\vec{H}$?

$$\nabla \times \vec{\tilde{E}} = -j\omega\mu \vec{\tilde{H}}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 e^{-jkz} & 0 & 0 \end{vmatrix} = -j\omega\mu (\hat{x} \tilde{H}_x + \hat{y} \tilde{H}_y + \hat{z} \tilde{H}_z)$$

$$\hat{x} 0 - \hat{y} \left[0 - \frac{\partial}{\partial z} (E_0 e^{-jkz}) \right] + \hat{z} \left[0 - \frac{\partial}{\partial y} (E_0 e^{-jkz}) \right] = -j\omega\mu \vec{\tilde{H}}$$

$$\hat{y} (-jk E_0 e^{-jkz}) = -j\omega\mu (\hat{x} \tilde{H}_x + \hat{y} \tilde{H}_y + \hat{z} \tilde{H}_z)$$

$$\therefore \tilde{H}_x = 0, \quad \tilde{H}_z = 0$$

$$-jk E_0 e^{-jkz} = -j\omega\mu \tilde{H}_y$$

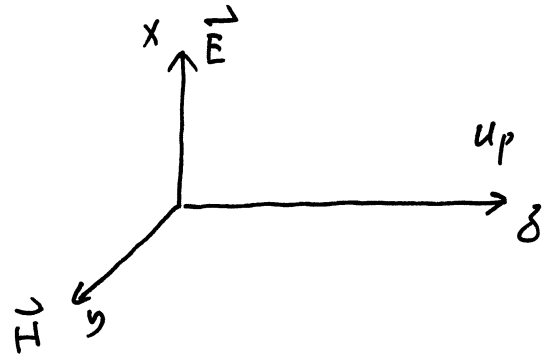
$$\therefore \tilde{H}_y = \frac{k}{\omega\mu} E_0 e^{-jkz}$$

$$\text{i.e. } \vec{\tilde{H}} = \hat{y} \frac{k}{\omega\mu} E_0 e^{-jkz}$$

$$\therefore (\vec{E} \times \vec{H}) \parallel \vec{u}_p.$$

Transverse wave.

(for $\vec{E}$ and $\vec{H}$)



$$\bullet \quad \frac{E}{H} = \frac{\omega \mu}{k} \equiv \eta \quad \text{— intrinsic impedance}$$

$$\text{since } k = \omega \sqrt{\mu \epsilon},$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}}.$$

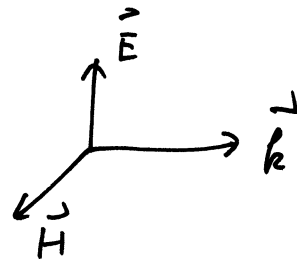
$$\text{In vacuum: } \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \, \Omega.$$

$$\bullet \quad \text{Wave vector } \vec{k} : \quad k = |\vec{k}| = \omega \sqrt{\mu \epsilon} = \frac{2\pi}{\lambda}$$

$$\hat{k} \parallel \vec{u}_p \quad (\text{direction of propagation})$$

$$\text{Then: } \vec{H} = \frac{1}{\eta} \hat{k} \times \vec{E}$$

$$\vec{E} = -\eta \hat{k} \times \vec{H}$$



• Polarization.

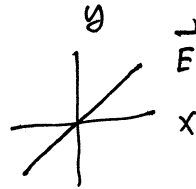
In general, for a plane wave propagating in z direction,
 $\vec{E}$ can have x - and y - components.

$$\vec{E}(z) = (\hat{x} a_x + \hat{y} a_y e^{j\delta}) e^{-jkz}$$

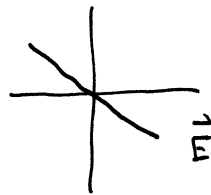
There may be a phase difference δ between the two components.

[Note: If there is only one component — linearly polarized]

① If $\delta = 0$, in phase



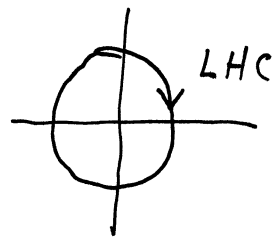
If $\delta = \pi$, out of phase



both are linearly polarized.

② If $\delta = \pm \frac{\pi}{2}$, and $a_x = a_y$.

Then, circularly polarized.



+ : Left handed

- : RHC.

(see page 219. figure 7-15 for the Rules).

③ All others: elliptically polarized.

see figure 7-19 on page 224 for a summary.