

Assignment #3. Solutions.

4.15. Find $\vec{E}$ at $P(0, 0, h)$ due to a circular disk charge in the x - y plane with a surface charge density $P_s = P_0 r^2$

$$\vec{E} = \hat{z} E_z \quad (\text{by symmetry})$$

$$E_z = \frac{1}{4\pi\epsilon_0} \int_0^a \frac{P_s 2\pi r dr \cdot \cos\theta}{r^2 + h^2}$$

$$= \frac{1}{2\epsilon_0} \int_0^a \frac{P_0 r^3 h dr}{(r^2 + h^2)^{3/2}}$$

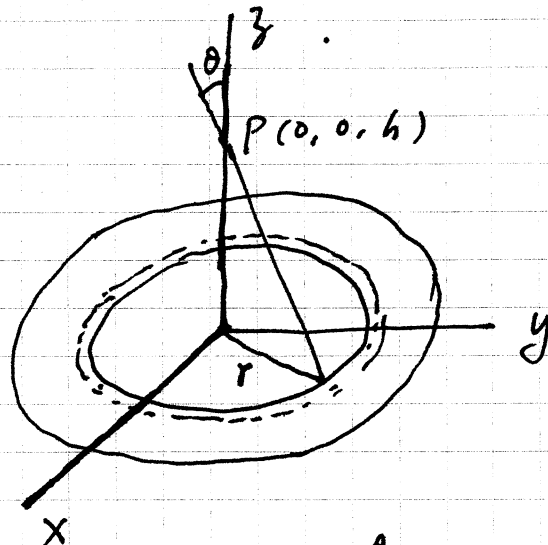
$$= \frac{P_0 h}{2\epsilon_0} \int_0^a \frac{r^3 dr}{(r^2 + h^2)^{3/2}}$$

$$= \frac{P_0 h}{4\epsilon_0} \int_{h^2}^{a^2+h^2} \frac{(u - h^2) du}{u^{3/2}}$$

$$= \frac{P_0 h}{4\epsilon_0} \left[2u^{1/2} + 2h^2 u^{-1/2} \right]_{h^2}^{a^2+h^2}$$

$$= \frac{P_0 h}{2\epsilon_0} \left[\sqrt{a^2+h^2} - h + \frac{h^2}{\sqrt{a^2+h^2}} - h \right]$$

$$\therefore \vec{E} = \hat{z} \frac{P_0 h}{2\epsilon_0} \left(\sqrt{a^2+h^2} + \frac{h^2}{\sqrt{a^2+h^2}} - 2h \right)$$



$$\cos\theta = \frac{h}{\sqrt{h^2 + r^2}}$$

let $u = r^2 + h^2$. When $r=0$, $u=h^2$.
 $du = 2r dr$

When $r=a$, $u=a^2+h^2$

4.25 · Gauss's law.

$$\oint \vec{D} \cdot d\vec{s} = Q$$

By symmetry · $\vec{D} = \hat{r} D_r$.On side of G.S.: $D_r = \text{const.}$ On top and bottom of G.S.: $\vec{D} \cdot d\vec{s} = 0$.

$$\therefore \int_{\text{lateral}} \vec{D} \cdot d\vec{s} = Q$$

$$D_r \cdot 2\pi r \cdot h = Q$$

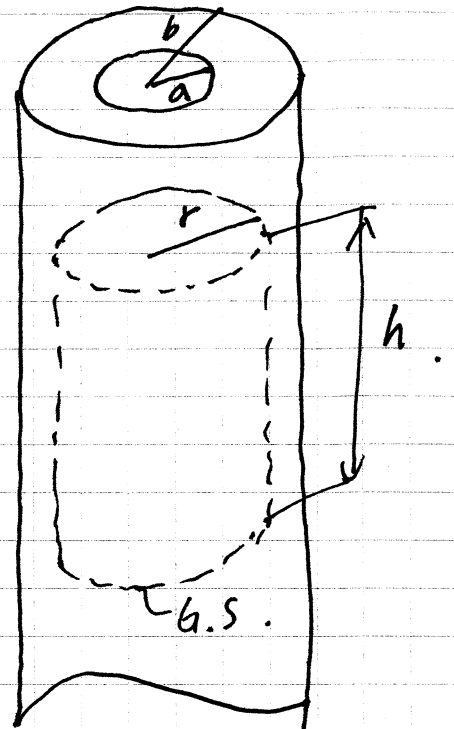
$$D_r = \frac{Q}{2\pi r h}$$

$$Q = \pi(r^2 - a^2) h \cdot \rho_{vo}$$

$$r < a: Q = 0 \quad \vec{D} = 0$$

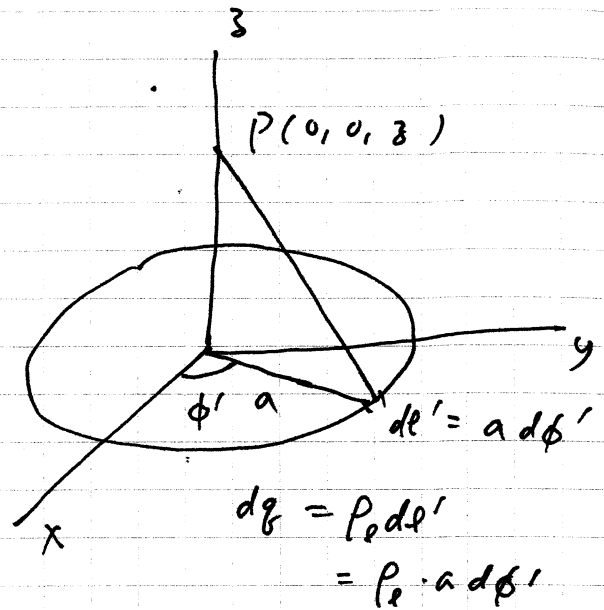
$$a < r < b: Q = \pi(r^2 - a^2) h \cdot \rho_{vo}, \quad \vec{D} = \hat{r} \frac{\rho_{vo}(r^2 - a^2)}{2r} = \hat{r} \frac{\rho_{vo}(r^2 - 1)}{2r}$$

$$r > b: Q = \pi(b^2 - a^2) h \rho_{vo}, \quad \vec{D} = \hat{r} \frac{\rho_{vo}(b^2 - a^2)}{2r} = \hat{r} \frac{4\rho_{vo}}{r}$$



4.29. Assume: $V=0$ at $r \rightarrow \infty$

$$\begin{aligned}
 a) \quad V &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\rho_e a \cdot d\phi'}{\sqrt{a^2 + z^2}} \\
 &= \frac{1}{4\pi\epsilon_0} \frac{\rho_e a}{\sqrt{a^2 + z^2}} \int_0^{2\pi} d\phi' \\
 &= \frac{\rho_e \cdot a}{2\epsilon_0 \sqrt{a^2 + z^2}}
 \end{aligned}$$



b) by symmetry: $\vec{E} = \hat{z} E_z = -\hat{z} \frac{\partial V}{\partial z}$

$$\vec{E}(0, 0, z) = -\hat{z} \frac{\partial V(0, 0, z)}{\partial z}$$

$$= -\frac{\rho_e a}{2\epsilon_0} \cdot \left(-\frac{1}{2}\right) \frac{2z}{(a^2 + z^2)^{3/2}} \cdot \hat{z}$$

$$= \hat{z} \frac{\rho_e a \cdot z}{2\epsilon_0 (a^2 + z^2)^{3/2}}$$

2(5/15 marks). An electric dipole consists of two charges q and $-q$ located at $(0, 0, d/2)$ and $(0, 0, -d/2)$ respectively.

(a) Determine the electric potential V at any point $P(x, y, z)$ in free space, given that P is far away from the dipole.

(b) Determine the electric field E (both magnitude and direction) at point P .

$$V = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{R_1} - \frac{q}{R_2} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \frac{R_2 - R_1}{R_1 R_2} \quad (V(\infty) = 0)$$

a). when $R \gg d$.

$$V \approx \frac{q}{4\pi\epsilon_0} \frac{d \cos \theta}{R^2}$$

OR:

$$V = \frac{q d \cdot z}{4\pi\epsilon_0 (x^2 + y^2 + z^2)^{3/2}}$$

b). $\vec{E} = -\nabla V$.

$$= - \left[\hat{R} \frac{\partial V}{\partial R} + \hat{\theta} \frac{1}{R} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi} \right]$$

$$= - \hat{R} \frac{\partial V}{\partial R} - \hat{\theta} \frac{1}{R} \frac{\partial V}{\partial \theta}$$

$$E_R = - \frac{\partial V}{\partial R} = \frac{q}{2\pi\epsilon_0} \frac{d \cos \theta}{R^3}$$

$$E_\theta = - \frac{1}{R} \frac{\partial V}{\partial \theta} = \frac{1}{R} \frac{q \cdot d \cdot \sin \theta}{4\pi\epsilon_0 R^2} = \frac{q d \cdot \sin \theta}{4\pi\epsilon_0 R^3}$$

$$E_\phi = 0$$

$$\therefore \vec{E} = \hat{R} \frac{q d \cos \theta}{2\pi\epsilon_0 R^3} + \hat{\theta} \frac{q d \sin \theta}{4\pi\epsilon_0 R^3} \quad (R \gg d)$$

